Instability of constant mean curvature surfaces of revolution in spherically symmetric spaces

Nahid Sultana

Abstract. We study the stability properties of constant mean curvature (CMC) surfaces of revolution in general simply-connected spherically symmetric 3-spaces, and in the particular case of a positive-definite 3-dimensional slice of Schwarzschild space. We derive their Jacobi operators, and then prove that closed CMC tori of revolution in such spaces are unstable, and finally numerically compute the Morse index of some minimal and closed non-minimal CMC surfaces of revolution in the slice of Schwarzschild space.

M.S.C. 2000: 53C42; 53A10, 53A35.
Key words: Morse index, spherically symmetric 3-space, Schwarzschild space.

1 Introduction

A spherically symmetric 3-manifold is one whose isometry group contains a subgroup which is isomorphic to the (rotation) group SO(3). These isometries are then interpreted as rotations, and so a spherically symmetric Riemannian manifold is often described as one whose metric is invariant under rotation.

The *Morse index* of a constant mean curvature (CMC) surface is defined as the sum of the multiplicities of the negative eigenvalues of its *Jacobi operator* ([21], [22] and [26]), and a CMC surface is *stable* when its area is minimal with respect to all variations preserving volume on each side of the surface. When it is stable, then the Morse index is ≤ 1. So conversely, to show that some such surface is unstable, it is sufficient to show its Morse index is ≥ 2.

We will study the Morse index and stability of CMC surfaces in spherically symmetric spaces, such spaces including Euclidean 3-space $\mathbb{R}^3$, the unit 3-sphere $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$ as special cases.

The index of both minimal and non-minimal CMC surfaces in $\mathbb{R}^3$ has been well studied. It is known that the only stable complete minimal surface is a plane [9], and minimal surfaces have finite index if and only if they have finite total curvature [12].

Balkan Journal of Geometry and Its Applications, Vol.13, No.1, 2008, pp. 120-133.

And the index for many minimal surfaces which have finite total curvature has been found ([10], [12], [17], [18]). It is also reported that a CMC surface is stable if and only if it is a round sphere [2], and CMC surfaces without boundary have finite index if and only if they are compact ([15], [24]). Furthermore, the Morse index of Wente tori is described in [19].

In $\mathbb{S}^3$, the totally geodesic spheres have index 1 ([23]). F. Urbano [26] proved that the minimal Clifford torus has index 5 and any other closed (compact without boundary) minimal surface has index ≥ 6. For the closed CMC surfaces in $\mathbb{S}^3$, a similar result to that in [26] is not yet known. But W. Rossman and the author [21] found relatively sharp lower bounds for the index of closed CMC surfaces of revolution, and a numerical method to compute the index of such surfaces is described in [22]. The index and stability of minimal and CMC surfaces in $\mathbb{H}^3$ have also been studied (see [7], [3], [14]).

However, the Morse index and stability properties of CMC surfaces in ambient spaces other than $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ have not yet been well studied. Therefore, the purpose of this paper is to study these for CMC surfaces of revolution in an ambient space which is spherically symmetric, and in one particular case that is related to general relativity. We will show that for a closed CMC torus the index is at least 2 and thus we have the following theorem:

Theorem 1.1. *Any closed immersed CMC torus of revolution (either minimal or non-minimal) in a spherically symmetric* 3*-space is unstable.*

Spherically symmetry is a characteristic of many solutions of Einstein's field equations of general relativity, including the Schwarzschild solution. Thus, as an application we consider a 3-dimensional positive-definite slice of the Schwarzschild space. We describe the surfaces of revolution in this space, and show graphics of some such minimal and non-minimal CMC surfaces. We also derive their Jacobi operators, and numerically compute the Morse index for some of those surfaces in Schwarzschild space.

2 Spherically symmetric spaces and their surfaces

We begin by considering Euclidean 3-space

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}\} ,$$

but with a metric

$$ds^2 = \lambda^2(dx_1^2 + dx_2^2 + dx_3^2) , \quad \lambda = \lambda(x_1, x_2, x_3) > 0$$

that is only conformal to the Euclidean metric $ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2$. Let $\mathrm{X}(x, y) : \Sigma \to (\mathbb{R}^3, ds^2)$ be an immersion, where (x, y) is a local coordinate chart of a 2-dimensional manifold Σ. The following well-known lemma gives the mean and Gauss curvatures of this surface in $(\mathbb{R}^3, ds^2)$, and we include a proof in the appendix (Section 8).

Lemma 2.1. *The mean curvature H and the Gauss curvature K of a surface in the ambient space $\mathbb{R}^3$ with metric $ds^2 = \lambda^2(dx_1^2 + dx_2^2 + dx_3^2)$ are given by*

$$H = \frac{\hat{H}}{\lambda} - \frac{D_{\hat{N}}\lambda}{\lambda^2} \quad \text{and} \quad K = \frac{\hat{K}}{\lambda^2} - 2\frac{\hat{H}D_{\hat{N}}\lambda}{\lambda^3} + \left(\frac{D_{\hat{N}}\lambda}{\lambda^2}\right)^2 \tag{2.1}$$

respectively, where

1. *(x_1, x_2, x_3) are the usual rectangular coordinates for $\mathbb{R}^3$,*
2. *the function $\lambda = \lambda(x_1, x_2, x_3) > 0$ is the metric factor,*
3. *$\hat{H}$ and $\hat{K}$ are the corresponding Euclidean mean and Gauss curvatures of the surface with respect to the Euclidean metric ds_0^2,*
4. *$D_{\hat{N}}\lambda$ is the derivative of λ with respect to $\hat{N}$, where $\hat{N}$ is the unit normal vector of the surface with respect to ds_0^2.*

We now define what it means for $(\mathbb{R}^3, ds^2)$ to be spherically symmetric, and we assume this symmetry throughout the remainder of this paper.

Definition 2.1. *The* 3*-manifold* $(\mathbb{R}^3, ds^2)$ *is called* spherically symmetric *if λ is spherically symmetric, that is, if*

$$\lambda(x_1, x_2, x_3) = \lambda(y_1, y_2, y_3)$$

whenever $x_1^2 + x_2^3 + x_3^2 = y_1^2 + y_2^3 + y_3^2$.

Definition 2.2. *Suppose $(\mathbb{R}^3, ds^2)$ is spherically symmetric, and suppose $X : \Sigma \to (\mathbb{R}^3, ds^2)$ is an immersion. If, after a possible change of coordinate of Σ and a possible rotation $\mathbb{R}^3 \ni X \mapsto AX \in \mathbb{R}^3$ for some $A \in SO(3)$, the surface X can be written as*

$$X(x, y) = (\phi(x)\cos y, \phi(x)\sin y, \psi(x))$$

for some C^∞ functions ϕ and ψ, then we call X a surface of revolution.

3 The Schwarzschild space

Coordinates for the Schwarzschild solution can be given in terms of the coordinates (r, θ, ϕ, t) in $\mathbb{R}^4$, where (r, θ, ϕ) are the usual spherical coordinates for Euclidean 3-space $\mathbb{R}^3$, and t is a coordinate in a time-like direction. In these coordinates, taking a positive constant m to represent mass, the Schwarzschild metric for the region $r > 2m$ is

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 - \left(1 - \frac{2m}{r}\right) dt^2 \ . \tag{3.1}$$

The above metric appears to have a singularity at $r = 2m$. However, this is not a true singularity. The Schwarzschild solution can be analytically continued through the surface $r = 2m$, and the sphere $r = 2m$ corresponds to the surface of a black hole.

Each element of the rotation group SO(3) for $\mathbb{R}^3$ induces an isometric motion of the Schwarzschild space. Namely, given $\psi \in$SO(3), a rigid motion $\bar{\psi}$ of Schwarzschild space-time may be defined by setting $\bar{\psi}(r, \theta, \phi, t) = (\psi(r, \theta, \phi), t)$. Thus for each fixed t, the Schwarzschild space-time is spherically symmetric.

Using the coordinate transformation $r = \rho\left(1 + \frac{m}{2\rho}\right)^2$, from Equation (3.1) we have the following isotropic form of the Schwarzschild metric in coordinates (ρ, θ, ϕ, t):

$$ds^2 = \left(1 + \frac{m}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2) - \frac{\left(1 - \frac{m}{2\rho}\right)^2}{\left(1 + \frac{m}{2\rho}\right)^2} dt^2 \quad .$$

The coordinates (ρ, θ, ϕ) are called the isotropic polar coordinates. The advantage of this form of the metric is that we can replace $d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2$ by the standard Euclidean metric on $\mathbb{R}^3$ in rectangular coordinates. This is useful when one considers the solar system as a 3-dimensional vector space, with some choice of metric and coordinates with respect to one fixed observer and one fixed time for that observer. The corresponding isotropic rectangular coordinates are obtained by setting $x_1 = \rho \sin\theta \cos\phi$, $x_2 = \rho \sin\theta \sin\phi$, $x_3 = \rho \cos\theta$, which gives

$$ds^2 = \left(1 + \frac{m}{2\rho}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) - \frac{\left(1 - \frac{m}{2\rho}\right)^2}{\left(1 + \frac{m}{2\rho}\right)^2} dt^2 \quad ,$$

with $x_1^2 + x_2^2 + x_3^2 = \rho^2$. Let us assume t is constant, and then we have

$$ds^2 = \left(1 + \frac{m}{2\rho}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) \quad . \tag{3.2}$$

The 3-dimensional Riemannian manifold $\mathcal{M}^3 = \mathbb{R}^3 \setminus \{\vec{0}\}$ with metric (3.2) is a 3-dimensional positive definite slice of Schwarzschild space.

4 Surfaces of revolution in Schwarzschild space

Let $\mathrm{X}(x, y) : \Sigma \to \mathcal{M}^3$ be a conformal immersion, where (x, y) is a local coordinate of a 2-dimensional manifold Σ, and assume $\mathrm{X}(x, y)$ is of the form

$$\mathrm{X}(x, y) = (\phi(x) \cos y, \phi(x) \sin y, \psi(x)) \tag{4.1}$$

for functions $\phi(x)$ and $\psi(x)$, where $\phi(x) > 0$, i.e. $\mathrm{X}(x, y)$ is a surface of revolution in the Schwarzschild space $\mathcal{M}^3$.

Using Equation (2.1), a direct computation gives the following proposition.

Proposition 4.1. *The mean curvature H and Gauss curvature K of the surface X as in Equation* (4.1) *are given by*

$$H = \frac{1}{\lambda\phi}\left(\frac{\phi - \phi''}{2\psi'} - \frac{m(\phi\psi' - \phi'\psi)}{\sqrt{\lambda}(\phi^2 + \psi^2)^{3/2}}\right) \tag{4.2}$$

and

$$K = \frac{1}{\lambda^2\phi^2}\left\{\frac{(\phi')^2 - \phi\phi''}{\phi^2} - \frac{m(\phi - \phi'')(\phi\psi' - \phi'\psi)}{\sqrt{\lambda}\psi'(\phi^2 + \psi^2)^{3/2}} + \frac{m^2(\phi\psi' - \phi'\psi)^2}{\lambda(\phi^2 + \psi^2)^3}\right\}, \tag{4.3}$$

respectively, where

$$\lambda = \lambda(x) = \left(1 + \frac{m}{2\sqrt{\phi^2 + \psi^2}}\right)^2 . \tag{4.4}$$

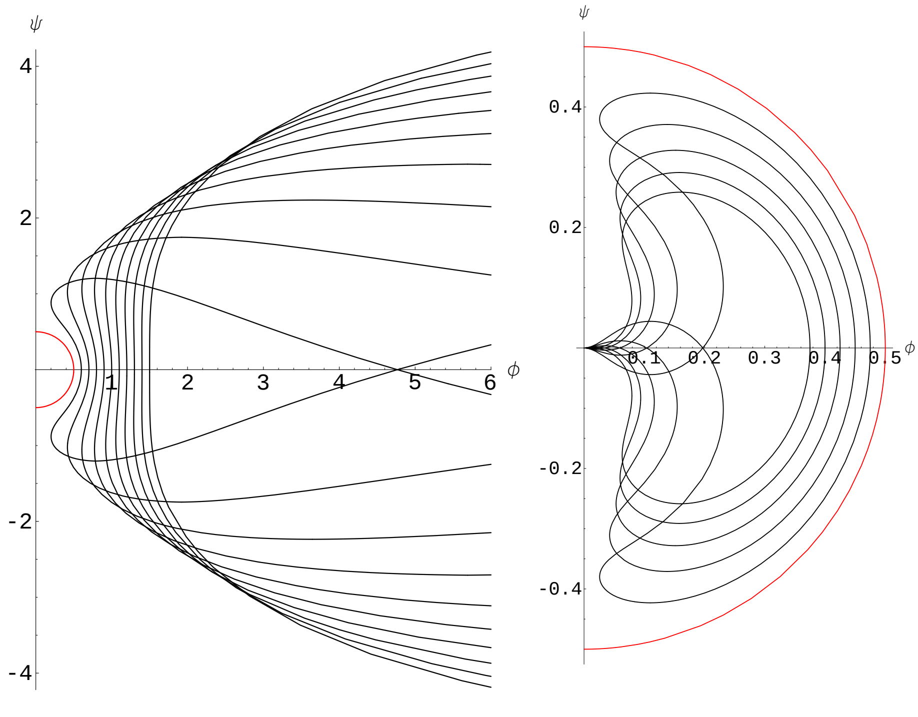

Figure 1: Profile curves of various minimal surfaces of revolution outside (left picture) and inside (right picture) the black hole.

We have numerically constructed some minimal and non-minimal CMC surfaces of revolution, and the profile curves of these surfaces are shown in Figure 1 and Figure 2, respectively. To find these CMC surfaces, we numerically solve the ordinary

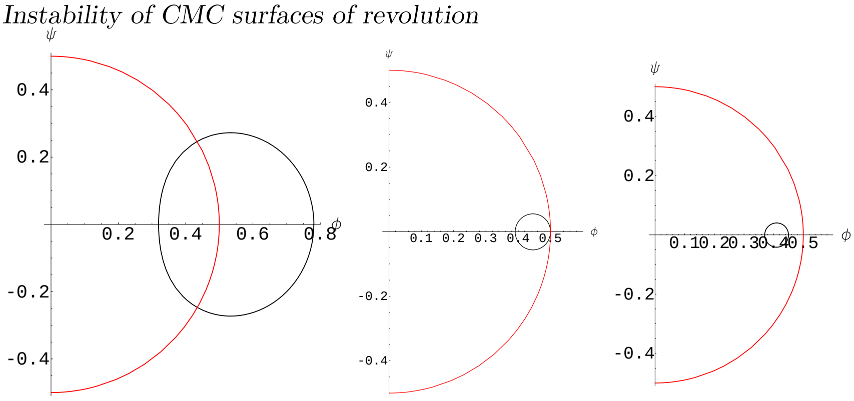

Figure 2: Profile curves of some closed non-minimal CMC surfaces of revolution. The left-most surface is partially inside and partially outside the black hole, the center surface is inside the black hole and touches the black hole's boundary, and the right-most surface is inside the interior of the black hole.

differential equation (ODE) (4.2) for various constant values of H. For this, we assume $m = 1$, and initial conditions $\phi(0) = c$ and $\phi'(0) = 0$, where c is a real constant. Here the profile curves are symmetric with respect to reflection across the x_1-axis in the x_1x_3-plane.

Remark 4.1. *When we take the constant $c = 0.5$, $m = 1$, and the mean curvature $H = 0$, then we have a surface whose profile curve is a half-circle with radius 0.5, and the resulting surface is the boundary of a black hole.*

5 The Jacobi operator and Morse index

Consider $(\mathbb{R}^3, ds^2)$ with $ds^2 = \lambda^2 ds_0^2$ conformally equivalent to the Euclidean metric ds_0^2. Suppose $\mathrm{X}(x, y) : \Sigma \to (\mathbb{R}^3, ds^2)$ is a conformal C^∞ immersion with mean curvature H and Gauss curvature K, respectively. When H is constant, X is critical for a variation problem ([1], [4], [6], [13], [24]) whose associated *Jacobi operator* is

$$\mathcal{L} = -\Delta - \left(2\,\mathrm{Ricc}(N) + 4H^2 - 2K\right), \tag{5.1}$$

where Δ is the Laplace-Beltrami operator of the induced metric $ds^2 = \rho(dx^2 + dy^2)$ for C^∞ function $\rho = \rho(x, y) : \Sigma \to \mathbb{R}^+$, and $\mathrm{Ricc}(N)$ is the (normalized) Ricci curvature of $(\mathbb{R}^3, ds^2)$ in the direction N (unit normal vector of the surface with respect to ds^2).

As $-\mathcal{L}$ is elliptic, it is well known ([5], [8], [16], [23], [25]) that the eigenvalues are real, discrete with finite multiplicities, and diverge to $+\infty$.

Definition 5.1. *The* index *$Ind(X)$ of X is the sum of the multiplicities of the negative eigenvalues of $\mathcal{L}$.*

Define $\hat{\mathcal{L}} = \rho\mathcal{L} = -\partial_x\partial_x - \partial_y\partial_y - \rho(2\,\mathrm{Ricc}(N) + 4H^2 - 2K)$, then like as for $\mathcal{L}$, the eigenvalues of $\hat{\mathcal{L}}$ form a discrete sequence

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \uparrow +\infty$$

whose first eigenvalue λ_1 is simple, and the corresponding eigenfunctions

$$\phi_1, \phi_2, \phi_3, \ldots \in C^\infty(\Sigma)\ , \quad \hat{\mathcal{L}}\phi_j = \lambda_j\phi_j\ , \quad j = 1, 2, 3, \ldots\ ,$$

can be chosen to form an orthonormal basis for the standard L^2 norm over Σ with respect to the Euclidean metric $dx^2 + dy^2$. $\hat{\mathcal{L}}$ and $\mathcal{L}$ will have different eigenvalues, but by Rayleigh quotient characterizations ([5], [16], [25]) for the eigenvalues, we know that these two operators will give the same index Ind(X).

6 The Jacobi operator and Morse index for surfaces of revolution in spherically symmetric spaces

We now additionally suppose that $(\mathbb{R}^3, ds^2)$ is spherically symmetric, and that $\mathrm{X}(x, y) : \Sigma \to (\mathbb{R}^3, ds^2)$ is a surface of revolution of the form

$$\mathrm{X} = (\phi(x)\cos y, \phi(x)\sin y, \psi(x))\ .$$

Any C^∞ function $f = f(x, y) : \Sigma \to \mathbb{R}$ can be decomposed into a series of spherical harmonics as follows:

$$f = \sum_{j=0}^{\infty} f_{j,1}(x)\cos(jy) + f_{j,2}(x)\sin(jy)\ , \tag{6.1}$$

where $f_{j,1}, f_{j,2}$ are functions of x only. We define the operator $\hat{\mathcal{L}}_0$ with the function space of C^∞ real-valued functions depending only on x by

$$\hat{\mathcal{L}}_0 = -\partial_x\partial_x - \rho(2\,\mathrm{Ricc}(N) + 4H^2 - 2K)\ , \tag{6.2}$$

and the spectrum

$$\lambda_{1,0} < \lambda_{2,0} \leq \lambda_{3,0} \leq \ldots \uparrow +\infty$$

of $\hat{\mathcal{L}}_0$ has again all the same properties as those for $\hat{\mathcal{L}}$. Furthermore, by uniqueness of the spherical harmonics decomposition, the following lemma holds ([22]):

Lemma 6.1. *We have* $Ind(X) = \sum_{j\in\mathbb{N}} \lambda_{j,0} \cdot \ell(\lambda_{j,0})$, *where*

$$\ell(\lambda) = \begin{cases} 0 & \text{if } \lambda \geq 0\ , \\ 2i - 1 & \text{if } \lambda \in [-i^2, -(i-1)^2) \text{ for } i \in \mathbb{N}\ . \end{cases}$$

In the special case that $(\mathbb{R}^3, ds^2)$ equals the Schwarzchild space $\mathcal{M}^3$, the mean curvature H and Gauss curvature K of X are given by Equations (4.2) and (4.3), respectively, and H is constant. Furthermore, the term $\text{Ricc}(N)$ in $\mathcal{L}$ satisfies

$$\text{Ricc}(N) = -\frac{m}{2\lambda^3\phi^2(\phi^2+\psi^2)^{5/2}}\left\{-\phi^2(\phi^2+\psi^2)+3(\phi'\psi-\phi\psi')^2\right\} . \tag{6.3}$$

We will return to this special case later.

7 Instability of CMC tori of revolution in spherically symmetric spaces

In this section, we prove our primary result (Theorem 1.1), and then compute the spectra of the Jacobi operator $\hat{\mathcal{L}}_0$ for minimal annuli of revolution and closed non-minimal CMC tori of revolution when $(\mathbb{R}^3, ds^2)$ is the slice of Schwarzchild space $\mathcal{M}^3$.

Lemma 7.1. *Let X be a closed immersed CMC torus of revolution in a spherically symmetric* 3*-space* $(\mathbb{R}^3, ds^2)$*. Then* -1 *is an eigenvalue of the operator* $\hat{\mathcal{L}}_0$ *and* $\lambda_{1,0} < -1$ *and Ind(X)* ≥ 2*.*

Proof. We may take

$$\text{X} = (\phi(x)\cos y, \phi(x)\sin y, \psi(x)) ,$$

and the unit normal vector is

$$N = \frac{1}{\phi(x)\lambda}(-\psi'(x)\cos y, -\psi'(x)\sin y, \phi'(x)) .$$

Let $\mathcal{K} = (0, \psi(x), -\phi(x)\sin y)$ be the restriction of the Killing field, to this surface, produced from rotation about the x_1-axis, so then

$$f := \langle \mathcal{K}, N\rangle_{\mathcal{M}^3} = u(x)\cdot \sin y ,$$

with $u(x) = \frac{\lambda}{\phi(x)}(\phi(x)\phi'(x) + \psi(x)\psi'(x))$, satisfies $\hat{\mathcal{L}}(f) = 0$ ([4], [10], [21]). Because $\hat{\mathcal{L}}_0 = \hat{\mathcal{L}} + \partial_y\partial_y$, we have $\hat{\mathcal{L}}_0(u(x)) = -u(x)$, i.e. -1 is an eigenvalue of $\hat{\mathcal{L}}_0$. Then, because the nodal set of $u(x)$ disconnects Σ, and because any eigenfunction associated to $\lambda_{1,0}$ must have an empty nodal set, we conclude that $\lambda_{1,0} < -1$.

Thus $\hat{\mathcal{L}}_0$ has at least two negative eigenvalues, and therefore $\mathcal{L}$ does as well, so $\text{Ind}(\text{X}) \geq 2$. ∎

Because any stable surface X could only have index either 0 or 1, this proves Theorem 1.1.

Remark 7.1. *When* λ *is as in Equation* (4.4)*, numerical evidence suggests that closed minimal tori of revolution do not exist.*

A numerical method for computing the spectra of $\hat{\mathcal{L}}_0$ on the function space over a closed loop was reported in [22]. Using this method, we can find the negative eigenvalues of the operator $\hat{\mathcal{L}}_0$ for closed non-minimal CMC surfaces (Figure 2). Furthermore, we have an analogous numerical method for the function space on an interval with Dirichlet boundary conditions, which can be applied to compute the negative eigenvalues of the operator $\hat{\mathcal{L}}_0$ for a minimal surface outside the black hole (Figure 1), as follows:

> **Algorithm for computing spectra.** We can numerically solve the ODE $\hat{\mathcal{L}}_0(f) = \lambda f$ for f with the initial conditions $f(0) = 0$, $(\frac{d}{dx} f)(0) = 1$ by a numerical ODE solver ([22]), and search for the values of λ that give solutions f in the function space with Dirichlet boundary conditions. Such values of λ are amongst the $\lambda_{j,0}$. And the eigenspace associated to each eigenvalue λ is 1-dimensional ([22]). Any eigenfunction f corresponding to the j'th eigenvalue $\lambda_{j,0}$ of $\hat{\mathcal{L}}_0$ has exactly $j+1$ nodes ([22]). So the value of j is determined simply by counting the number of nodes of f. Because we can determine j, we will know when we have found all $\lambda_j \leq M$ for any given $M \in \mathbb{R}$.

After finding the negative eigenvalues, we can compute the Morse index of a surface by using Lemma 6.1. As examples, we do this for various closed non-minimal CMC surfaces C_1, C_2, C_3, C_4, C_5 shown in Figure 3 and minimal surfaces $M_1 \supseteq M_2 \supseteq M_3$, $M_4 \supseteq M_5 \supseteq M_6$ shown in Figure 4. Table 1 shows the numerical results for all of these surfaces.

surface	H	a	eigenvalues $\lambda_{i,0}$, $i = 1, 2, 3, ...$ for the operator $\hat{\mathcal{L}}_0$	numerical value for Ind(X)
C_1	0.5	-	-4.063 , -1, 0.5, 11.1 , 11.1	6
C_2	1	-	-16.015, -1, 1, 47, 47	10
C_3	1.5	-	-36.008 , -1, 1, 108, 108	14
C_4	2	-	-64.4929, -1, 1, 194, 194	18
C_5	2.5	-	-102.097, -1, 1, 309, 309	22
M_1	0	4.03	-0.668 ,-0.55, 1.1	2
M_2	0	2.8	-.185 ,0.14, 2.26	1
M_3	0	1.8	0.98, 2.72, 6	0
M_4	0	5	-0.6397 ,-0.255, 0.66	2
M_5	0	3.1	-0.543 , 0.02, 1.83	1
M_6	0	1.8	0.1, 1.95, 6	0

Table 1: Numerical estimates for the eigenvalues of the operator $\hat{\mathcal{L}}_0$ and index for the surfaces $C_1, C_2, C_3, C_4, C_5, M_1, M_2, M_3, M_4, M_5$ and M_6.

For the minimal surfaces M_1, M_2 and M_3, we take x in the interval $[-a, a]$ where $a > 0$, for three different values of a and use the same initial conditions to solve the ODE (4.2), i.e. we only change the size of the domain, and we can then verify that Lemma 7.2 is satisfied, as must be the case. We can also check this property (Lemma 7.2) for the minimal surfaces M_4, M_5 and M_6 (see Table 1).

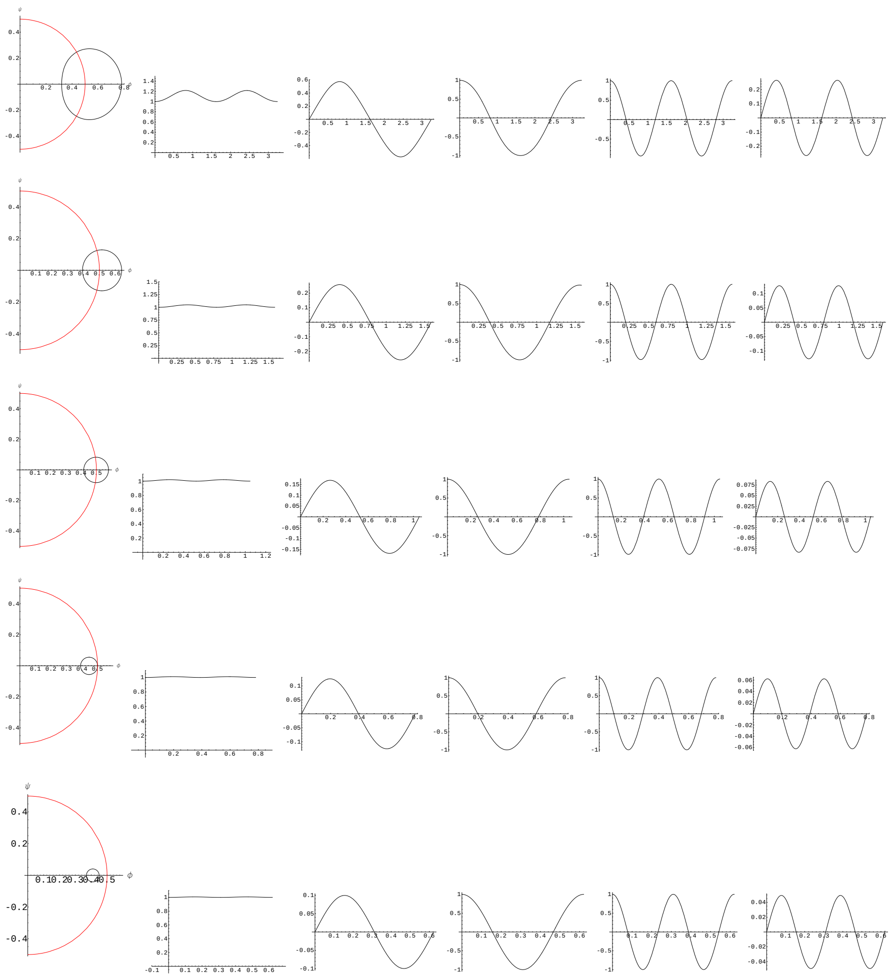

Figure 3: Profile curves (1st column) and the eigenfunctions (2nd, 3rd, 4th, 5th and 6th columns) associated to the eigenvalues $\lambda_{i,0}$, $i = 1, 2, 3, 4, 5$ of the surfaces C_1, C_2, C_3, C_4 and C_5.

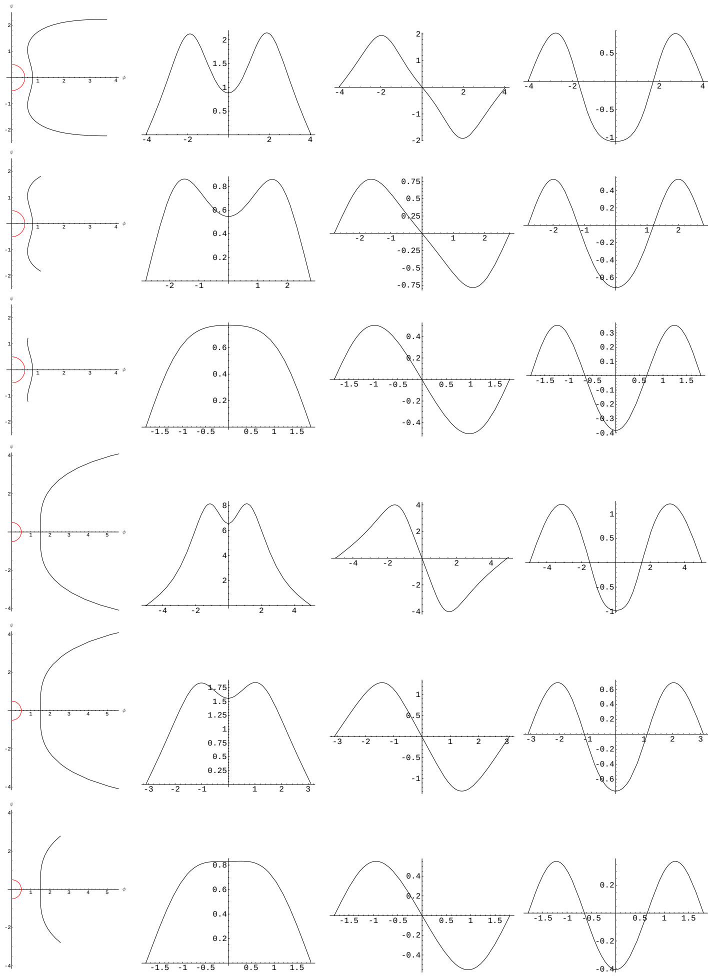

Figure 4: Profile curves (1st column) and the eigenfunctions (2nd, 3rd and 4th columns) associated to the eigenvalues $\lambda_{i,0}$, $i = 1, 2, 3$ of the surfaces $M_1, M_2, ..., M_6$. Note that $M_3 \subseteq M_2 \subseteq M_1$ and $M_6 \subseteq M_5 \subseteq M_4$.

Lemma 7.2. *If the domain Σ increases in size, then the eigenvalues $\lambda_{j,0}$ either decrease or stay the same.*

Proof. Let $\Sigma_1 \subseteq \Sigma_2$ be two domains with boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$. We will show that for all j,

$$\lambda_{j,0}(\Sigma_1) \geq \lambda_{j,0}(\Sigma_2) \ .$$

Let $\lambda_{1,0}(\Sigma_1) < \lambda_{2,0}(\Sigma_1) \leq ... \leq \lambda_{j-1,0}(\Sigma_1)$ be the first $j-1$ eigenvalues of the operator $\hat{\mathcal{L}}_0$ with corresponding orthonormal eigenfunctions $\phi_1, \phi_2, ..., \phi_{j-1}$ defined on Σ_1, and set $\mathcal{V}^{\circ}_{j-1} = \text{span}\{\phi_1, \phi_2, ..., \phi_{j-1}\}$. Here $\Sigma_1 \subseteq \Sigma_2$, and we extend the functions ϕ_k to the domain Σ_2 by setting them equal to zero on $\Sigma_2 \setminus \Sigma_1$. The functions ϕ_k thus defined now have domain in Σ_2, and the value of the L^2 inner product is the same over the two domains for any functions that are zero on $\Sigma_2 \setminus \Sigma_1$. Also, $L^2(\Sigma_1) \subseteq L^2(\Sigma_2)$, and it follows that

$$\inf_{\psi \in L^2(\Sigma_1)} R(\psi) \geq \inf_{\psi \in L^2(\Sigma_2)} R(\psi) \ ,$$

where $R(\psi) = \frac{\langle \hat{\mathcal{L}}_0 \psi, \psi \rangle_{L^2}}{\langle \psi, \psi \rangle_{L^2}}$ is the Rayleigh quotient. Therefore, by variational characterizations as in [5], the j'th eigenvalues of $\hat{\mathcal{L}}_0$ for the domains Σ_1 and Σ_2 are given by

$$\lambda_{j,0}(\Sigma_1) = \inf_{\psi \in (\mathcal{V}^{\circ}_{j-1})^{\perp} \cap L^2(\Sigma_1)} R(\psi) \geq \inf_{\psi \in (\mathcal{V}^{\circ}_{j-1})^{\perp} \cap L^2(\Sigma_2)} R(\psi) = \lambda_{j,0}(\Sigma_2) \ .$$

■

Remark 7.2. *The numerical results shown in Table 1 demonstrate that Ind(X) of minimal surfaces of revolution outside the black hole (left picture in Figure 1) is ≤ 2, which is also true for any compact portion of a minimal catenoid in $\mathbb{R}^3$ [15].*

8 Appendix

Here we prove Lemma 2.1, using the moving frame method. Let $\hat{H}$ and $\hat{K}$ be the mean and Gauss curvatures of the surface X with respect to the Euclidean metric ${ds_0}^2$, and let $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ be an orthonormal frame of the surface X such that $\hat{e}_1, \hat{e}_2$ are tangent to X and $\hat{e}_3 = \hat{N}$ is normal to X. We define 1-forms $\hat{\omega}^i$ and $\hat{\omega}^j_i$ by

$$\hat{\omega}^i(\hat{e}_j) = \delta^i_j \quad , \quad \nabla_{\hat{e}_i} = \sum_{j=1}^{3} \hat{\omega}^j_i \hat{e}_j \ .$$

Here the $\hat{\omega}^j_i$ are skew symmetric, i.e. $\hat{\omega}^j_i = -\hat{\omega}^i_j$. Furthermore, we have the structure equations $d\hat{\omega}^i = \sum_{j=1}^3 \hat{\omega}^j \wedge \hat{\omega}^i_j$, and can then define $\hat{H}$ and $\hat{K}$ as follows:

$$\hat{H} = \frac{1}{2}(\hat{h}_{11} + \hat{h}_{22}) \quad \text{and} \quad \hat{K} = \hat{h}_{11}\hat{h}_{22} - \hat{h}^2_{12} \ ,$$

where $\hat{h}_{ij} = \langle \nabla_{\hat{e}_i} \hat{e}_j, \hat{e}_3 \rangle_{(\mathbb{R}^3, ds_0^2)} = \hat{\omega}^3_j(\hat{e}_i)$.

Now consider the same surface X, but with the ambient metric ds^2, and can define an orthonormal moving frame in the same way as above, now with respect to ds^2. In this case, we denote the orthogonal vectors by e_i, and 1-forms by ω^i and $\omega_i{}^j$. Here $e_i = \frac{\hat{e}_i}{\lambda}$ and $\omega^i = \lambda\hat{\omega}^i$, and also the mean curvature is $H = \frac{1}{2}(h_{11} + h_{22})$ and Gauss curvature is $K = h_{11}h_{22} - h_{12}^2$. Using the fact that $\omega^i \wedge \hat{\omega}^i = 0$, we have

$$\begin{aligned}
\sum_{j=1}^{3} \omega^j \wedge \omega_j^i &= d\omega^i = d(\lambda\hat{\omega}^i) = d\lambda \wedge \hat{\omega}^i + \lambda d\hat{\omega}^i \\
&= \sum_{j=1}^{3} \left(\lambda_j \hat{\omega}^j \wedge \hat{\omega}^i + \lambda\hat{\omega}^j \wedge \hat{\omega}_j^i\right) = \sum_{j=1}^{3} \left(\lambda\hat{\omega}^j \wedge \left(\frac{\lambda_j}{\lambda}\hat{\omega}^i + \hat{\omega}_j^i\right)\right) \\
&= \sum_{j=1}^{3} \left(\omega^j \wedge \left(\frac{\lambda_j}{\lambda}\hat{\omega}^i - \frac{\lambda_i}{\lambda}\hat{\omega}^j + \hat{\omega}_j^i\right)\right) .
\end{aligned}$$

Here $\frac{\lambda_j}{\lambda}\hat{\omega}^i - \frac{\lambda_i}{\lambda}\hat{\omega}^j + \hat{\omega}_j^i$ is skew symmetric, so we have $\omega_j^i = \frac{\lambda_j}{\lambda}\hat{\omega}^i - \frac{\lambda_i}{\lambda}\hat{\omega}^j + \hat{\omega}_j^i$. Thus for $i, j \leq 2$, we have

$$h_{ij} = \omega_j^3(e_i) = \left(\frac{\lambda_j}{\lambda}\hat{\omega}^3 - \frac{\lambda_3}{\lambda}\hat{\omega}^j + \hat{\omega}_j^3\right)\left(\frac{\hat{e}_i}{\lambda}\right) = \frac{\hat{h}_{ij}}{\lambda} - \frac{\lambda_3}{\lambda^2}\delta_j^i .$$

Therefore, $H = \frac{\hat{H}}{\lambda} - \frac{\lambda_3}{\lambda^2}$ and $K = \frac{\hat{K}}{\lambda^2} - \frac{2\hat{H}\lambda_3}{\lambda^3} + \left(\frac{\lambda_3}{\lambda^2}\right)^2$, where $\lambda_3 = D_{\hat{N}}(\lambda)$ is the derivative of λ with respect to $\hat{N} = \hat{e}_3$.

Acknowledgement: The author thanks W. Rossman for helpful discussions.

References

[1] L. Barbosa, P. Berard, *Eigenvalue and "twisted" eigenvalue problems, applications to CMC surfaces*, J. Math. Pures Appl. (9) 79(5) (2000), 427-450.

[2] L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. 185 (1984), 339-353.

[3] L. Barbosa and M. do Carmo, *Stability of minimal surfaces in a 3-dimensional hyperbolic space*, Arch. Math. (Basel) 36(6) (1981), 554–557.

[4] L. Barbosa, M. do Carmo and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. 197 (1988), 123-138.

[5] P. Berard, *Spectral Geometry: Direct and Inverse Problems*, Lect. Notes in Math. 1207 (1986).

[6] P. Berard, L. L. de Lima and W. Rossman, *Index growth of hypersurfaces with constant mean curvature*, Math Z. 239 (2002), 99-115.

[7] P. Berard, M. do Carmo and W. Santos, *The index of constant mean curvature surfaces in hyperbolic 3-space*, Math Z. 224(2)(1997), 313-326

[8] M. Berger, P. Gauduchon, E. Mazet, *Le Spectre d'une Variete Riemannienne*, Lect. Notes in Math. 194 (1971).

[9] M. Do Carmo and C. K. Peng, *Stable complete minimal surfaces in* $\mathbb{R}^3$ *are planes*, Bull. Amer. Math. Soc. (NS) 1 (1979), 903-906.

[10] J. Choe, *Index, vision number and stability of complete minimal surfaces*, Arch. Rat. Mech. Anal. 109 (1990), 195-212.

[11] N. Ejiri and M. Kotani, *Index and flat ends of minimal surfaces*, Tokyo J. of Math. 16 (1993), 37-48.

[12] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three manifolds*, Invent. Math. 82 (1985), 121-132.

[13] M. Koiso and B. Palmer, *Geometry and stability of bubbles with gravity*, Indiana Univ. Math. J. 54 (2005), 65-98.

[14] L. L. de Lima and Wayne Rossman, *On the Index of Mean Curvature 1 Surfaces in hyperbolic space*, Indiana Univ. Math. J. 47(2) (1998), 685-723.

[15] F. Lopez and A. Ros, *Complete minimal surfaces with index one and stable constant mean curvature surfaces*, Comment. Math. Helv. 64(1) (1989), 34-43.

[16] S. G. Mikhlin, *Variational methods in mathematical physics*, Pergamon Press (1964).

[17] S. Montiel and A. Ros, *Schrödinger operators associated to a holomorphic map*, Lect. Notes in Math. 1481 (1991), 147-174.

[18] S. Nayatani, *Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space*, Comment. Math. Helv. 68 (1993), 511-537.

[19] W. Rossman, *The Morse index of Wente tori*, Geometria Dedicata 86 (2001), 129-151.

[20] W. Rossman, *The first bifurcation point for Delaunay nodoids*, J. Exp. Math. 14(3) (2005), 331-342.

[21] W. Rossman and N. Sultana, *Morse index of constant mean curvature tori of revolution in the 3-sphere*, to appear in Illinois J. Math.

[22] W. Rossman and N. Sultana, *The spectra of Jacobi operators for constant mean curvature tori of revolution in the 3-sphere*, to appear in Tokyo J. Math.

[23] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. 88 (1968), 62-105.

[24] A. da Silveira, *Stability of complete noncompact surfaces with constant mean curvature*, Math. Ann. 277 (1987), 629-638.

[25] H. Urakawa, *Geometry of Laplace-Beltrami operator on a complete Riemannian manifold*, Progress on Diff. Geometry, Adv. Stud. Pure Math. 22 (1993), 347-406.

[26] F. Urbano, *Minimal surfaces with low index in three-dimensional sphere*, Proc. A.M.S. 108(4) (1990), 989-992.

Authors' addresses:

Nahid Sultana
Department of Mathematics, Faculty of Science,
Kobe University, Rokko, Kobe 657-8501, Japan.
E-mail: nahid@math.kobe-u.ac.jp