Optimal control of electromagnetic energy

Monica Pîrvan and Constantin Udrişte

Dedicated to the 70-th anniversary
of Professor Constantin Udriste

Abstract. We establish a multitime maximum principle for a multiple integral functional constrained by nonhomogeneous linear PDEs. Applying this result to the linear-quadratic electromagnetic regulator problem based on electromagnetic energy (multiple integral functional), the electric field as control and Maxwell PDE as constraints, we rediscover the Stokes representations of the electric field and of the magnetic field.

M.S.C. 2000: 35Q60, 49K20, 49N10.
Key words: Multitime maximum principle; Maxwell PDEs; Stokes representations; electromagnetic energy; regulator problem.

Introduction

The optimization problems where the objective is a multiple integral functional and the constraints are PDEs model many natural phenomenons. That is the reason why diverse optimal control problem, with PDE constraints, appear in aerodynamics [13], finance [2], medicine [6], environmental engineering [7], etc. Generally, the complexity and infinite dimensional nature of optimal control problems with PDE constraints stimulated the scientific researchers in the last time [1]-[21].

Several conferences and work-shops all around the world with the main topic multivariable optimization constrained by PDEs took place in Europe, America, Asia, etc. Recent scientific papers[13]-[21], proved that the Pontryaguin single-time maximum principle has as correspondent a multitime maximum principle.

Section 1 formulates and proves multitime maximum principle for a multiple integral functional and nonhomogeneous linear PDE constraints, similar to the multitime maximum principle for a multiple cost integral functional constrained by an m-flow PDE [13]-[21].

Section 2 presents Maxwell PDE as closeness conditions of the electromagnetic 2-form and as extremals of a multiple functional, the Lagrangian function being the modified electromagnetic energy (see also, [15],[5]).

In Section 3 and 4, we consider the multitime optimal control problem of electromagnetic energy, with electric field as control vector and magnetic field as state vector, subject to Maxwell simplified PDE, respectively Maxwell PDE. Using the maximum

Balkan Journal of Geometry and Its Applications, Vol.15, No.1, 2010, pp. 131-141.

principle proved in Section 1, the optimal conditions give the Stokes representations of the electric and the magnetic field.

Section 5 analyzes the optimal control of electromagnetic energy through the electric field subject to a PDE relating the partial derivatives of the electric field E and of the magnetic field H.

1 Multitime maximum principle for a multiple integral functional and nonhomogenenous linear PDE constraints

Let us consider the following multitime optimal control problem with a cost functional described by a multiple integral and linear PDE constraints:

$$\max_{u(\cdot)} I(u(\cdot)) = \int_{\Omega_{t_0,t_f}} X(t, x(t), u(t))dt, \tag{1.1}$$

with the constraints

$$\begin{aligned} &a_i^{r\alpha}\frac{\partial x^i}{\partial t^\alpha}(t) + b_a^{r\alpha}\frac{\partial u^a}{\partial t^\alpha}(t) = F^r(t),\\ &u(t) \in U(t),\ \forall t \in \Omega_{t_0,t_f},\ x(t_0) = x_0,\ x(t_f) = x_f,\\ &i = \overline{1,n},\ a = \overline{1,q},\ r = \overline{1,N},\ \alpha = \overline{1,m} \end{aligned} \tag{1.2}$$

where $t = (t^\alpha)_{\alpha=\overline{1,m}} \in R^m$ is the multitime variable, $dt = dt^1dt^2 \dots dt^m$ is the volume element, Ω_{t_0,t_f} is the parallelipiped fixed by the opposite diagonal points $t_0 = (t_0^1, t_0^2, \dots, t_0^m)$ and $t_f = (t_f^1, t_f^2, \dots, t_f^m)$, $(a_i^{r\alpha})_{i,r,\alpha}$ and $(b_a^{r\alpha})_{a,r,\alpha}$ are real constants matrices, $(F^r(t))_{r=\overline{1,N}}$ are C^1 functions with respect to the multitime variable t, $x(t) = (x^i(t))_{i=\overline{1,n}}$ is an C^2 state vector, $u(t) = (u^a(t))_{a=\overline{1,q}}$ is a C^1 control vector and the scalar function $X(t, x(t), u(t))$ represents the current cost.

We apply the theory from [13]-[21] for the new Lagrangian

$$L(t, x(t), u(t), p(t)) = X(t, x(t), u(t)) + p_r(t)\left(a_i^{r\alpha}\frac{\partial x^i}{\partial t^\alpha}(t) + b_a^{r\alpha}\frac{\partial u^a}{\partial t^\alpha}(t) - F^r(t)\right),$$

where $p(t) = (p_r(t))_{r=\overline{1,N}}$ is C^1 co-state vector (Lagrange multipliers).

Thus, the initial multitime optimal control problem is transformed into a new optimal control problem

$$\max_{u(\cdot)} \int_{\Omega_{t_0,t_f}} L(t, x(t), u(t), p(t))dt,$$

$$u(t) \in U(t),\ p(t) \in P(t),\ \forall t \in \Omega_{t_0,t_f},\ x(t_0) = x_0,\ x(t_f) = x_f,$$

where the set of suitable co-state variables $P(t)$ will be defined later.

Let us consider there exists an interior optimal C^1 control vector $u^*(t) \in \mathrm{Int}(U(t))$. Because $u^*(t)$ is continuous on a compact set, for any arbitrary continuous vector function $h(t)$, it exists $\theta_h > 0$ so that $u(t, \theta) = u^*(t) + \theta h(t) \in \mathrm{Int}\,(u(t)),\quad \forall |\theta| < \theta_h$.

On the domain $|\theta| < \theta_h$, we define the integral function

$$I(\theta) = \int_{\Omega_{t_0,t_f}} L(t, x(t,\theta), u(t,\theta), p(t))dt,$$

where $x(t,\theta)$ is the state variable corresponding to the variation $u(t,\theta)$ of the control function.

We suppose that the integral function $I(\theta)$ admits a maximum point $\theta = 0$. Using the total derivative, the integral function $I(\theta)$ takes the form

$$\begin{aligned} I(\theta) &= \int_{\Omega_{t_0,t_f}} \bigg(X(t, x(t,\theta), u(t,\theta)) - a_i^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) x^i(t,\theta) - \\ &- b_a^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) u^a(t,\theta) + p_r F^r(t) \bigg) dt + \\ &+ \int_{\Omega_{t_0,t_f}} \frac{\partial}{\partial t^\alpha} \left(a_i^{r\alpha} p_r(t) x^i(t,\theta) + b_a^{r\alpha} p_r(t) u^a(t,\theta) \right) dt. \end{aligned}$$

Integral divergence formula applied to the integral

$$\int_{\Omega_{t_0,t_f}} \frac{\partial}{\partial t^\alpha} \left(a_i^{r\alpha} p_r(t) x^i(t,\theta) + b_a^{r\alpha} p_r(t) u^a(t,\theta) \right) dt,$$

allows a new form of integral function $I(\theta)$,

$$\begin{aligned} I(\theta) &= \int_{\Omega_{t_0,t_f}} \bigg(X(t, x(t,\theta), u(t,\theta)) - a_i^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) x^i(t,\theta) - \\ &- b_a^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) u^a(t,\theta) + p_r F^r(t) \bigg) dt + \\ &+ \int_{\partial\Omega_{t_0,t_f}} \delta_{\alpha,\beta} \left(a_i^{r\alpha} p_r(t) x^i(t,\theta) + b_a^{r\alpha} p_r(t) u^a(t,\theta) \right) n^\beta(t) dt, \end{aligned}$$

where $n(t) = (n^\alpha(t))_{\alpha=\overline{1,m}}$ is the normal unit vector of the boundary $\partial\Omega_{t_0,t_f}$.

Deriving with respect to the variable θ, it follows

$$\begin{aligned} I'(\theta) &= \int_{\Omega_{t_0,t_f}} \bigg[\frac{\partial X}{\partial x^i}(t, x(t,\theta), u(t,\theta)) \frac{\partial x^i}{\partial \theta}(t,\theta) + \\ &+ \frac{\partial X}{\partial u^a}(t, x(t,\theta), u(t,\theta)) h^a(t) - \\ &- a_i^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) \frac{\partial x^i}{\partial \theta}(t,\theta) - b_a^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) h^a(t) \bigg] dt + \\ &+ \int_{\partial\Omega_{t_0,t_f}} \delta_{\alpha\beta} \left(a_i^{r\alpha} p_r(t) \frac{\partial x^i}{\partial \theta}(t,\theta) + b_a^{r\alpha} p_r(t) h^a(t) \right) n^\beta(t) d\sigma. \end{aligned}$$

Consequently,

$$\begin{aligned} I'(0) &= \int_{\Omega_{t_0,t_f}} \left[\frac{\partial X}{\partial x^i}(t, x(t,0), u(t,0)) \frac{\partial x^i}{\partial \theta}(t,0) + \right. \\ &+ \frac{\partial X}{\partial u^a}(t, x(t,0), u(t,0)) h^a(t) - \\ &- \left. a_i^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t,0) \frac{\partial x^i}{\partial \theta}(t,0) - b_a^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) h^a(t) \right] dt + \\ &+ \int_{\partial\Omega_{t_0,t_f}} \delta_{\alpha\beta} \left(a_i^{r\alpha} p_r(t) \frac{\partial x^i}{\partial \theta}(t,0) + b_a^{r\alpha} p_r(t) h^a(t) \right) n^\beta(t) d\sigma, \end{aligned}$$

or

$$\begin{aligned} I'(0) &= \int_{\Omega_{t_0,t_f}} \left[\frac{\partial}{\partial x^i} X(t, x(t,0), u(t,0)) - a_i^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) \right] \frac{\partial x^i}{\partial \theta}(t,0) dt + \\ &+ \int_{\Omega_{t_0,t_f}} \left[\frac{\partial}{\partial u^a} X(t, x(t,0), u(t,0)) - b_a^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) \right] h^a(t) dt + \\ &+ \int_{\partial\Omega_{t_0,t_f}} \delta_{\alpha\beta} \left(a_i^{r\alpha} p_r(t) \frac{\partial x^i}{\partial \theta}(t,0) \right) n^\beta(t) d\sigma + \\ &+ \int_{\partial\Omega_{t_0,t_f}} \delta_{\alpha\beta} \left(b_a^{r\alpha} p_r(t) h^a(t) \right) n^\beta(t) d\sigma. \end{aligned}$$

The condition $I'(0) = 0$ is necessary to be accomplished for any arbitrary vector function $h(t)$. To eliminate the functions $\frac{\partial x^i}{\partial \theta}(t,0)$ that depend on $h(t)$ we define the set of admissible co-states $P(t)$ as the set of solutions for the boundary value problem

$$\frac{\partial X}{\partial x^i}(t, x^*(t), u^*(t)) - a_i^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) = 0,\ \forall t \in \Omega_{t_0,t_f},\ i = \overline{1,n},\ \text{(adjoint PDEs)} \tag{1.3}$$

$$\delta_{\alpha\beta} b_a^{r\alpha} p_r(t) n^\beta(t)|_{\partial\Omega_{t_0,t_f}} = 0, \quad \text{(orthogonality condition)},\ i = \overline{1,n}. \tag{1.4}$$

It follows that

$$\begin{gathered} \frac{\partial X}{\partial u^a}(t, x^*(t), u^*(t)) - b_a^{r\alpha} \frac{\partial p_r}{\partial t^\alpha}(t) = 0,\ \forall t \in \Omega_{t_0,t_f},\ a = \overline{1,q}, \\ \text{(critical point or adjoint PDE)} \end{gathered} \tag{1.5}$$

$$\delta_{\alpha\beta} b_a^{r\alpha} p_r(t) n^\beta(t)|_{\partial\Omega_{t_0,t_f}} = 0, \quad \text{(orthogonality condition)},\ a = \overline{1,q}. \tag{1.6}$$

We are able now to formulate the multitime maximum principle.

Theorem 1. *If the multitime optimal control problem (1.1), with nonhomogeneous linear PDE (1.2) constraints, admits an interior optimal control $u^*(t)$ and $x^*(t)$ is the corresponding state variable, then there exists a C^1 co-state $p(t)$ so that the relations (1.2),(1.3),(1.4),(1.5),(1.6) to be true.*

2 Maxwell PDEs as closeness conditions and as Euler-Lagrange PDEs

Here we recall that two of the Maxwell PDEs represent the closeness conditions of the electromagnetic 2-form and the other are Euler-Lagrange PDEs. Let E be the electric field strength, H be the magnetic field strength, J be the electric current density, ρ be the density of charge, B be the magnetic induction, D be the electric displacement, ε be the permitivity (electric constant) and μ be the permeability (magnetic constant).

In a linear homogeneous isotropic media, Maxwell PDE reflects the relations between magnetic field component and electric field component of the electromagnetic field, and are described by

$$\operatorname{div} D = \rho, \quad \text{(Gauss law for electric field)}, \tag{2.1}$$

$$\operatorname{div} B = 0, \quad \text{(Gauss law for magnetic field)}, \tag{2.2}$$

$$\operatorname{curl} H = J + \partial_t D, \quad \text{(Ampere law with Maxwell correction)}, \tag{2.3}$$

$$\operatorname{curl} E = -\partial_t B, \quad \text{(Faraday induction law)}, \tag{2.4}$$

with the constitutive equations $B = \mu H$, $D = \varepsilon E$. The Maxwell PDE system (2.1)-(2.4) contains six dependent variables, namely, the components of the electric field $E = (E^1, E^2, E^3)$ and the magnetic field $H = (H^1, H^2, H^3)$ and eight PDEs, i.e., it is over determined. This system cannot have a Lagrangian since the number of Euler-Lagrangian PDEs must be equal to the number of dependent variables [21].

The electromagnetic energy is the quadratic form

$$\mathcal{H} = \frac{1}{2}(\mu||H||^2 + \varepsilon||E||^2). \tag{2.5}$$

The electromagnetic field is generated by a real 1-form $\Phi = A_I dx^I$, $x^I = (x^i, t)_{i=\overline{1,3}}$, where $A = (A_i)_{i=\overline{1,3}}$ represents the magnetic potential co-vector. The field strength of Φ is defined as $F = d\Phi = F_{IJ} dx^I \wedge dx^J$. The electric field E and the magnetic field B can be extracted from the field strength writting

$$F = \left(\sum_{i=1}^{3} \delta_{ij} E^i dx^j \right) \wedge dt + B \rfloor (dx^1 \wedge dx^2 \wedge dx^3),$$

where $\rfloor$ is the inner product with the vector field $B = B^i \dfrac{\partial}{\partial x^i}$.

The closeness of the electromagnetic field 2-form F is equivalent to two of Maxwell equations

$$\operatorname{curl} E = -\partial_t B, \quad \operatorname{div} B = 0.$$

Because $B = \operatorname{curl} A$, it exists a scalar vector V so that the electric field strength is $E = -\operatorname{grad} V - \partial_t A$. Considering the constitutive equations, it follows that $H = \dfrac{1}{\mu}\operatorname{curl}(A)$. The electromagnetic energy (2.5) takes the form

$$\mathcal{H} = \frac{1}{2}\left(\frac{1}{\mu}||\operatorname{curl} A||^2 + \varepsilon||(\operatorname{grad} V + \partial_t A)||^2 \right).$$

Let Ω be a domain in R^4 and the modified electromagnetic energy functional

$$I(A,V) = \int_\Omega (\mathcal{H} - AJ + \rho V)\, dx^1 dx^2 dx^3 dt,$$

where $(x^1, x^2, x^3, t) \in \Omega$.

The Euler-Lagrange PDEs produced by the Lagragian function

$$L\left(A, V, \frac{\partial A_j}{\partial x^i}, \frac{\partial A_j}{\partial t}, \frac{\partial V}{\partial x^i}\right) = \frac{1}{2}\left(\frac{1}{\mu}||\text{curl}\, A||^2 + \varepsilon||(\text{grad}\, V + \partial_t A)||^2\right) - AJ + \rho V,$$

are

$$\frac{\partial L}{\partial A_j} - \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial\left(\frac{\partial A_j}{\partial x^i}\right)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\left(\frac{\partial A_j}{\partial t}\right)}\right) = 0, \quad j = \overline{1,3}, \tag{2.6}$$

$$\frac{\partial L}{\partial V} - \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial\left(\frac{\partial V}{\partial x^i}\right)}\right) = 0. \tag{2.7}$$

The PDEs (2.6) is equivalent to Ampere law with Maxwell correction, curl $H = J + \partial_t D$, and PDE (2.7) gives Gauss law for electric field, div $D = \rho$.

3 Stokes representation for the solutions of simplified Maxwell PDE

We consider that the electromagnetic field does not depend on the time variable, obtaining in this way a simplified form for Maxwell PDE

$$\text{div}\, E(x) = \frac{1}{\varepsilon}\rho(x),\ \text{div}\, H(x) = 0,\ \text{curl}\,(H(x)) = J(x),\ \text{curl}\,(E(x)) = 0, \tag{3.1}$$

where $x = (x^i)_{i=\overline{1,3}}$.

Theorem 2. *The solutions of Maxwell simplified PDE (3.1) admit the Stokes representation*

$$E(x) = \frac{1}{\varepsilon}(curl\, q(x) - grad\, \beta(x)), \quad (adjoint\ PDEs) \tag{3.2}$$

$$H(x) = \frac{1}{\mu}(curl\, p(x) - grad\, \alpha(x)), \quad (adjoint\ PDEs) \tag{3.3}$$

with the boundary conditions

$$q(x) \times n(x) + \beta(x) n(x)|_{\partial\Omega_{x_0,x_f}} = 0, \tag{3.4}$$

$$p(x) \times n(x) + \alpha(x)n(x)|_{\partial\Omega_{x_0,x_f}} = 0, \tag{3.5}$$

where $q(x), p(x), \beta(x)$ are Stokes potentials, Ω_{x_0,x_f} is the parallelipiped fixed by two diagonal points x_0, x_f and $n(x)$ is the unit normal vector of the boundary $\partial\Omega_{x_0,x_f}$.

Proof. We apply the result of Section 1, i.e., we look for optimal control problem

$$\max_{E(\cdot)} I(E(\cdot) = -\frac{1}{2}\int_{\Omega_{x_0,x_f}} \left(\mu||H(x)||^2 + \varepsilon||E(x)||^2\right) dx^1dx^2dx^3, \tag{3.6}$$

subject to Maxwell simplified PDE

$$\text{div}\, E(x) = \frac{1}{\varepsilon}\rho(x),\ \text{div}\, H(x) = 0,\ \text{curl}\,(H(x)) = J(x),\ \text{curl}\,(E(x)) = 0,$$

$$H(x_0) = H_0,\ H(x_f) = H_f,$$

where $H(x) = (H^i(x))_{i=\overline{1,3}}$ is the magnetic state vector and $E(x) = (E^i(x))_{i=\overline{1,3}}$ is the C^1 electric control vector.

Let $p(x) = (p_i(x))_{i=\overline{1,3}} \in P(x)$, $q(x) = (q_i(x))_{i=\overline{1,3}} \in Q(x)$, $\alpha(x) \in R(x)$ and $\beta(x) \in S(x)$ be C^1 functions, considered as co-state variables (Lagrange multipliers), and the Lagrange function

$$L_1(x, H(x), E(x), p(x), q(x), \alpha(x), \beta(x)) = -\frac{1}{2}\left(\varepsilon||E(x)||^2 + \mu||H(x)||^2\right) +$$

$$+\langle p(x), \text{curl}\,(H(x)) - J(x)\rangle + \langle q(x), \text{curl}\,(E(x))\rangle +$$

$$+\alpha(x)\text{div}\, H(x) + \beta(x)\left(\text{div}\, E(x) - \frac{1}{\varepsilon}\rho(x)\right).$$

Necessary conditions for the optimal multitime problem (3.6), with simplified Maxwell PDE (3.1) as constraints, are obtained from Theorem 1.

Relations (1.3),(1.4),(1.5),(1.6) are equivalent with Stokes representations for simplified Maxwell PDE solution

$$E(x) = \frac{1}{\varepsilon}(\text{curl}\, q(x) - \text{grad}\,\beta(x)),\quad q(x) \times n(x) + \beta(x)n(x)|_{\partial\Omega_{x_0,x_f}} = 0,$$

$$H(x) = \frac{1}{\mu}(\text{curl}\, p(x) - \text{grad}\,\alpha(x)),\quad p(x) \times n(x) + \alpha(x)n(x)|_{\partial\Omega_{x_0,x_f}} = 0.$$

4 Extended Stokes representation for the solutions of Maxwell PDE

Let us consider the general case of Maxwell PDE (2.1),(2.2),(2.3),(2.4).

Theorem 3. *The solutions of Maxwell PDE admit the extended Stokes representation*

$$E(x,t) = \frac{1}{\varepsilon}\left(curl\, q(x,t) - grad\,\alpha(x,t) + \varepsilon\frac{\partial p}{\partial t}\right),\quad (adjoint\ PDEs) \tag{4.1}$$

$$(4.2)\qquad H(x,t)=\frac{1}{\mu}\left(curl\, p(x,t)-grad\,\beta(x,t)-\mu\frac{\partial q}{\partial t}\right),\quad (adjoint\ PDEs)$$

with boundary conditions

$$(4.3)\qquad \beta(x,t)N(x,t)+p(x,t)\times N(x,t)+\mu q(x,t)n^4(x,t)|_{\partial\Omega_{(x_0,t_0),(x_f,t_f)}}=0,$$

$$(4.4)\qquad \alpha(x,t)N(x,t)+q(x,t)\times N(x,t)-\varepsilon p(x,t)n^4(x,t)|_{\partial\Omega_{(x_0,t_0),(x_f,t_f)}}=0,$$

where $p(x,t),q(x,t),\alpha(x,t),\beta(x,t)$ *are Stokes potentials,* $n(x,t)=(n^i(x,t))_{i=\overline{1,4}}$ *is the unit normal vector of the boundary* $\partial\Omega_{(x_0,t_0),(x_f,t_f)}$ *and* $N(x,t)=(n^i(x,t))_{i=\overline{1,3}}$.

Proof. We apply the results proved in Section 1, i.e., we refer to the multitime maximum control problem

$$(4.5)\quad \max_{E(\cdot,\cdot)} I(E(\cdot,\cdot))=\frac{-1}{2}\int_{\Omega_{(x_0,t_0),(x_f,t_f)}}\left(\mu||H(x,t)||^2+\varepsilon||E(x,t)||^2\right)dx^1dx^2dx^3dt,$$

subject to Maxwell PDE

$$\mathrm{div}\,(E(x,t))=\frac{1}{\varepsilon}\rho(x,t),\quad \mathrm{curl}\,(E(x,t))=-\mu\frac{\partial H}{\partial t}(x,t),$$

$$\mathrm{div}\,(H(x,t))=0,\quad \mathrm{curl}\,(H(x,t))=J(x,t)+\varepsilon\frac{\partial E}{\partial t}(x,t),$$

$$E(x,t)\in U(x,t),\quad \forall(x,t)\in\Omega_{(x_0,t_0),(x_f,t_f)},\quad H(x_0,t_0)=H_0,\quad H(x_f,t_f)=H_f.$$

Considering the C^1 co-state variables $p(x,t)=(p_i(x,t))_{i=\overline{1,3}}\in P(x,t)$, $\alpha(x,t)\in R(x,t)$, $q(x,t)=(q_i(x,t))_{i=\overline{1,3}}\in Q(x,t)$, $\beta(x,t)\in S(x,t)$ (Lagrangian multipliers) and the Lagrange function

$$\begin{aligned}&L_2(x,t,H(x,t),E(x,t),p(x,t),q(x,t),\alpha(x,t),\beta(x,t))=\\&=-\frac{1}{2}(\varepsilon||E(x,t)||^2+\mu||H(x,t)||^2)+\\&+\langle p(x),\mathrm{curl}\left(H(x,t)-J(x,t)-\varepsilon\frac{\partial E}{\partial t}(x,t)\right)\rangle+\\&+\langle q(x,t),\mathrm{curl}\left(E(x,t)+\mu\frac{\partial H}{\partial t}(x,t)\right)\rangle+\\&+\alpha(x,t)\mathrm{div}\,H(x,t)+\beta(x,t)\left(\mathrm{div}\,E(x,t)-\frac{1}{\varepsilon}\rho(x,t)\right),\end{aligned}$$

the optimal control problem (4.5), with Maxwell PDE as constraints, is transformed into a new multitime optimal problem

$$\max_{E(\cdot,\cdot)}\int_{\Omega_{(x_0,t_0),(x_f,t_f)}} L_2((x,t),H(x,t),E(x,t),p(x,t),q(x,t),\alpha(x,t),\beta(x,t))\cdot$$

$$\cdot dx^1dx^2dx^3dt,$$

$$p(x,t) \in P(x,t),\ q(x,t) \in Q(x,t),\ \alpha(x,t) \in R(x,t),\ \beta(x,t) \in S(x,t),$$

$$E(x,t) \in U(x,t),\ \forall (x,t) \in \Omega_{(x_0,t_0),(x_f,t_f)},\ H(x_0,t_0) = H_0,\ H(x_f,t_f) = H_f.$$

Optimality conditions given by Theorem 1 are the extended Stokes representations for the solutions of Maxwell PDEs

$$E(x,t) = \frac{1}{\varepsilon}\left(\operatorname{curl} q(x,t) - \operatorname{grad} \alpha(x,t) + \varepsilon \frac{\partial p}{\partial t}\right),\ \text{(dual PDEs)}$$

$$H(x,t) = \frac{1}{\mu}\left(\operatorname{curl} p(x,t) - \operatorname{grad} \beta(x,t) - \mu \frac{\partial p}{\partial t}\right),$$

$$\beta(x,t)N(x,t) + p(x,t) \times N(x,t) + \mu q(x,t) n^4(x,t)|_{\partial\Omega_{(x_0,t_0),(x_f,t_f)}} = 0,$$

$$\alpha(x,t)N(x,t) + q(x,t) \times N(x,t) - \varepsilon p(x,t) n^4(x,t)|_{\partial\Omega_{(x_0,t_0),(x_f,t_f)}} = 0.$$

5 Multitime optimal control of electromagnetic energy subject to a PDE relating the partial derivatives of electric field E and magnetic field H

We consider the multitime optimal control problem

$$\text{(5.1)} \quad \max_{E(\cdot,\cdot)} I(E(\cdot,\cdot)) = \frac{-1}{2}\int_{\Omega_{(x_0,t_0),(x_f,t_f)}} (\mu||H(x,t)||^2 + \varepsilon||E(x,t)||^2) dx^1 dx^2 dx^3 dt,$$

with linear PDE constraint (inspired from Maxwell PDEs)

$$\text{(5.2)} \quad A^i_j \frac{\partial H^j}{\partial x^i} + a_j \frac{\partial H^j}{\partial t} + B^i_j \frac{\partial E^j}{\partial x^i} + b_j \frac{\partial E^j}{\partial t} = 0,$$

$$E(x,t) \in U(x,t),\ \forall (x,t) \in \Omega_{(x_0,t_0),(x_f,t_f)},\ H(x_0,t_0) = H_0,\ H(x_f,t_f) = H_f,$$

$i, j = \overline{1,3}$, where $x = (x^i)_{i=\overline{1,3}}$, the magnetic field $H(x,t) = (H^j(x,t))_{j=\overline{1,3}}$ is the state vector, the electric field $E(x,t) = (E^j(x,t))_{j=\overline{1,3}}$ is the control vector, $(A^i_j)_{i,j=\overline{1,3}}$, $(B^i_j)_{i,j=\overline{1,3}}$, $(a_j)_{j=\overline{1,3}}$, $(b_j)_{j=\overline{1,3}}$ are real matrices.

Theorem 4. *If the optimal control problem (5.1) with constraints (5.2) admits an interior optimal electric control, then the adjoint PDEs are*

$$-\mu H^j(x,t) - A^i_j \frac{\partial p}{\partial x^i}(x,t) - a_j \frac{\partial p}{\partial t} = 0, \quad j = \overline{1,3},$$

with boundary conditions

$$p(x,t)\left(\delta_{ik} A^i_j n^k(x,t) + a_j n^4(x,t)\right)|_{\partial\Omega_{(x_0,t_0),(x_f,t_f)}} = 0, \quad j = \overline{1,3}$$

and optimality conditions

$$-\varepsilon E^j(x,t) - B^i_j \frac{\partial p}{\partial x^i}(x,t) - b_j \frac{\partial p}{\partial t} = 0, \quad j = \overline{1,3},$$

together with boundary conditions

$$p(x,t)\left(\delta_{ik}B^i_j n^k(x,t) + b_j n^4(x,t)\right)|_{\partial\Omega_{(x_0,t_0),(x_f,t_f)}} = 0, \quad j=\overline{1,3}$$

where $n(x,t) = (n^i(x,t))_{i=\overline{1,4}}$ *is the unit normal vector of the boundary* $\partial\Omega_{(x_0,t_0),(x_f,t_f)}$ *and* $p(x,t)$ *is a* C^1 *co-state variable.*

Proof. The proof is a consequence of theorem 1, section 1, where the current cost is $\frac{-1}{2}\left(\mu||H(x,t)||^2 + \varepsilon||E(x,t)||^2\right)$ and nonhomogeneous linear PDE (1.2) are replaced with linear PDE (5.2). □

References

[1] N. Barron, R. Jensen, *The Pontryaguin Maximum Principle from Dynamic Programming and Viscosity Solutions to First-Order Partial Differential Equations*, Transactions AMS 298, 1986, 653-641.

[2] I. Bouchouev, V. Isakov, *Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets*, Inverse Problems, 15 (1999), 95-116 .

[3] F. Cardin, C. Viterbo, *Commuting Hamiltonian and Hamilton-Jacobi multi-time equations*, arXiv.math/0507418v2[math.DG], 10 Sep. (2007).

[4] F. Giannessi, A. Maugeri, *Variational Analysis and Applications*, Springer 2008.

[5] N.H. Ibragimov, *Integrating factors, adjoint equations and Lagrangians*, Journal of Mathematical Analysis and Applications 318, 2 (2006), 742-757.

[6] M.V. Klibanov, T.R. Lucas, *Numerical solution of a parabolic inverse problem in optical tomography using experimental data*, SIAM J. Appl. Math. 59 (1999), 1763-1789.

[7] C.D. Laird, L.T. Biegler, B. Waanders, R.A. Bartlett, *Time dependent contaminant source determination for municipal water networks using large scale optimization*, ASCE J. Water Res. Mgt. Plan, 131 (2005), 125-134.

[8] P.L. Lions, J.C. Rochet, *Hopf formula and multi-time Hamilton-Jacobi equations*, Proc. Am. Math. Soc. 96 (1986), 79-84.

[9] M. Motta, F. Rampazzo, *Nonsmooth multi-time Hamilton-Jacobi systems*, Indiana University Mathematics Journal 55, 5 (2006), 1573-1614.

[10] A. Pitea, C. Udrişte, Ş. Mititelu, *PDI&PDE-constrained optimization problems with curvilinear functional quotients as objective vectors*, Balkan J. Geom. Appl. 14, 2 (2009), 75-88.

[11] S. Plaskacz, M. Quincampoix, *Oleinik-Lax formulas and multi-time Hamilton-Jacobi systems*, Nonlinear Analysis TMA 51 (2002), 957-969.

[12] J.C. Rochet, *The taxation principle and multi-time Hamilton-Jacobi equation*, J. Math. Ecom. 14 (1985), 113-128.

[13] A. Shenoy, M. Heinkenshloss, E.M. Cliff, *Airfoil design by an all-once method*, Int. J. Comput. Fluid Dyn. 11 (1998), 3-25.

[14] C. Udrişte, O. Dogaru, I. Ţevy, *Extrema with Nonholonomic Constraints*, Monographs and Textbooks 4, Geometry Balkan Press, Bucharest, 2002.

[15] C. Udrişte, *Maxwell geometric dynamics*, European Computing Conference, Vouliagmeni Beach, Athens, Greece, September 24-26, 2007, Chapter in a Springer book, 2008, Part 15, Theoretical Approaches in Information Science, Chapter 5, 1597-1609.

[16] C. Udrişte, *Multi-time maximum principle*, Short Communication, International Congress of Mathematicians, Madrid, August 22-30, ICM Abstracts, 2006, p. 47, Plenary Lecture at 6-th WSEAS International Conference on Circuits, Systems, Electronics, Control&Signal Processing (CSECS'07), p. 10-11 and 12-th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, December 29-31, 2007, p. ii.

[17] C. Udrişte, L. Matei, *Lagrange-Hamilton Theories* (in Romanian), Monographs and Textbooks 8, Geometry Balkan Press, Bucharest, 2008.

[18] C. Udrişte, *Multitime controllability, observability and bang-bang principle*, Journal of Optimization Theory and Applications 139, 1 (2008), 141-156.

[19] C. Udrişte, *Simplified multitime maximum principle*, Balkan J. Geom. Appl. 14, 1 (2009), 102-119.

[20] C. Udrişte, *Nonholonomic approach of multitime maximum principle*, Balkan J. Geom. Appl. 14, 2 (2009), 111-126.

[21] C. Udrişte, I. Ţevy, *Multitime linear-quadratic regulator problem based on curvilinear integral*, Balkan J. Geom. Appl. 14, 2 (2009), 127-137.

Authors' address:

Monica Pîrvan and Constantin Udrişte
University "Politehnica" of Bucharest,
Faculty of Applied Sciences, Splaiul Independenţei no. 313,
RO-060042, Bucharest, Romania.
E-mail: monicapirvan@yahoo.com , udriste@mathem.pub.ro