Notes on some classes of 3-dimensional contact metric manifolds

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Abstract. A review of the geometry of 3-dimensional contact metric manifolds shows that generalized Sasakian manifolds and η-Einstein manifolds are deeply interrelated. For example, it is known that a 3-dimensional Sasakian manifold is η-Einstein. In this paper, we discuss the relationships between several special classes of 3-dimensional contact metric manifolds which are generalizations of 3-dimensional Sasakian manifolds. We also provide examples illustrating our result in this paper.

M.S.C. 2010: 53D10, 53C25.
Key words: Sasakian manifolds; contact metric manifolds.

1 Introduction

It is well-known that any 3-dimensional compact oriented manifold admits a contact structure [21], and hence, it admits an associated contact metric structure. Therefore, it is natural to investigate 3-dimensional compact oriented manifolds from the contact metric view point. We shall give a brief review of contact metric manifolds focusing on the interrelationships between the generalizations of Sasakian manifolds and η-Einstein contact metric manifolds. It is well known that a Sasakian manifold is characterized as a contact metric manifold $M = (M, \phi, \xi, \eta, g)$ whose curvature tensor R satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

for any X, $Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. As a generalization of the Sasakian manifold, Blair, Koufogiorgos and Papantoniou [2] introduced the notion of a contact metric manifold called a (κ, μ)-*contact metric manifold* satisfying the condition

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1.2}$$

for any X, $Y \in \mathfrak{X}(M)$, where κ and μ are constants on M and $h = \frac{1}{2}\pounds_\xi \phi$ (here, $\pounds_\xi$ is the Lie derivative in the direction of ξ). (κ, μ)-contact metric manifolds have attracted

Balkan Journal of Geometry and Its Applications, Vol.17, No.2, 2012, pp. 54-65.

by many authors [4, 5, 9, 10, 11, 18, 20]. (κ, μ)-contact metric manifolds include Sasakian manifolds ($\kappa = 1$ and $h = 0$), and also many examples of non-Sasakian (κ, μ)-contact metric manifolds have been provided. Koufogiorgos and Tsichlias [12] generalized the notion of a (κ, μ)-contact metric manifold by regarding the constants κ and μ in (1.2) to be smooth functions on M, called a *generalized* (κ, μ)*-contact metric manifold*. Further, the same authors [11] studied 3-dimensional generalized (κ, μ)-contact metric manifolds with $\xi\mu = 0$ (this condition means the function μ is constant along each integral curve of the characteristic vector field ξ) and showed that it is possible to construct two families of such manifolds in $\mathbb{R}^3$, for any smooth function κ ($\kappa < 1$) of one variable. We shall introduce an example belonging to such families in §5, which illustrates Theorem B in the present paper. Koufogiorgos, Markellas and Papantoniou [10] introduced the notion of a (κ, μ, ν)*-contact metric manifold* which is a generalization of the generalized (κ, μ)-contact metric manifold, defined as a contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying

$$\begin{aligned} R(X,Y)\xi = {} & \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \\ & + \nu(\eta(Y)\phi hX - \eta(X)\phi hY). \end{aligned} \tag{1.3}$$

for any $X,\ Y \in \mathfrak{X}(M)$, where $\kappa,\ \mu,\ \nu$ are smooth functions on M. In the same paper [10], they proved that a (κ, μ, ν)-contact metric manifold is necessarily a (κ, μ)-contact metric manifold if the dimension of M is greater than or equal to 5. They also proved that the condition (1.3) is invariant under the D-homothetic deformations, and further that, if $\dim M = 3$, then the condition (1.3) is equivalent to the following condition

$$Q = \Big(\frac{r}{2} - \kappa\Big) I + \Big(-\frac{r}{2} + 3\kappa\Big) \eta \otimes \xi + \mu h + \nu\phi h \tag{1.4}$$

holding on an open and dense subset of M, where Q is the Ricci operator and r is the scalar curvature of M ([10], Proposition 3.1). We note that $\kappa \leq 1$ on 3-dimensional (κ, μ, ν)-contact metric manifold (see(3.13)). A contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called η*-Einstein* if the Ricci operator Q takes the following form

$$Q = \alpha I + \beta\eta \otimes \xi, \tag{1.5}$$

where α and β are some smooth functions on M. From (1.3) and (1.4), taking account of (1.5), we may observe that the geometry of (κ, μ, ν)-contact metric manifolds and of generalized (κ, μ)-contact metric manifolds is deeply interrelated with the generalization of the η-Einstein contact metric manifold in the 3-dimensional case. On the other hand, a contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is said to be *H-contact* if the characteristic vector field ξ is a harmonic vector field. We remark that (κ, μ, ν)-contact metric manifold is H-contact. Koufogiorgos, Markellas and Papantoniou [10] proved that a 3-dimensional H-contact manifold is a (κ, μ, ν)-contact metric manifold on an open and dense subset of M ([10], Theorem 1.1). The last two of the present authors worked on the H-contact unit tangent sphere bundles [6, 7, 14]. Concerning 3-dimensional (κ, μ, ν)-contact metric manifolds, the present authors previously proved the following theorem.

Theorem A [8] *Let $M = (M, \phi, \xi, \eta, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold. If the functions μ and ν are constant on M, then M is either Sasakian* *or a non-Sasakian (κ, μ)-contact metric manifold with constant scalar curvature $r = 2\kappa - 2\mu$.*

In this paper, we shall prove the following theorem.

Theorem B *Let $M = (M, \phi, \xi, \eta, g)$ be a 3-dimensional compact (κ, μ, ν)-contact metric manifold with $\xi\mu = \xi\nu = 0$ and let r be the scalar curvature. If either (the inequality) $r + \frac{\mu^2}{2} \geq 0$ or $r + \frac{\mu^2}{2} \leq 0$ holds everywhere on M, then M is a Sasakian manifold or a non-Sasakian (κ, μ)-contact metric manifold with $\kappa = \mu - \frac{\mu^2}{4}$ and $r = -\frac{\mu^2}{2}$.*

We here remark that the hypothesis "$M = (M, \phi, \xi, \eta, g)$ is a 3-dimensional (κ, μ, ν)-contact metric manifold with $\xi\mu = \xi\nu = 0$" is preserved under any D-homothetic transformation [10] of the contact metric structure (ϕ, ξ, η, g) on M. Unless otherwise specified, the manifolds to be considered in this paper will be assumed to be connected.

2 Preliminaries

In this section, we present some basic facts about contact metric manifolds. We refer to [1] for more details. A $(2n+1)$-dimensional smooth manifold M is called a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. We call η a *contact form* of M. It is well known that given a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X on M. A Riemannian metric g is said to be an *associated metric* to a contact form η if there exists a $(1,1)$-tensor field ϕ satisfying

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1), one can easily obtain

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

The structure (ϕ, ξ, η, g) is called a *contact metric structure*, and a manifold M with a contact metric structure (ϕ, ξ, η, g) is said to be a *contact metric manifold* and is denoted by $M = (M, \phi, \xi, \eta, g)$. Let ∇ be the Levi-Civita connection and let R be the corresponding Riemann curvature tensor field given by $R(X,Y)=[\nabla_X, \nabla_Y]- \nabla_{[X,Y]}$ for all vector fields X,Y on M. We denote by S the Ricci tensor field of type (0,2), by Q the Ricci operator, and by r the scalar curvature. We define on M the operators h, l by setting

$$hX = \frac{1}{2}(\pounds_\xi \phi)X, \quad lX = R(X, \xi)\xi, \tag{2.3}$$

where $\pounds_\xi$ is the Lie derivative in the direction of ξ. It is easily checked that h and l are symmetric operators and satisfy the following equalities

$$h\xi = 0, \quad l\xi = 0, \quad h\phi = -\phi h. \tag{2.4}$$

We also have the following formulas for a contact metric manifold:

$$
\begin{aligned}
&\nabla_X \xi = -\phi X - \phi h X, \quad \text{(and hence } \nabla_\xi \xi = 0) \\
&\nabla_\xi \phi = 0, \qquad\qquad\qquad Trl = g(Q\xi, \xi) = 2n - tr(h^2), \\
&\phi l \phi - l = 2(\phi^2 + h^2), \qquad \nabla_\xi h = \phi - \phi l - \phi h^2.
\end{aligned} \tag{2.5}
$$

On the other hand, a contact metric manifold for which ξ is a Killing vector field is called a *K-contact manifold*. It is well known that a contact metric manifold is K-contact if and only if $h = 0$. It is well known that Sasakian manifolds are necessarily K-contact but the converse is generally not true except in the 3-dimensional case ([1], pp.70 and pp.76). Here, we note that on any $(2n+1)(n > 1)$-dimensional η-Einstein K-contact manifold, the functions α and β in the defining equation (1.5) are both constant. We may also note that any 3-dimensional Sasakian manifold is η-Einstein ((1.4), [17]) and $\alpha + \beta$ is constant [3]. Hence, it is natural to ask whether there exists a 3-dimensional Sasakian manifold with non-constant coefficient functions α and β as a η-Einstein or not. Concerning this question, to our knowledge, it seems that any explicit example of a 3-dimensional Sasakian manifold with non-constant coefficient functions α and β as an η-Einstein manifold has not yet appeared in any literature. In the last section, we shall provide an explicit example of such a 3-dimensional Sasakian manifold. Based on the above arguments, it seems worthwhile to discuss the coefficient functions in the equation (1.4) for a 3-dimensional (κ, μ, ν)-contact metric manifold, along with the generalizations of a 3-dimensional Sasakian manifold introduced in the §1.

3 Fundamental formulas

In this section, we shall prepare some fundamental formulas which we need in the proof of the Theorem B.

Let $M = (M, \phi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold, and h, l be the $(1,1)$ tensor fields defined by (2.3). First, we recall the following formula by [19]:

$$
(\nabla_X \phi) Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{3.1}
$$

for any $X, Y \in \mathfrak{X}(M)$. Next, we recall that the curvature tensor R of a 3-dimensional Riemannian manifold satisfies the following identity

$$
\begin{aligned}
R(X,Y)Z = &\ g(Y,Z)QX - g(X,Z)QY - g(QX,Z)Y \\
&+ g(QY,Z)X - \frac{r}{2}\left(g(Y,Z)X - g(X,Z)Y\right),
\end{aligned} \tag{3.2}
$$

for any $X,\ Y,\ Z \in \mathfrak{X}(M)$. Now, let U be the open subset of M on which $h \neq 0$, and V be the open subset of points $m \in M$ such that $h = 0$ on a neighborhood of m. Then, we may easily check that $U \cup V$ is an open and dense subset of M. If U is not empty, for any $m \in U$, we may choose a local orthonormal frame field $\{\xi,\ e_1,\ e_2 = \phi e_1\}$ on a neighborhood of m in such a way that

$$
he_1 = \lambda e_1, \quad he_2 = -\lambda e_2, \tag{3.3}
$$

where λ is a smooth positive function on U. We may also note that, if V is not empty, then V becomes a Sasakian manifold (see §2).
Now, we assume that U is not empty. Then, by making use of (2.4), (2.5), (3.2) and (3.3), we have the following basic formulas on U:

(3.4)
$$\nabla_\xi e_1 = -ae_2, \quad \nabla_\xi e_2 = ae_1, \quad \nabla_{e_1}\xi = -(\lambda+1)e_2, \quad \nabla_{e_2}\xi = -(\lambda-1)e_1,$$
$$\nabla_{e_1}e_1 = \frac{1}{2\lambda}(e_2\lambda + A)e_2, \qquad \nabla_{e_1}e_2 = -\frac{1}{2\lambda}(e_2\lambda + A)e_1 + (\lambda+1)\xi,$$
$$\nabla_{e_2}e_2 = \frac{1}{2\lambda}(e_1\lambda + B)e_1, \qquad \nabla_{e_2}e_1 = -\frac{1}{2\lambda}(e_1\lambda + B)e_2 + (\lambda-1)\xi,$$

and we have

$$[e_1, e_2] = -\frac{1}{2\lambda}(e_2\lambda + A)e_1 + \frac{1}{2\lambda}(e_1\lambda + B)e_2 + 2\xi, \tag{3.5}$$

where $A = S(\xi, e_1), B = S(\xi, e_2)$ and a is a smooth function. Further, the Ricci operator Q [16] on U is given by

$$\begin{aligned} Q\xi &= 2(1-\lambda^2)\xi + Ae_1 + Be_2, \\ Qe_1 &= A\xi + \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right) e_1 + \xi(\lambda)e_2, \\ Qe_2 &= B\xi + \xi(\lambda)e_1 + \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right) e_2. \end{aligned} \tag{3.6}$$

Thus, from (3.2) and (3.6), we get that the components of the curvature tensor are given by

(3.7)
$$\begin{aligned} R(e_1,e_2)e_1 &= \left(2 - \frac{r}{2} - 2\lambda^2\right) e_2 - B\xi, & R(e_1,e_2)e_2 &= \left(\frac{r}{2} - 2 + 2\lambda^2\right) e_1 + A\xi, \\ R(e_1,e_2)\xi &= Be_1 - Ae_2, & R(e_1,\xi)e_1 &= -Be_2 + (\lambda^2 - 1 - 2a\lambda)\xi, \\ R(e_1,\xi)e_2 &= Be_1 - \xi(\lambda)\xi, & R(e_1,\xi)\xi &= (2a\lambda + 1 - \lambda^2)e_1 + \xi(\lambda)e_2, \\ R(e_2,\xi)e_1 &= Ae_2 - \xi(\lambda)\xi, & R(e_2,\xi)e_2 &= Be_2 + (-1 + \lambda^2 + 2a\lambda)\xi, \\ R(e_2,\xi)\xi &= \xi(\lambda)e_1 + (1 - 2a\lambda - \lambda^2)e_2. \end{aligned}$$

We have noted that $Trl = 2(1-\lambda^2)$ by (2.5). In the remaining section, we assume that M (under consideration) is a (κ, μ, ν)-contact metric manifold. Then, from (1.3), we have

(3.8)
$$R(e_1,e_2)\xi = 0, \quad R(e_1,\xi)\xi = (\kappa + \lambda\mu)e_1 + \lambda\nu e_2, \quad R(e_2,\xi)\xi = \lambda\nu e_1 + (\kappa - \lambda\mu)e_2.$$

Thus, comparing (3.7) and (3.8), we have

$$A = B = 0, \tag{3.9}$$

$$\xi\lambda = \lambda\nu, \tag{3.10}$$

$$1-\lambda^2+2a\lambda=\kappa+\lambda\mu,\qquad 1-\lambda^2-2a\lambda=\kappa-\lambda\mu, \tag{3.11}$$

Thus, from (1.4), (2.5), (3.6), (3.9), and (3.11), we have further

$$\mu=2a, \tag{3.12}$$

$$\kappa=\frac{1}{2}S(\xi,\xi)=1-\frac{1}{2}Tr(h^2)=1-\lambda^2. \tag{3.13}$$

On the other hand, from (2.4) and (3.3), taking account of (3.4), (3.9), (3.10) and (3.12), we have

$$\begin{aligned}
&(\nabla_{e_1}\eta)(e_2)=-(\lambda+1), && (\nabla_{e_1}\eta)(\xi)=0, && (\nabla_{e_2}\eta)(e_1)=-(\lambda-1),\\
&(\nabla_{e_2}\eta)(\xi)=0, && (\nabla_{\xi}\eta)(e_1)=0, && (\nabla_{\xi}\eta)(e_2)=0,
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&(\nabla_{e_1}h)(e_2)=-(e_1\lambda)e_2+(e_2\lambda)e_1-\lambda(\lambda+1)\xi,\\
&(\nabla_{e_2}h)(e_1)=-(e_1\lambda)e_2+(e_2\lambda)e_1+\lambda(\lambda-1)\xi,\\
&(\nabla_{e_1}h)(\xi)=-\lambda(\lambda+1)e_2, && (\nabla_{e_2}h)(\xi)=\lambda(\lambda-1)e_1,\\
&(\nabla_{\xi}h)(e_1)=\lambda\nu e_1-\lambda\mu e_2, && (\nabla_{\xi}h)(e_2)=-\lambda\nu e_2-\lambda\mu e_1,\\
&(\nabla_{e_1}\phi h)(e_2)=(e_1\lambda)e_1+(e_2\lambda)e_2, && (\nabla_{e_1}\phi h)(\xi)=\lambda(\lambda+1)e_1,\\
&(\nabla_{e_2}\phi h)(e_1)=(e_1\lambda)e_1+(e_2\lambda)e_2, && (\nabla_{e_2}\phi h)\xi=\lambda(\lambda-1)e_2,\\
&(\nabla_{\xi}\phi h)e_1=\lambda\nu e_2+\lambda\mu e_1, && (\nabla_{\xi}\phi h)e_2=\lambda\nu e_1-\lambda\mu e_2.
\end{aligned}$$

From (1.3), taking account of the second Bianchi identity, we get

$$\begin{aligned}
&\underset{X,Y,Z}{\mathfrak{S}} R(X,Y)\nabla_Z\xi\\
&=\underset{X,Y,Z}{\mathfrak{S}}\{(Z\kappa)(\eta(Y)X-\eta(X)Y)+\kappa((\nabla_Z\eta)(Y)X-(\nabla_Z\eta)(X)Y)\\
&\quad+(Z\mu)(\eta(Y)hX-\eta(X)hY)+\mu((\nabla_Z\eta)(Y)hX+\eta(Y)(\nabla_Zh)X\\
&\quad-(\nabla_Z\eta)(X)hY-\eta(X)(\nabla_Zh)Y)+(Z\nu)(\eta(Y)\phi hX-\eta(X)\phi hY)\\
&\quad+\nu((\nabla_Z\eta)(Y)\phi hX+\eta(Y)(\nabla_Z\phi h)X-(\nabla_Z\eta)(X)\phi hY-\eta(X)(\nabla_Z\phi h)Y)\}
\end{aligned} \tag{3.15}$$

for any $X,\ Y,\ Z\in\mathfrak{X}(M)$, where $\underset{X,Y,Z}{\mathfrak{S}}$ denotes the cycle sum with respect to the vector fields X, Y and Z. Setting $X=e_1$, $Y=e_2$ and $Z=\xi$ in (3.15), and taking account of (3.4), (3.7) and (3.14), we have

$$-2(\lambda^2-1+\lambda^2\mu)\xi=2(\kappa-\lambda^2\mu)\xi+(\lambda e_1\nu-\lambda e_2\mu-e_2\kappa)e_1+(e_1\kappa-\lambda e_1\mu-\lambda e_2\nu)e_2,$$

and hence, we have

$$e_1\kappa=\lambda(e_1\mu+e_2\nu),\quad e_2\kappa=\lambda(e_1\nu-e_2\mu). \tag{3.16}$$

Thus, from (3.16), taking account of (3.13), we have also

$$e_1\lambda=-\frac{1}{2}\left(e_1\mu+e_2\nu\right),\quad e_2\lambda=\frac{1}{2}\left(e_2\mu-e_1\nu\right). \tag{3.17}$$

By the second Bianchi identity, we have further

$$\underset{\xi,e_1,e_2}{\mathfrak{S}}(\nabla_\xi R)(e_1,\ e_2)e_1 = 0, \tag{3.18}$$

Taking account of (3.4) and (3.7) with (3.9), (3.10), (3.12) and (3.13), we have

$$\begin{aligned}(\nabla_\xi R)(e_1,e_2)e_1 &= -\left(\frac{1}{2}\xi r + 4\lambda^2\nu\right)e_2,\\ (\nabla_{e_1}R)(e_2,\xi)e_1 &= -(e_1(\lambda\nu)+\mu e_2\lambda)\xi + \lambda(\lambda+1)\nu e_2,\\ (\nabla_{e_2}R)(\xi,e_1)e_1 &= (e_2(\lambda\mu) - 2\lambda e_2\lambda + \nu e_1\lambda)\xi + \lambda(\lambda-1)\nu e_2.\end{aligned} \tag{3.19}$$

Thus, from (3.18) and (3.19), we have

$$\xi r = -4\lambda^2\nu. \tag{3.20}$$

From (3.10) and (3.13), we have also

$$\xi\kappa = -2\lambda^2\nu. \tag{3.21}$$

Now, from (3.4), (3.9), (3.12) and (3.13), we obtain

$$\begin{aligned}&R(e_1,e_2)e_1\\ &= \nabla_{e_1}(\nabla_{e_2}e_1) - \nabla_{e_2}(\nabla_{e_1}e_1) - \nabla_{[e_1,e_2]}e_1\\ &= \left\{-\frac{1}{2}e_1(e_1\log\lambda) - \frac{1}{2}e_2(e_2\log\lambda) + \frac{1}{4}(e_2\log\lambda)^2 + \frac{1}{4}(e_1\log\lambda)^2 + \kappa + \mu\right\}e_2.\end{aligned} \tag{3.22}$$

On one hand, taking account of (2.5) and (3.4), we also obtain

$$\begin{aligned}&-\frac{1}{2}\triangle\log\lambda\\ &= -\frac{1}{2}\left\{e_1(e_1\log\lambda) + e_2(e_2\log\lambda) + \xi(\xi\log\lambda) - \frac{1}{2}(e_2\log\lambda)^2 - \frac{1}{2}(e_1\log\lambda)^2\right\}.\end{aligned} \tag{3.23}$$

Thus, from the first equality in (3.7), (3.22) and (3.23), we have

$$r = \triangle\log\lambda + 2\kappa - 2\mu - \xi\nu. \tag{3.24}$$

4 Proof of Theorem B

Let $M = (M,\phi,\xi,\eta,g)$ be a 3-dimensional compact (κ,μ,ν)-contact metric manifold with $\xi\mu = \xi\nu = 0$ on M. Now, we assume that the open subset U of M on which $h \neq 0$, is not empty. We set

$$\begin{aligned}F_{min} &= \{m\in M \mid \kappa \text{ takes into minimum at } m\},\\ F_{max} &= \{m\in M \mid \kappa \text{ takes into maximum at } m\}.\end{aligned} \tag{4.1}$$

Then, we may easily check that F_{min} and F_{max} are both non-empty closed (and hence, compact) subsets of M such that $F_{min} \subset U$. And, we see that each integral curve of ξ is a geodesic in M. We denote by $\gamma(t) = \gamma(t; m)$ the integral curve of ξ though $m \in U$ with the arc-length parameter t. Then, from (3.10) and hypothesis $\xi\nu = 0$, we have

$$\lambda(t) \equiv \lambda(\gamma(t)) = \lambda(m)e^{\nu(m)t}. \tag{4.2}$$

for $|t| < \epsilon$, where ϵ is a certain positive real number. From (3.13), (4.2), we see that $\kappa(t) = \kappa(\gamma(t))$ is given by

$$\kappa(t) = 1 - \lambda(m)^2 e^{2\nu(m)t}, \tag{4.3}$$

for $|t| < \epsilon$. Thus from (4.3), we see that, for each point $m \in U$, $\gamma(t) \in U$ for all $t \in \mathbb{R}$. Now, we suppose that there exists a point $m \in U$ with $\nu(m) > 0$. Then, from (4.3), we have

$$\lim_{t \to +\infty} \kappa(t) = -\infty. \tag{4.4}$$

Similarly, if there exists a point $m \in U$ with $\nu(m) < 0$. Then from (4.3), we have also

$$\lim_{t \to -\infty} \kappa(t) = -\infty. \tag{4.5}$$

Since M is compact, we see that κ (≤ 1) must bounded on M. But, from (4.4) and (4.5), this is a contradiction. Therefore, it follows that $\nu = 0$ on U. Since V is Sasakian, it follows immediately $\nu = 0$ on V. Since $U \cup V$ is an open and dense subset in M, we see that ν vanishes on M and hence, the (κ, μ, ν)-contact metric manifold M under consideration reduces to a generalized (κ, μ)-contact metric manifold with $\xi\mu = 0$. Since $\nu = 0$ on M, from (3.17), we have on U.

$$A_1 = -\frac{1}{2}B_1, \quad A_2 = \frac{1}{2}B_2, \tag{4.6}$$

where $A_1 = e_1\lambda$, $B_1 = e_1\mu$, $A_2 = e_2\lambda$, $B_2 = e_2\mu$. From (3.4) and (3.5), we have

$$[e_1, \xi] = \left(\frac{\mu}{2} - \lambda - 1\right) e_2, \quad [e_2, \xi] = -\left(\frac{\mu}{2} + \lambda - 1\right) e_1. \tag{4.7}$$

Since $\nu = 0$, from (3.10), we have also

$$\xi\lambda = 0. \tag{4.8}$$

Thus, from (4.7), taking account of (4.6) and (4.8), we obtain

$$\xi A_1 = \left(\lambda + 1 - \frac{\mu}{2}\right) A_2, \quad \xi A_2 = \left(\lambda - 1 + \frac{\mu}{2}\right) A_1. \tag{4.9}$$

Similarly, from (4.7), taking account of (4.6) and $\xi\mu = 0$, we obtain

$$\xi A_1 = -\left(\lambda + 1 - \frac{\mu}{2}\right) A_2, \quad \xi A_2 = -\left(\lambda - 1 + \frac{\mu}{2}\right) A_1. \tag{4.10}$$

Thus, from (4.9) and (4.10), we have

$$\left(\lambda + 1 - \frac{\mu}{2}\right) A_2 = 0. \tag{4.11}$$

$$\left(\lambda - 1 + \frac{\mu}{2}\right) A_1 = 0. \tag{4.12}$$

Lemma 4.1. *$A_1 = 0$ or $A_2 = 0$ at each point of U.*

Proof. We assume that $A_1 \neq 0$ and $A_2 \neq 0$ at some point $m \in U$. Then, from (4.11) and (4.12), it follows that $\lambda + 1 - \frac{\mu}{2} = 0$ and $\lambda - 1 + \frac{\mu}{2} = 0$ at the point m, and hence, $\lambda = 0$ at m. But, this is a contradiction. □

Now, we define subsets F_1, F_2, G_1, G_2 and F of U by

$$\begin{aligned}
G_1 &= \{m \in U | A_1 \neq 0 \ (i.e.\ A_2 = 0) \ at\ m\},\\
G_2 &= \{m \in U | A_2 \neq 0 \ (i.e.\ A_1 = 0) \ at\ m\},\\
F_1 &= \{m \in U | \lambda - 1 + \frac{\mu}{2} = 0 \ at\ m\},\\
F_2 &= \{m \in U | \lambda + 1 - \frac{\mu}{2} = 0 \ at\ m\},\\
F &= \{m \in U | A_1 = A_2 = 0 \ (i.e.\ B_1 = B_2 = 0) \ at\ m\}.
\end{aligned}$$

Then, taking account of (4.11) and (4.12) and Lemma 4.1, we have the following relations.

$$\begin{aligned}
&G_1 \subset F_1,\ G_2 \subset F_2,\ F_1 \cap F_2 = \varnothing, \text{ and}\\
&U = G_1 \cup G_2 \cup F = F_1 \cup F_2 \quad \text{(disjoint union).}
\end{aligned} \tag{4.13}$$

We have denoted by $F_{(i)}$ the interior of F in U. Then, taking account of (4.9), we may observe that, if $F_{(i)} \neq \varnothing$, then λ (and hence, κ) is constant on $F_{(i)}$. From (4.13), we see that $G_1 \cup G_2 \cup F_{(i)}$ is an open and dense subset in U. First, we assume that the inequality $r + \frac{\mu^2}{2} \geq 0$ holds on M. If $G_1 \neq \varnothing$, then from (3.24), taking account of (4.12), we have

$$\triangle \log \lambda = r - 2(1 - \lambda^2) - 4(\lambda - 1) = r + 2(\lambda - 1)^2 = r + \frac{\mu^2}{2} \geq 0 \tag{4.14}$$

on G_1. Similarly, if $G_2 \neq \varnothing$, then, from (3.24), taking account of (4.11), we have

$$\triangle \log \lambda = r - 2(1 - \lambda^2) + 4(\lambda + 1) = r + 2(\lambda + 1)^2 = r + \frac{\mu^2}{2} \geq 0 \tag{4.15}$$

on G_2. Therefore, we have the following inequality

$$\triangle \log \lambda \geq 0 \tag{4.16}$$

on $G_1 \cup G_2$. By direct calculation, we get

$$\triangle \log \lambda = -\frac{1}{\lambda^2} |\mathrm{g}\, rad \lambda|^2 + \frac{1}{\lambda} \triangle \lambda \tag{4.17}$$

on $G_1 \cup G_2$. Further, since $\kappa = 1 - \lambda^2$ on U, we get also

$$\triangle\kappa = -2|\,\mathrm{g}\,rad\lambda|^2 - 2\lambda \triangle \lambda \tag{4.18}$$

on $G_1 \cup G_2$. Thus, from (4.17) and (4.18), we have

$$\triangle\kappa = -4|\,\mathrm{g}\,rad\lambda|^2 - 2\lambda^2 \triangle \log\lambda \leq 0 \tag{4.19}$$

on $G_1 \cup G_2$. On the other hand, κ=const on $F_{(i)}$. Since $G_1 \cup G_2 \cup F_{(i)}$ is an open and everywhere dense subset of U, from (4.19), we have the inequality $\triangle\kappa \leq 0$ on U. If $V \neq \varnothing$, V is Sasakian (and has $\kappa = 1$ on V), since $\kappa = 1$ on V, it is evident that $\triangle\kappa = 0$ holds on V. Since $U \cup V$ is open and everywhere dense in M, we see finally that

$$\triangle\kappa \leq 0 \tag{4.20}$$

holds on M. On the other hand, the function κ takes its minimum on the non-empty subset F_{min}. Therefore, by Hopf's theorem, we see that κ is constant on M, and hence, μ is also constant on M. Next, we assume that the inequality $r + \frac{\mu^2}{2} \leq 0$ holds everywhere on M. Then, applying the similar arguments as in the previous case where $r + \frac{\mu^2}{2} \geq 0$, we have $\triangle\kappa \geq 0$ holds on M. Since the function κ takes its maximum on the non-empty subset F_{max}. Therefore, by Hopf's theorem, we see also that κ and μ are both constant on M.

As the result, we see that M is a non-Sasakian (κ, μ)-contact metric manifold with $\kappa = \mu - \frac{\mu^2}{4}$ and hence $r = -\frac{\mu^2}{2}$ by virtue of (3.24) if $U \neq \varnothing$. On the other hand, it is evident that M is Sasakian ($\kappa = 1$ and $\mu = \nu = 0$) if $U = \varnothing$. This completes the proof of Theorem B.

5 Examples

In this section, we shall provide an example of the 3-dimensional Sasakian manifold $M = (M, \phi, \xi, \eta, g)$ with non-constant coefficient functions α and β in the defining equation (1.5) of an η-Einstein manifold are both non-constant (see Example 1), and also an example of the 3-dimensional generalized (κ, μ)-contact metric manifold which illustrates as well as supports Theorem B (see Example 2). Example 1 below is a special case of the example introduced in Blair's book [1].

Example 1 Let M=$\mathbb{R}^3$ and set

$$\xi = 2\frac{\partial}{\partial z}, \quad e_1 = 2\frac{\partial}{\partial y}, \quad e_2 = 2(\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}). \tag{5.1}$$

Let η be the 1-form dual to ξ, and define $(1,1)$-tenser field ϕ by $\phi\xi = 0$, $\phi e_1 = e_2$ and $\phi e_2 = -e_1$. Further, let g be the Riemannian metric defined by $g(\xi, \xi) = 1, g(\xi, e_i) = 0$ and $g(e_i, e_i) = \delta_{ij}$ for $1 \leq i, j \leq 2$. Then, by direct calculation, we may check that (M, ϕ, ξ, η, g) is a 3-dimensional Sasakian manifold and the Ricci transformation Q is given by

$$Q = -(2 + 24y^2)I + (4 + 24y^2)\eta \otimes \xi \tag{5.2}$$

on M. Therefore, from (5.2), we see that the 3-dimensional Sasakian manifold M provides an explicit example of the η-Einstein manifold with non-constant coefficient functions α and β in (1.5) which is mentioned in §2.

The following example which is constructed by Koufogiorgos and Tsichlias [11], which illustrates Theorem B.

Example 2 Let $M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ and set

$$\xi = \frac{\partial}{\partial x}, \quad e_1 = -2y\frac{\partial}{\partial x} + (2\sqrt{z}x - \frac{1}{4z}y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}. \tag{5.3}$$

Let η be the 1-form dual to ξ, and define $(1,1)$-tenser field ϕ by $\phi\xi = 0$, $\phi e_1 = e_2$ and $\phi e_2 = -e_1$. Further, let g be the Riemannian metric defined by $g(\xi, \xi) = 1, g(\xi, e_i) = 0$ and $g(e_i, e_i) = \delta_{ij}$ for $1 \leq i, j \leq 2$. Then, by direct calculation, we may check that (M, ϕ, ξ, η, g) is a 3-dimensional generalized (κ, μ, ν)-contact metric manifold with $\kappa = 1 - z$, $\mu = 2(1 - \sqrt{z})$ (and $\nu = 0$) and $r + \frac{\mu^2}{2} = -\frac{5}{8z^2} < 0$ on M.

Thus, Example 2 shows that the compactness assumption in Theorem B plays an essential role.

It is well-known that a 3-dimensional Lie group G admits a discrete subgroup Γ such that the space of right cosets $\Gamma\backslash G$ is compact if and only if G is unimodular [13]. Let G be one of the following simply connected unimodular Lie groups: $\tilde{E}(2)$, $E(1,1)$. Then, from the proof of the Theorem B and ([2,§4], [15]), we may check that $M = \Gamma\backslash G$ with a suitable discrete subgroup Γ of G, provides an example illustrating Theorem B for non-Sasakian case. **Acknowledgement.** This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (2011-0012987).

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