On three dimensional affine Szabó manifolds

A. S. Diallo, S. Longwap, F. Massamba

Abstract. We consider the cyclic parallel Ricci tensor condition, which is a necessary condition for an affine manifold to be Szabó. We show that, in three dimension, there are affine manifolds which satisfy the cyclic parallel Ricci tensor but are not Szabó. Conversely, it is known that in two dimension, the cyclic parallel Ricci tensor forces the affine manifold to be Szabó. Examples of 3-dimensional affine Szabo manifolds are also given. We prove that an affine surface with skew-symmetric Ricci tensor is affine Szabó. Finally, we give some properties of Riemann extensions defined on the cotangent bundle over an affine Szabó manifold.

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1 Introduction

The theory of connection is a classical topic in differential geometry. It was initially developed to solve pure geometrical problems. It provides an extremely important tool to study geometric structures on manifolds and, as such, has been applied with great success in many different settings. For instance, Opozda in [15] classified locally homogeneous connection on 2-dimensional manifolds equipped with torsion free affine connection. Arias-Marco and Kowalski [1] classified locally homogeneous connections with arbitrary torsion on 2-dimensional manifolds. García-Rio *et al.* [8] introduced the notion of the affine Osserman connections. The affine Osserman connections are well understood in two dimension (see [5, 8] for more details and references therein).

A (pseudo-)Riemannian manifold (M, g) is said to be Szabó if the eigenvalues of the Szabó operator given by

$$S(X) : Y \to (\nabla_X R)(Y, X)X,$$

are constants on the unit (pseudo-)sphere bundle, where R denoting the curvature tensor (see [2] and [9] for details). The Szabó operator is a self adjoint operator with $S(X)X = 0$. It plays an important role in the study of totally isotropic manifolds

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[10]. Szabó in [17] used techniques from algebraic topology to show, in the Riemannian setting, that any such a metric is locally symmetric. He used this observation to prove that any two point homogeneous space is either flat or is a rank one symmetric space. Subsequently Gilkey and Stravrov [11] extended this result to show that any Szabó Lorentzian manifold has constant sectional curvature. However, for metrics of higher signature the situation is different. Indeed it was showed in [10] the existence of Szabó Pseudo-Riemannian manifolds endowed with metrics of signature (p, q) with $p \geq 2$ and $q \geq 2$ which are not locally symmetric .

In [6], the authors introduced the so-called *affine Szabó connections.* They proved, in two dimension, that an affine connection ∇ is affine Szabó if and only if the Ricci tensor of ∇ is cyclic parallel while in dimension 3 the concept seems to be very challenging by giving only partial results.

The aim of this paper is to give an explicit form of two families of affine connections which are affine Szabó on 3-dimensional manifolds. Moreover, although both results provide examples of affine Szabó connections, they are essentially different in nature since, in the first family, the affine Szabó condition coincides with the cyclic parallelism of the Ricci tensor, whereas the second one is not. For any affine connection ∇ on M, there exist a technique called *Riemann extension*, which relates affine and pseudo-Riemannian geometries. This technique is very powerful in constructing new examples of pseudo-Riemannian metrics. The relation between affine Szabó manifolds and pseudo-Riemannian Szabó manifolds are investigated by using Riemann extensions.

The paper is organized as follows. In section 2, we recall some basic definitions and geometric objects, namely, torsion tensor, curvature tensor, Ricci tensor and affine Szabó operator on an affine manifold. In section 3, we study the cyclic parallelism of the Ricci tensor for two particular cases of affine connections in 3-dimensional affine manifolds. We establish geometrical configurations of affine manifolds admitting a cyclic parallel Ricci tensor (Propositions 3.2 and 3.4). In section 4, we study the Szabó condition on two particular affine connections (Theorems 4.5 and 4.7). Affine surfaces with skew symmetric Ricci tensor are also studied. Finally, we end the paper in section 5 by investigating the Riemann extensions defined on the cotangent bundle over an affine Szabó manifold.

2 Preliminaries

Let M be an n-dimensional smooth manifold and ∇ be an affine connection on M. We consider a system of coordinates $(x_1, x_2, \cdots, x_n)$ in a neighborhood $\mathcal{U}$ of a point p in M. In $\mathcal{U}$ the affine connection is given by

$$\nabla_{\partial_i}\partial_j = f_{ij}^k \partial_k, \tag{2.1}$$

where $\{\partial_i = \frac{\partial}{\partial x_i}\}_{1\leq i\leq n}$ is a basis of the tangent space T_pM and the functions f_{ij}^k $(i, j, k = 1, 2, 3, \cdots, n)$ are called the *Christoffel symbols* of the affine connection. We shall call the pair (M, ∇) *affine manifold.* Some tensor fields associated with the given affine connection ∇ are defined below.

The *torsion tensor* field T^∇ is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]}, \tag{2.2}$$

for any vector fields X and Y on M. The components of the torsion tensor T^∇ in local coordinates are

$$T_{ij}^k = f_{ij}^k - f_{ji}^k. \tag{2.3}$$

If the torsion tensor of a given affine connection ∇ vanishes, we say that ∇ is torsion-free. The *curvature tensor field* $\mathcal{R}^\nabla$ is defined by

$$\mathcal{R}^\nabla(X,Y) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.4}$$

for any vector field X, Y and Z on M. The components in local coordinates are

$$\mathcal{R}^\nabla(\partial_k, \partial_l) = \sum_i R_{jkl}^i \partial_i. \tag{2.5}$$

We shall assume that ∇ is torsion-free. If $\mathcal{R}^\nabla = 0$ on M, we say that ∇ is flat affine connection. It is known that ∇ is flat if and only if around a point p there exist a local coordinate system such that $f_{ij}^k = 0$ for all i, j, k.

We define *Ricci tensor* Ric^∇ by

$$Ric^\nabla(X,Y) = \mathrm{trace}\{Z \mapsto \mathcal{R}^\nabla(Z,X)Y\}. \tag{2.6}$$

The components in local coordinates are given by

$$Ric^\nabla(\partial_j, \partial_k) = \sum_i R_{kij}^i. \tag{2.7}$$

It is known that in Riemannian geometry the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is $Ric^\nabla(X,Y) = R^\nabla(Y,X)$. But this property is not true for an arbitrary torsion-free affine connection. In fact, the property is closely related to the concept of parallel volume element. (See [14] for more details).

The covariant derivative of the curvature tensor $\mathcal{R}^\nabla$ is given by

$$\begin{aligned}(\nabla_X\mathcal{R}^\nabla)(Y,Z)W &= \nabla_X\mathcal{R}^\nabla(Y,Z)W - \mathcal{R}^\nabla(\nabla_X Y, Z)W \\ &\quad - \mathcal{R}^\nabla(Y,\nabla_X Z)W - \mathcal{R}^\nabla(Y,Z)\nabla_X W.\end{aligned}$$

The covariant derivative of the Ricci tensor Ric^∇ is defined by

$$(\nabla_X Ric^\nabla)(Z,W) = X(Ric^\nabla(Z,W)) - Ric^\nabla(\nabla_X Z, W) - Ric^\nabla(Z, \nabla_X W). \tag{2.8}$$

For $X \in \Gamma(T_pM)$, we define the *affine Szabó operator* $S^\nabla(X) : T_pM \to T_pM$ with respect to X by

$$S^\nabla(X)Y := (\nabla_X\mathcal{R}^\nabla)(Y,X)X, \tag{2.9}$$

for any vector field Y. The affine Szabó operator satisfies $S^\nabla(X)X = 0$ and $S^\nabla(\beta X) = \beta^3 S^\nabla(X)$ for $\beta \in \mathbb{R} - \{0\}$ and $X \in T_pM$. If $Y = \partial_m$, for $m = 1, 2, \cdots, n$ and $X = \sum_i \alpha_i\partial_i$, one gets

$$S^\nabla(X)\partial_m = \sum_{i,j,k=1}^n \alpha_i\alpha_j\alpha_k(\nabla_{\partial_i}\mathcal{R}^\nabla)(\partial_m, \partial_j)\partial_k. \tag{2.10}$$

Note that, by definition of the Ricci tensor, one has

$$\mathrm{trace}(Y \mapsto (\nabla_X\mathcal{R}^\nabla)(Y,X)X) = (\nabla_X Ric^\nabla)(X,X). \tag{2.11}$$

3 Affine connections with cyclic parallel Ricci tensor

In this section, we investigate affine connections whose Ricci tensors are cyclic parallel. We shall consider two cases of 3-dimensional smooth manifolds with specific affine connections. We start with a formal definition.

Definition 3.1. The Ricci tensor Ric^{∇} of an affine manifold (M, ∇) is *cyclic parallel* if

$$(\nabla_X Ric^{\nabla})(X, X) = 0, \tag{3.1}$$

for any vector field X tangent to M or, equivalently, if

$$\mathfrak{S}_{X,Y,Z}(\nabla_X Ric^{\nabla})(Y, Z) = 0,$$

for any vector fields X, Y, and Z tangent to M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y and Z.

Locally, the equation (3.1) takes the form

$$(\nabla_{\partial_i} Ric^{\nabla})_{jk} = 0, \tag{3.2}$$

or can be written out without the symmetrizing brackets

$$(\nabla_{\partial_i} Ric^{\nabla})_{jk} + (\nabla_{\partial_j} Ric^{\nabla})_{ki} + (\nabla_{\partial_k} Ric^{\nabla})_{ij} = 0. \tag{3.3}$$

For $X = \sum_i \alpha_i \partial_i$, it is easy to show that

$$(\nabla_X Ric^{\nabla})(X, X) = \sum_{i,j,k} \alpha_i \alpha_j \alpha_k (\nabla_{\partial_i} Ric^{\nabla})_{jk}. \tag{3.4}$$

Now, we are going to present two cases of affine connections in which we investigate the cyclic parallelism of the Ricci tensor.

Case 1: Let M be a 3-dimensional smooth manifold and ∇ be an affine torsion-free connection. Suppose that the action of the affine connection ∇ on the basis $\{\partial_i\}_{1\leq i\leq 3}$ is given by

$$\nabla_{\partial_1}\partial_1 = f_1\partial_1, \;\; \nabla_{\partial_1}\partial_2 = f_2\partial_1 \;\text{ and }\; \nabla_{\partial_1}\partial_3 = f_3\partial_1, \tag{3.5}$$

where the smooth functions$f_i = f_i(x_1, x_2, x_3)$ are the Christoffel symbols. The non-zero components of the curvature tensor $\mathcal{R}^{\nabla}$ of the affine connection (3.5) are given by

$$\begin{aligned}
&\mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_1 = (\partial_1 f_2 - \partial_2 f_1)\partial_1, \quad \mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_2 = -(\partial_2 f_2 + f_2^2)\partial_1,\\
&\mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_3 = -(\partial_2 f_3 + f_2 f_3)\partial_1, \quad \mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_1 = (\partial_1 f_3 - \partial_3 f_1)\partial_1,\\
&\mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_2 = -(\partial_3 f_2 + f_2 f_3)\partial_1, \quad \mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_3 = -(\partial_3 f_3 + f_3^2)\partial_1,\\
&\mathcal{R}^{\nabla}(\partial_2, \partial_3)\partial_1 = (\partial_2 f_3 - \partial_3 f_2)\partial_1.
\end{aligned}$$

From (2.7), the non-zero components of the Ricci tensor Ric^{∇} of the affine connection (3.5) are given by

$$\begin{aligned}
&Ric^{\nabla}(\partial_2,\partial_1)=\partial_1 f_2-\partial_2 f_1,\quad Ric^{\nabla}(\partial_2,\partial_2)=-(\partial_2 f_2+f_2^2),\\
&Ric^{\nabla}(\partial_2,\partial_3)=-(\partial_2 f_3+f_2 f_3),\quad Ric^{\nabla}(\partial_3,\partial_1)=\partial_1 f_3-\partial_3 f_1,\\
&Ric^{\nabla}(\partial_3,\partial_2)=-(\partial_3 f_2+f_2 f_3),\quad Ric^{\nabla}(\partial_3,\partial_3)=-(\partial_3 f_3+f_3^2).
\end{aligned}$$

Proposition 3.1. *On $\mathbb{R}^3$, the affine connection ∇ defined in (3.5) satisfies the relation (3.1) if the functions $f_i=f_i(x_1,x_2,x_3)$, for $i=1,2,3$, satisfy the following partial differential equations:*

$$\begin{aligned}
&\partial_3^2 f_3+2f_3\partial_3 f_3=0,\\
&\partial_2^2 f_2+2f_2\partial_2 f_2=0,\\
&\partial_3^2 f_1+4f_3\partial_1 f_3-2f_3\partial_3 f_1=0,\\
&\partial_2^2 f_1+4f_2\partial_1 f_2-2f_2\partial_2 f_1=0,\\
&\partial_1^2 f_3-\partial_1\partial_3 f_1-f_1\partial_1 f_3+f_1\partial_3 f_1=0,\\
&\partial_1^2 f_2-\partial_1\partial_2 f_1-f_1\partial_1 f_2+f_1\partial_2 f_1=0,\\
&\partial_2^2 f_3+2\partial_3\partial_2 f_2+2f_2\partial_3 f_2+2f_3\partial_2 f_2+2f_2\partial_2 f_3=0,\\
&\partial_3^2 f_2+2\partial_3\partial_2 f_3+2f_3\partial_2 f_3+2f_3\partial_3 f_2+2f_2\partial_3 f_3=0,\\
&4f_3\partial_1 f_2+4f_2\partial_1 f_3-2f_3\partial_2 f_1-2f_2\partial_3 f_1+2\partial_3\partial_2 f_1=0.
\end{aligned}\tag{3.6}$$

Proof. The proof follows straightforward from (2.8) and (3.4). □

As an example to the Proposition 3.1, we have the following.

Example 3.2. The Ricci tensors of the affine connections defined in (3.5) on $\mathbb{R}^3$ with (1) $f_1=0, f_2=-x_3$ and $f_3=x_2$; (2) $f_1=x_1, f_2=2x_3$ and $f_3=-2x_2$ are cyclic parallel.

Case 2: Let M be a 3-dimensional smooth manifold and ∇ be an affine torsion-free connection. Suppose that the action of the affine connection ∇ on the basis $\{\partial_i\}_{1\leq i\leq 3}$ is given by

$$\nabla_{\partial_1}\partial_1=f_1\partial_2,\ \ \nabla_{\partial_2}\partial_2=f_2\partial_3\ \text{ and }\ \nabla_{\partial_3}\partial_3=f_3\partial_1,\tag{3.7}$$

where the smooth functions $f_i=f_i(x_1,x_2,x_3)$ are the Christoffel symbols. The non-zero components of the curvature tensor $\mathcal{R}^{\nabla}$ of the affine connection (3.7) are given by

$$\begin{aligned}
&\mathcal{R}^{\nabla}(\partial_1,\partial_2)\partial_1=-(\partial_2 f_1\partial_2+f_1f_2\partial_3),\quad \mathcal{R}^{\nabla}(\partial_1,\partial_2)\partial_2=\partial_1 f_2\partial_3,\\
&\mathcal{R}^{\nabla}(\partial_1,\partial_3)\partial_1=-\partial_3 f_1\partial_2,\quad \mathcal{R}^{\nabla}(\partial_1,\partial_3)\partial_3=\partial_1 f_3\partial_1+f_1f_3\partial_2,\\
&\mathcal{R}^{\nabla}(\partial_2,\partial_3)\partial_2=-(\partial_3 f_2\partial_3+f_3f_2\partial_1),\quad \mathcal{R}^{\nabla}(\partial_2,\partial_3)\partial_3=\partial_2 f_3\partial_1.
\end{aligned}$$

From (2.7), the non-zero components of the Ricci tensor Ric^{∇} of the affine connection (3.7) are given by $Ric^{\nabla}(\partial_1,\partial_1)=\partial_2 f_1$, $Ric^{\nabla}(\partial_2,\partial_2)=\partial_3 f_2$, $Ric^{\nabla}(\partial_3,\partial_3)=\partial_1 f_3$.

Proposition 3.2. *The affine connection ∇ defined on $\mathbb{R}^3$ by (3.7) satisfies (3.1) if the functions $f_i = f_i(x_1, x_2, x_3)$, for $i = 1, 2, 3$, have the following forms: $f_1 = f(x_1) + g(x_3)$, $f_2 = h(x_1) + u(x_2)$, $f_3 = v(x_2) + t(x_3)$, where f, g, h, u, v and t are smooth functions on $\mathbb{R}^3$.*

Proof. From a straightforward calculation, using (2.8) and (3.4), one obtains the following partial differential equations: $\partial_1\partial_2 f_1 = 0$, $\partial_3\partial_2 f_1 = 0$, $\partial_1\partial_3 f_2 = 0$, $\partial_2\partial_3 f_2 = 0$, $\partial_2\partial_1 f_3 = 0$, $\partial_3\partial_1 f_3 = 0$, $\partial_2^2 f_1 - 2f_1\partial_3 f_2 = 0$, $\partial_1^2 f_3 - 2f_3\partial_2 f_1 = 0$, $\partial_3^2 f_2 - 2f_2\partial_1 f_3 = 0$, and the result follows. □

As an application to this proposition, we have:

Example 3.3. The Ricci tensor of the following affine connection defined on $\mathbb{R}^3$ by $\nabla_{\partial_1}\partial_1 = x_1^2\partial_2$, $\nabla_{\partial_2}\partial_2 = (x_1 + x_2)\partial_3$ and $\nabla_{\partial_3}\partial_3 = (x_2 + x_3^2)\partial_1$ is cyclic parallel.

The manifolds with cyclic parallel Ricci tensor, known as L_3-spaces, are well-developed in Riemannian geometry. The cyclic parallelism of the Ricci tensor is sometime called the "*First Ledger condition*" [16]. In [18], for instance, the author proved that a smooth Riemannian manifold satisfying the first Ledger condition is real analytic. These Riemannian manifolds were introduced by A. Gray in ([12]) as a special subclass of (connected) Riemannian manifolds (M, g), called Einstein-like spaces, all of which have constant scalar curvature. Also, Riemannian manifolds of three dimension with cyclic parallel Ricci tensor are locally homogeneous naturally reductive ([16]). Tod in [19] used the same condition to characterize the 4-dimensional Kähler manifolds which are not Einstein. It has also enriched the D'Atri spaces (see [13, 16] for more details).

4 The affine Szabó manifolds

Let (M, ∇) be an n-dimensional affine manifold, i.e., ∇ is a torsion free connection on the tangent bundle of a smooth manifold M of n-dimension. Let $\mathcal{R}^\nabla$ be the associated curvature operator. We define the *affine Szabó operator* $S^\nabla(X) : T_pM \to T_pM$ with respect to a vector $X \in T_pM$ by

$$S^\nabla(X)Y := (\nabla_X \mathcal{R}^\nabla)(Y, X)X.$$

Definition 4.1. Let (M, ∇) be a smooth affine manifold and $p \in M$.

(1) (M, ∇) is called *affine Szabó* at $p \in M$ if the affine Szabó operator $S^\nabla(X)$ has the same characteristic polynomial for every vector field X on M.

(2) Also, (M, ∇) is called affine Szabó if (M, ∇) is affine Szabó at each point $p \in M$.

Theorem 4.1. *Let (M, ∇) be an n-dimensional affine manifold and $p \in M$. Then (M, ∇) is affine Szabó at $p \in M$ if and only if the characteristic polynomial of the affine Szabó operator $S^\nabla(X)$ is $P_\lambda(S^\nabla(X)) = \lambda^n$, for every $X \in T_pM$.*

Corollary 4.2. *We say that (M, ∇) is affine Szabó if the affine Szabó operators are nilpotent, i.e., 0 is the eigenvalue of $S^\nabla(X)$ on the tangent bundle.*

Corollary 4.3. *If (M, ∇) is affine Szabó at $p \in M$, then the Ricci tensor is cyclic parallel.*

Affine Szabó connections are well-understood in 2-dimension, due to the fact that an affine connection is Szabó if and only if its Ricci tensor is cyclic parallel [6]. The situation is however more involved in higher dimensions where the cyclic parallelism is a necessary but not sufficient condition for an affine connection to be Szabó.

4.1 Affine surface with skew-symmetric Ricci tensor

The curvature of an affine surface is encoded by its Ricci tensor. Here, we investigate affine surfaces whose Ricci tensor are skew-symmetric.

Theorem 4.4. *Let ∇ be a torsion-free affine connection on a surface Σ such that the Ricci tensor of ∇ is skew-symmetric and nonzero everywhere. Then (Σ, ∇) is affine Szabó.*

Proof. Fixing coordinates (x_1, x_2) on Σ and let ∇ be a torsion-free affine connection given by $\nabla_{\partial_i}\partial_j = f_{ij}^k \partial_k$, for $i, j, k = 1, 2$, where $f_{ij}^k = f_{ij}^k(x_1, x_2)$. Let $X = \alpha_1\partial_1 + \alpha_2\partial_2$ be a vector field on Σ. It is easy to check that the affine Szabó operator $\mathcal{S}(X)$ is expressed, with respect to the basis $\{\partial_1, \partial_2\}$, as

$$(\mathcal{S}^\nabla(X)) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

where the coefficients A, B, C and D are given by

$$\begin{aligned}
A &= \alpha_1^2\alpha_2[\partial_1\rho_{21} - a(f_{11}^1 + f_{12}^2)\rho_{21} - f_{12}^1\rho_{11} - f_{11}^2\rho_{22}] \\
&\quad + \alpha_1\alpha_2^2[\partial_2\rho_{21} + \partial_1\rho_{22} - (f_{12}^1 + f_{22}^2)\rho_{21} - (\rho_{12} + \rho_{21})f_{12}^1 - f_{22}^1\rho_{11} - 3f_{12}^2\rho_{22}] \\
&\quad + \alpha_2^3[\partial_2\rho_{22} - 2f_{22}^2\rho_{22} - (\rho_{12} + \rho_{21})f_{22}^1], \\
B &= \alpha_1^2\alpha_2[-\partial_1\rho_{11} + 2f_{11}^1\rho_{11} + (\rho_{12} + \rho_{21})f_{11}^2] \\
&\quad + \alpha_1\alpha_2^2[-\partial_2\rho_{11} - \partial_1\rho_{12} + 3f_{12}^1\rho_{11} + f_{11}^2\rho_{22} + (\rho_{12} + \rho_{21})f_{12}^2 + (f_{11}^1 + f_{12}^2)\rho_{12}] \\
&\quad + \alpha_2^3[-\partial_2\rho_{12} + f_{22}^1\rho_{11} + f_{12}^2\rho_{22} + (f_{12}^1 + f_{22}^2)\rho_{12}], \\
C &= \alpha_1^3[-\partial_1\rho_{21} + (f_{11}^1 + f_{12}^2)\rho_{21} + f_{12}^1\rho_{11}] \\
&\quad + \alpha_1^2\alpha_2[-\partial_2\rho_{21} - \partial_1\rho_{22} + (f_{12}^1 + f_{22}^2)\rho_{21} + f_{22}^1\rho_{11} + 3f_{12}^2\rho_{22} + (\rho_{12} + \rho_{21})f_{12}^1] \\
&\quad + \alpha_1\alpha_2^2[-\partial_2\rho_{22} + 2f_{22}^2\rho_{22} + (\rho_{12} + \rho_{21})f_{22}^1], \\
D &= \alpha_1^3[\partial_1\rho_{11} - 2f_{11}^1\rho_{11} - (\rho_{12} + \rho_{21})f_{11}^2] \\
&\quad + \alpha_1^2\alpha_2[\partial_2\rho_{11} + \partial_1\rho_{12} - 3f_{12}^1\rho_{11} - f_{11}^2\rho_{22} - (f_{11}^1 + f_{12}^2)\rho_{12} - (\rho_{12} + \rho_{21})f_{12}^2] \\
&\quad + \alpha_1\alpha_2^2[\partial_2\rho_{12} - f_{22}^1\rho_{11} - f_{12}^2\rho_{22} - (f_{12}^1 + f_{22}^2)\rho_{12}].
\end{aligned}$$

Hence, the characteristic polynomial of $\mathcal{S}^\nabla(X)$ is given by $P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^2 - \lambda(A + D) + (AD - BC)$. If the Ricci tensor of ∇ is skew-symmetric, that is, $\rho_{11} = \rho_{22} = 0$ and $\rho_{12} = -\rho_{21}$. Then the Szabó operator is nilpotent. □

The investigation of affine connections with skew-symmetric Ricci tensor on surfaces has been extremely attractive and fruitful over the recent years. We refer to the paper [4] by Derdzinski for further details. Taking into account the simplified Wong's theorem given in [4], we have the following.

Theorem 4.5. *If every point of an affine surface Σ has a neighborhood U with coordinates u_1, u_2 in which the component functions of a torsion-free affine connection ∇ are $f_{11}^1 = -\partial_1\varphi$, $f_{22}^2 = \partial_2\varphi$, for some function φ, $f_{jk}^l = 0$, unless $j = k = l$, then (Σ, ∇) is affine Szabó.*

4.2 Affine Szabó connections on three-dimensional affine manifolds

Let $X = \sum_{i=1}^3 \alpha_i\partial_i$ be a vector field on a 3-dimensional affine manifold M. Then the affine Szabó operator is given by

$$S^\nabla(X)(\partial_m) = \sum_{i,j,k=1}^{3} \alpha_i\alpha_j\alpha_k(\nabla_{\partial_i}\mathcal{R}^\nabla)(\partial_m, \partial_j)\partial_k, \quad m = 1, 2, 3.$$

First Family of affine Szabó connection.

Next, we give an example of a family of affine Szabó connection on a 3-dimensional manifold. Let us consider the affine connection defined in (3.5), i.e.,

$$\nabla_{\partial_1}\partial_1 = f_1\partial_1, \quad \nabla_{\partial_1}\partial_2 = f_2\partial_1 \quad \text{and} \quad \nabla_{\partial_1}\partial_3 = f_3\partial_1,$$

where the smooth functions $f_i = f_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ are Christoffel symbols. For $X = \sum_{i=1}^3 \alpha_i\partial_i$, the affine Szabó operator is given by

$$S^\nabla(X)(\partial_1) = a_{11}\partial_1, \quad S^\nabla(X)(\partial_2) = a_{12}\partial_1 \quad \text{and} \quad S^\nabla(X)(\partial_3) = a_{13}\partial_1,$$

with

$$\begin{aligned}
a_{11} &= \alpha_3^3\{\partial_3^2 f_3 + 2f_3\partial_3 f_3\} + \alpha_2^3\{\partial_2^2 f_2 + 2f_2\partial_2 f_2\} + \alpha_3^2\alpha_1\{\partial_3^2 f_1 + 4f_3\partial_1 f_3 - 2f_3\partial_3 f_1\}\\
&\quad + \alpha_2^2\alpha_1\{\partial_2^2 f_1 + 4f_2\partial_1 f_2 - 2f_2\partial_2 f_1\} + \alpha_1^2\alpha_3\{\partial_1^2 f_3 - \partial_1\partial_3 f_1 - f_1\partial_1 f_3 + f_1\partial_3 f_1\}\\
&\quad + \alpha_1^2\alpha_2\{\partial_1^2 f_2 - \partial_1\partial_2 f_1 - f_1\partial_1 f_2 + f_1\partial_2 f_1\}\\
&\quad + \alpha_2^2\alpha_3\{\partial_2^2 f_3 + 2\partial_3\partial_2 f_2 + 2f_2\partial_3 f_2 + 2f_3\partial_2 f_2 + 2f_2\partial_2 f_3\}\\
&\quad + \alpha_3^2\alpha_2\{\partial_3^2 f_2 + 2\partial_3\partial_2 f_3 + 2f_3\partial_2 f_3 + 2f_3\partial_3 f_2 + 2f_2\partial_3 f_3\}\\
&\quad + \alpha_1\alpha_2\alpha_3\{4f_3\partial_1 f_2 + 4f_2\partial_1 f_3 - 2f_3\partial_2 f_1 - 2f_2\partial_3 f_1 + 2\partial_3\partial_2 f_1\},\\
a_{12} &= \alpha_1^3\{\partial_1\partial_2 f_1 - \partial_1^2 f_2 - f_1\partial_2 f_1 + 2f_1\partial_1 f_2\} + \alpha_2^2\alpha_1\{\partial_2^2 f_2 + 2f_2\partial_2 f_2 - f_2^3\}\\
&\quad + \alpha_3^2\alpha_1\{-\partial_3^2 f_2 f_3 - 2f_3\partial_2 f_3 + 2f_3\partial_3 f_2 + 2f_2\partial_3 f_3 + 2\partial_3\partial_2 f_3\}\\
&\quad + \alpha_1^2\alpha_3\{2\partial_1\partial_2 f_3 - 2\partial_1\partial_3 f_2 + \partial_3\partial_2 f_1 + f_2\partial_3 f_1 - 3f_3\partial_2 f_1 + 4f_3\partial_1 f_2\}\\
&\quad + \alpha_1^2\alpha_2\{\partial_2^2 f_1 - 2f_2\partial_2 f_1 + 4f_2\partial_1 f_2\} + \alpha_1\alpha_2\alpha_3\{4f_2\partial_3 f_2 + 2\partial_2^2 f_3\},\\
a_{13} &= \alpha_1^3\{\partial_1\partial_3 f_1 - \partial_1^2 f_3 - f_1\partial_3 f_1 + f_1\partial_1 f_3\} + \alpha_1^2\alpha_3\{\partial_3^2 f_1 + 3f_3\partial_1 f_3 - f_3\partial_3 f_1\}\\
&\quad + \alpha_1^2\alpha_2\{\partial_2\partial_3 f_1 + 4f_2\partial_1 f_3 - 2\partial_2\partial_1 f_3 - 3f_2\partial_3 f_1 + f_3\partial_2 f_1\}\\
&\quad + \alpha_2^2\alpha_1\{2f_3\partial_2 f_2 + 2f_2\partial_2 f_3 + \partial_2\partial_3 f_2\} + \alpha_3^2\alpha_1\{\partial_3^2 f_3 - f_3\partial_3 f_3 - f_3^3 + 2f_3\partial_3 f_3\}\\
&\quad + \alpha_1^2\alpha_3\{4f_3\partial_2 f_3 - 2f_2\partial_3 f_3 - f_3\partial_3 f_2 + 2\partial_3^2 f_2 - f_3^2 f_2\}.
\end{aligned}$$

Since the Ricci tensor of any affine Szabó connection is cyclic parallel, it follows that $a_{11} = 0$. Thus the characteristic polynomial of the matrix associated to $S^{\nabla}(X)$ with respect to the basis $\{\partial_1, \partial_2, \partial_3\}$ is equal to:

$$P_\lambda(S^{\nabla}(X)) = -\lambda^3.$$

We have the following result.

Theorem 4.6. *Let $M = \mathbb{R}^3$ and ∇ be the torsion free affine connection, whose nonzero coefficients of the connection are given by $\nabla_{\partial_1}\partial_1 = f_1\partial_1$, $\nabla_{\partial_1}\partial_2 = f_2\partial_1$ and $\nabla_{\partial_1}\partial_3 = f_3\partial_1$. Then (M, ∇) is affine Szabó if and only if the Ricci tensor of (M, ∇) is cyclic parallel.*

From Theorem 4.6, one can construct examples of affine Szabó connections.

Example 4.2. The following affine connections on $\mathbb{R}^3$ whose non-zero Christoffel symbols are given by: (1) $\nabla_{\partial_1}\partial_1 = 0$, $\nabla_{\partial_1}\partial_2 = -x_3\partial_1$ and $\nabla_{\partial_1}\partial_3 = x_2\partial_1$; (2) $\nabla_{\partial_1}\partial_1 = x_1\partial_1$, $\nabla_{\partial_1}\partial_2 = 2x_3\partial_1$ and $\nabla_{\partial_1}\partial_3 = -2x_2\partial_1$ are affine Szabó.

Note that the result in Theorem 4.6 remains the same if the affine connection ∇ has non-zero components $\nabla_{\partial_1}\partial_1$, $\nabla_{\partial_1}\partial_2$ and $\nabla_{\partial_1}\partial_3$ in the same direction of the element of the basis $\{\partial_i\}_{i=1.2.3}$.

The affine manifolds in Theorem 4.6 are also called L_3-spaces, and Therefore, are d'Atri spaces. We refer to [13] for a further discussion of D'Atri spaces.

Second Family of affine Szabó connection

Let us consider the affine connection defined in (3.7), i.e.,

$$\nabla_{\partial_1}\partial_1 = f_1\partial_2, \quad \nabla_{\partial_2}\partial_2 = f_2\partial_3 \quad \text{and} \quad \nabla_{\partial_3}\partial_3 = f_3\partial_1,$$

where the smooth functions $f_i = f_i(x_1, x_2, x_3)$, for $i = 1, 2, 3$, are Christoffel symbols. Since the Ricci tensor of any affine Szabó connection is cyclic parallel, it follows from the Proposition 3.2, that the matrix associated to the affine Szabó operator with respect to the basis $\{\partial_1, \partial_2, \partial_3\}$ is reduced to

$$(S^{\nabla})(X) = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix},$$

with

$$\begin{aligned}
b_{12} &= \alpha_1^2\alpha_3(-\partial_1\partial_3 f_1) + \alpha_1\alpha_2^2(f_2\partial_3 f_1) + \alpha_1\alpha_3^2(f_3\partial_1 f_1 - \partial_3^2 f_1) \\
&\quad + \alpha_2^2\alpha_3(-2f_2f_3f_1) + \alpha_2\alpha_3^2(f_1\partial_2 f_3) + \alpha_3^3(2f_3\partial_3 f_1 + f_1\partial_3 f_3); \\
b_{13} &= \alpha_1^2\alpha_2(-2f_1\partial_1 f_2 - f_2\partial_1 f_1) + \alpha_1\alpha_2^2(\partial_1^2 f_2 - f_1\partial_2 f_2) \\
&\quad + \alpha_1\alpha_2\alpha_3(-2f_2\partial_3 f_1) + \alpha_2^3(\partial_2\partial_1 f_2) + \alpha_2\alpha_3^2(2f_2f_3f_1); \\
b_{21} &= \alpha_1^2\alpha_3(2f_3f_2f_1) + \alpha_1\alpha_2\alpha_3(-2f_3\partial_1 f_2) + \alpha_2^2\alpha_3(-2f_2\partial_2 f_3 - f_3\partial_2 f_2) \\
&\quad + \alpha_2\alpha_3^2(\partial_2^2 f_3 - f_2\partial_3 f_3) + \alpha_3^3(\partial_3\partial_2 f_3), \\
b_{23} &= \alpha_1^3(2f_1\partial_1 f_2 + f_2\partial_1 f_1) + \alpha_1^2\alpha_2(-\partial_1^2 f_2 + f_1\partial_2 f_2) + \alpha_1^2\alpha_3(f_2\partial_3 f_1) \\
&\quad + \alpha_1\alpha_2^2(-\partial_2\partial_1 f_2) + \alpha_1\alpha_3^2(-2f_2f_3f_1) + \alpha_2\alpha_3^2(f_3\partial_1 f_2)
\end{aligned}$$

$$\begin{aligned} b_{31} &= \alpha_1^2\alpha_2(-2f_1f_2f_3) + \alpha_1^2\alpha_3(f_1\partial_2 f_3) + \alpha_1\alpha_2^2(f_3\partial_1 f_2) \\ &\quad + \alpha_2^3(2f_2\partial_2 f_3 + f_3\partial_2 f_2) + \alpha_2^2\alpha_3(-\partial_2^2 f_3 + f_2\partial_3 f_3) + \alpha_2\alpha_3^2(-\partial_3\partial_2 f_3); \\ b_{32} &= \alpha_1^3(\partial_1\partial_3 f_1) + \alpha_1^2\alpha_3(-f_3\partial_1 f_1 + \partial_3^2 f_1) + \alpha_1\alpha_2^2(2f_1f_3f_2) \\ &\quad + \alpha_1\alpha_2\alpha_3(-2f_1\partial_2 f_3) + \alpha_1\alpha_3^2(-2f_3\partial_3 f_1 - f_1\partial_3 f_3). \end{aligned}$$

The characteristic polynomial of the affine Szabó operator is now seen to be:

$$P[S^\nabla(X)](\lambda) = -\lambda^3 + (b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31})\lambda + (b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}).$$

It follows that the affine connection given by (3.7) is affine Szabó if and only if:

$$b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31} = 0 \quad \text{and} \quad b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0.$$

A straightforward calculation shows that: $b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0$. Then $S^\nabla(X)$ has eigenvalue zero if and only if:

$$b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31} = 0. \tag{4.1}$$

1. Assume $f_1 = 0$. Then, the relation (4.1) reduces to:

$$b_{13}b_{31} = 0.$$

   (a) If $\partial_1 f_2 = 0$, then $f_2 = u(x_2)$ and $f_3 = v(x_2) + t(x_3)$.

   (b) If $\partial_1 f_2 \neq 0$, then $f_3 = 0$.

2. Assume $f_2 = 0$, then we have

$$b_{12}b_{21} = 0.$$

   (a) If $\partial_2 f_3 = 0$, then $f_3 = t(x_3)$ and $f_1 = f(x_1) + g(x_3)$.

   (b) If $\partial_2 f_3 \neq 0$, then $f_1 = 0$.

3. Assume $f_3 = 0$, then we have

$$b_{23}b_{32} = 0.$$

   (a) If $\partial_3 f_1 = 0$, then $f_1 = f(x_1)$ and $f_2 = h(x_1) + k(x_2)$.

   (b) If $\partial_3 f_1 \neq 0$, then $f_2 = 0$.

We have the following result.

Theorem 4.7. *Let $M = \mathbb{R}^3$ and ∇ be the torsion free affine connection, whose non-zero coefficients of the connection are given by*

$$\nabla_{\partial_1}\partial_1 = f_1\partial_2, \ \ \nabla_{\partial_2}\partial_2 = f_2\partial_3 \ \ \textit{and} \ \ \nabla_{\partial_3}\partial_3 = f_3\partial_1.$$

Then (M, ∇) is affine Szabó if at least one of the following conditions holds:

(1) $f_1 = 0$, $f_2 = u(x_2)$ and $f_3 = v(x_2) + t(x_3)$.

(2) $f_2 = 0$, $f_3 = t(x_3)$ and $f_1 = f(x_1) + g(x_3)$.

(3) $f_3 = 0$, $f_1 = f(x_1)$ *and* $f_2 = h(x_1) + u(x_2)$.

Or at least one of the following conditions holds:

(4) $f_1 = 0$, $f_2 = f(x_1) + g(x_2)$ *and* $f_3 = 0$.

(5) $f_2 = 0$, $f_3 = v(x_2) + t(x_3)$ *and* $f_1 = 0$.

(6) $f_3 = 0$, $f_1 = f(x_1) + g(x_3)$ *and* $f_2 = 0$.

From Theorem 4.7, one can construct examples of affine Szabó connections. As an example, we have the following.

Example 4.3. The following connections on $\mathbb{R}^3$ whose non-zero Christoffel symbols are given by: (1) $\nabla_{\partial_1}\partial_1 = 0$, $\nabla_{\partial_2}\partial_2 = x_2\partial_3$, $\nabla_{\partial_3}\partial_3 = (x_2 + x_3^2)\partial_1$; (2) $\nabla_{\partial_1}\partial_1 = x_1^2\partial_2$, $\nabla_{\partial_2}\partial_2 = (x_1 + x_2)\partial_3$, $\nabla_{\partial_3}\partial_3 = 0$ are affine Szabó.

Remark 4.4. The affine connection defined in Example 3.3 has a Ricci tensor which is cyclic parallel but it is not affine Szabó. This means that the manifold defined in Example 3.3 is an L_3-space but not an affine Szabó manifold.

One has also the following observation.

Theorem 4.8. *Let* (M_1, ∇_1) *be an affine Szabó at* $p_1 \in M_1$ *and* (M_2, ∇_2) *be an affine Szabó at* $p_2 \in M_2$*. Then the product manifold* $(M, \nabla) := (M_1 \times M_2, \nabla \oplus \nabla_2)$ *is affine Szabó at* $p = (p_1, p_2)$.

Proof. Let $X = (X_1, X_2) \in T_{(p_1,p_2)}(M_1 \times M_2)$ with $X_1 \in T_{p_1}M_1$ and $X_2 \in T_{p_2}M_2$. Then we have $S^{\nabla}(X) = S^{\nabla_1}(X_1) \oplus S^{\nabla_2}(X_2)$. So $Spect\{S^{\nabla}(X)\} = Spect\{S^{\nabla_1}(X_1)\} \cup Spect\{S^{\nabla_2}(X_2)\} = \{0\}$. This completes the proof. $\square$

Affine Szabó connections are of interest not only in affine geometry, but also in the study of Pseudo-Riemannian Szabó metrics since they provide some examples without Riemannian analogue by means of the Riemann extensions.

5 Riemann extensions

Let (M, ∇) be an n-dimensional affine manifold, Let T^*M be its cotangent bundle and let $\pi : T^*M \to M$ be the natural projection defined by $\pi(p, \omega) = p \in M$ and $(p, \omega) \in T^*M$. A system of local coordinates $(U, x_i), i = 1, \cdots, n$ around $p \in M$ induces a system of local coordinates $(\pi^{-1}(U), x_i, x_{i'} = \omega_i), i' = n + i = n + 1, \cdots, 2n$ around $(p, \omega) \in T^*M$, where $x_{i'} = \omega_i$ are components of covectors ω in each cotangent space T_p^*M, $p \in U$ with respect to the natural coframe $\{dx^i\}$. If we use the notation $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_{i'} = \frac{\partial}{\partial \omega_i}, i = i, \cdots, n$ then at each point $(p, \omega) \in T^*M$, its follows that

$$\{(\partial_1)_{(p,\omega)}, \cdots, (\partial_n)_{(p,\omega)}, (\partial_{1'})_{(p,\omega)}, \cdots, (\partial_{n'})_{(p,\omega)}\},$$

is a basis for the tangent space $(T^*M)_{(p,\omega)}$.

For each vector field X on M, define a function $\iota X : T^*M \longrightarrow \mathbb{R}$ by

$$\iota X(p, \omega) = \omega(X_p).$$

This function is locally expressed by, $\iota X(x_i, x_{i'}) = x_{i'}X^i$, for all $X = X^i\partial_i$. Vector fields on T^*M are characterized by their actions on functions ιX. The complete lift X^C of a vector field X on M to T^*M is characterized by the identity

$$X^C(\iota Z) = \iota[X, Z], \quad \text{for all} \quad Z \in \Gamma(TM).$$

Moreover, since a $(0, s)$-tensor field on M is characterized by its evaluation on complete lifts of vector fields on M, for each tensor field T of type $(1, 1)$ on M, we define a 1-form ιT on T^*M which is characterized by the identity

$$\iota T(X^C) = \iota(TX).$$

For more details on the geometry of cotangent bundle, we refer to the book of Yano and Ishihara [20].

Let ∇ be a torsion free affine connection on an n-dimensional affine manifold M. The *Riemann extension* g_∇ is the pseudo-Riemannian metric on T^*M of neutral signature (n, n) characterized by the identity [2, 7]

$$g_\nabla(X^C, Y^C) = -\iota(\nabla_X Y + \nabla_Y X).$$

In the locally induced coordinates $(x_i, x_{i'})$ on $\pi^{-1}(U) \subset T^*M$, the Riemann extension is expressed as

$$g_\nabla = \begin{pmatrix} -2x_{k'}\Gamma^k_{ij} & \delta^j_i \\ \delta^j_i & 0 \end{pmatrix}, \tag{5.1}$$

with respect to $\{\partial_1, \cdots, \partial_n, \partial_{1'}, \cdots, \partial_{n'}\}(i, j, k = 1, \cdots, n; k' = k + n)$, where Γ^k_{ij} are the Christoffel symbols of the torsion free affine connection ∇ with respect to (U, x_i) on M. Some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemann extension g_∇. For instance, (M, ∇) is locally symmetric if and only if (T^*M, g_∇) is locally symmetric [7]. Furthermore (M, ∇) is projectively flat if and only if (T^*M, g_∇) is locally conformally flat (see [3] for more details and references therein).

Let Γ^k_{ij} be the Christoffel symbols of ∇. The non-zero Christoffel symbols $\tilde{\Gamma}^\gamma_{\alpha\beta}$ of the Levi-Civita connection of g_∇ are given by

$$\begin{aligned} \tilde{\Gamma}^k_{ij} &= \Gamma^k_{ij}, \quad \tilde{\Gamma}^{k'}_{i'j} = -\Gamma^i_{jk} \quad \tilde{\Gamma}^{k'}_{ij'} = -\Gamma^j_{ik}, \\ \tilde{\Gamma}^{k'}_{ij} &= \sum_r x_{r'}\Big(\partial_k\Gamma^r_{ij} - \partial_i\Gamma^r_{jk} - \partial_j\Gamma^r_{ik} + 2\sum_l \Gamma^r_{kl}\Gamma^l_{ij}\Big), \end{aligned}$$

where $(i, j, k, l, r = 1, \cdots, n)$ and $(i' = i + n, j' = j + n, k' = k + n, r' = r + n)$. The non-zero components of the curvature tensor of (T^*M, g_∇) up to the usual symmetries are given as follows

$$\tilde{R}^h_{kji} = R^h_{kji}, \quad \tilde{R}^{h'}_{kji}, \quad \tilde{R}^{h'}_{kji'} = -R^i_{kjh}, \quad \tilde{R}^{h'}_{k'ji} = R^k_{hij},$$

where R^h_{kji} are the components of the curvature tensor of (M, ∇). Here we omit $\tilde{R}^{h'}_{kji}$, as it plays no role in our considerations. Let $\tilde{X} = \alpha_i\partial_i + \alpha_{i'}\partial_{i'}$ and $\tilde{Y} = \beta_i\partial_i + \beta_{i'}\partial_{i'}$ be vector fields on T^*M. Let $X = \alpha_i\partial_i$ and $Y = \beta_i\partial_i$ be the corresponding vector fields on M. Let $\mathcal{S}^{\nabla}(X)$ be the matrix of the affine Szabó operator on M relative to the basis $\{\partial_i\}$. Then the matrix of the Szabó operator $\tilde{\mathcal{S}}(\tilde{X})$ with respect to the basis $\{\partial_i, \partial_{i'}\}$ has the form

$$\tilde{\mathcal{S}}(\tilde{X}) = \begin{pmatrix} \mathcal{S}^{\nabla}(X) & 0 \\ * & {}^t\mathcal{S}^{\nabla}(X) \end{pmatrix}.$$

Lemma 5.1. *Let (M, ∇) be an n-dimensional affine manifold and (T^*M, g_{∇}) be the cotangent bundle with the Riemann extension. Then, we have*

$$Spect\{\tilde{\mathcal{S}}(\tilde{X})\} = Spect\{\mathcal{S}^{\nabla}(X)\}.$$

Proof. Let $\tilde{X} = \alpha_i\partial_i + \alpha_{i'}\partial_{i'}$ be a vector field on T^*M. Then the matrix of the Szabó operator $\tilde{S}(\tilde{X})$ with respect to the basis $\{\partial_i, \partial_{i'}\}$ is of the form

$$\tilde{\mathcal{S}}(\tilde{X}) = \begin{pmatrix} \mathcal{S}^{\nabla}(X) & 0 \\ * & {}^t\mathcal{S}^{\nabla}(X) \end{pmatrix}. \tag{5.2}$$

where $\mathcal{S}^{\nabla}(X)$ is the matrix of the affine Szabó operator on M relative to the basis $\{\partial_i\}$. It is easy to see that the characteristic polynomial $P_\lambda[\tilde{\mathcal{S}}(\tilde{X})]$ of $\tilde{\mathcal{S}}(\tilde{X})$ and $P_\lambda[\mathcal{S}^{\nabla}(X)]$ of $\mathcal{S}^{\nabla}(X)$ are related by $P_\lambda[\tilde{\mathcal{S}}(\tilde{X})] = P_\lambda[\mathcal{S}^{\nabla}(X)] \cdot P_\lambda[{}^t\mathcal{S}^{\nabla}(X)]$ □

We have the following results.

Theorem 5.2. *Let (M, ∇) be a smooth torsion-free affine manifold. Then the following statements are equivalent:*

(i) (M, ∇) *is affine Szabó.*

(ii) The Riemann extension (T^*M, g_{∇}) of (M, ∇) is a pseudo-Riemannian Szabó manifold.

Proof. Now, if the affine manifold (M, ∇) is assumed to be affine Szabó, then $\mathcal{S}^{\nabla}(X)$ has zero eigenvalues for each vector field X on M. Therefore, it follows from (5.2) that the eigenvalues of $\tilde{\mathcal{S}}(\tilde{X})$ vanish for every vector field $\tilde{X}$ on T^*M. Thus (T^*M, g_{∇}) is pseudo-Riemannian Szabó manifold.
Conversely, assume that (T^*M, g_{∇}) is a pseudo-Riemannian Szabó manifold. If $X = \alpha_i\partial_i$ with $\alpha_i \neq 0$, for any i, is a vector field on M, then $\tilde{X} = \alpha_i\partial_i + \frac{1}{2\alpha_i}\partial_{i'}$ is a unit vector field at every point of the zero section on T^*M. Then from (5.2), we see that, the characteristic polynomial $P_\lambda[\tilde{\mathcal{S}}(\tilde{X})]$ of $\tilde{\mathcal{S}}(\tilde{X})$ is the square of the characteristic polynomial $P_\lambda[\mathcal{S}^{\nabla}(X)]$ of $\mathcal{S}^{\nabla}(X)$. Since for every unit vector field $\tilde{X}$ on T^*M the characteristic polynomial $P_\lambda[\tilde{\mathcal{S}}(\tilde{X})]$ should be the same, it follows that for every vector field X on M the characteristic polynomial $P_\lambda[\mathcal{S}^{\nabla}(X)]$ is the same. Hence (M, ∇) is affine Szabó. □

5.1 Six-dimensional Riemann extensions

Let (M, ∇) be an 3-dimensional affine manifold. Let (x_1, x_2, x_3) be local coordinates on M. We expand $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$ for $i, j, k = 1, 2, 3$ to define the Christoffel symbols $\Gamma_{ij}^k$of ∇. If $\omega \in T^*M$, we expand $\omega = x_4 dx_i + x_5 dx_2 + x_6 dx_3$ to define the dual fiber coordinates (x_4, x_5, x_6) thereby obtain canonical local coordinates

$(x_1, x_2, x_3, x_4, x_5, x_6)$ on T^*M. The Riemann extension in the metric of neutral signature $(3, 3)$ on the cotangent bundle T^*M is given locally by

$$\begin{aligned}
g_\nabla(\partial_1, \partial_4) &= g_\nabla(\partial_2, \partial_5) = g_\nabla(\partial_3, \partial_6) = 1, \\
g_\nabla(\partial_1, \partial_1) &= -2x_4\Gamma^1_{11} - 2x_5\Gamma^2_{11} - 2x_6\Gamma^3_{11}, \\
g_\nabla(\partial_1, \partial_2) &= -2x_4\Gamma^1_{12} - 2x_5\Gamma^2_{12} - 2x_6\Gamma^3_{12}, \\
g_\nabla(\partial_1, \partial_3) &= -2x_4\Gamma^1_{13} - 2x_5\Gamma^2_{13} - 2x_6\Gamma^3_{13}, \\
g_\nabla(\partial_2, \partial_2) &= -2x_4\Gamma^1_{22} - 2x_5\Gamma^2_{22} - 2x_6\Gamma^3_{22}, \\
g_\nabla(\partial_2, \partial_3) &= -2x_4\Gamma^1_{23} - 2x_5\Gamma^2_{23} - 2x_6\Gamma^3_{23}, \\
g_\nabla(\partial_3, \partial_3) &= -2x_4\Gamma^1_{33} - 2x_5\Gamma^2_{33} - 2x_6\Gamma^3_{33}.
\end{aligned}$$

As an example, we have the following. The Riemann extension of the affine Szabó connection on $\mathbb{R}^3$ defined by

$$\nabla_{\partial_1}\partial_1 = x_1\partial_1, \quad \nabla_{\partial_1}\partial_2 = 2x_3\partial_1, \quad \nabla_{\partial_1}\partial_3 = -2x_2\partial_1,$$

is the pseudo-Riemannian metric of signature $(3, 3)$ given by

$$\begin{aligned}
g_\nabla &= 2dx_1 \otimes dx_4 + 2dx_2 \otimes dx_5 + 2dx_3 \otimes dx_6 \\
&\quad - 2x_1x_4dx_1 \otimes dx_1 - 4x_3x_4dx_1 \otimes dx_2 + 4x_2x_4dx_1 \otimes dx_3.
\end{aligned}$$

After, a straightforward calculation, it easy to see that this metric is Szabó.

The Riemann extensions provide a link between affine and pseudo-Riemannian geometries. Some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemann extension g_∇. For more details and information about Riemann extensions, see [2, 3, 7, 8] and references therein. For instance, it is known, in [2, 3] and references therein, that a Walker metric is a triple $(M, g, \mathcal{D})$, where M is an n-dimensional manifold, g is a pseudo-Riemannian metric on M and $\mathcal{D}$ is an r-dimensional parallel null distribution $(r > 0)$. In [3], the authors showed that any 4-dimensional Riemann extension is necessarily a self-dual Walker manifold, but for some particular cases, they proved that the converse holds.

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Authors' addresses:

Abdoul Salam Diallo
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Private Bag X01, Scottsville 3209, South Africa

Université Alioune Diop de Bambey
UFR SATIC, Département de Mathématiques
B. P. 30, Bambey, Sénégal.
E-mail: Diallo@ukzn.ac.za, abdoulsalam.diallo@uadb.edu.sn

Silas Longwap and Fortuné Massamba
School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal,
Private Bag X01, Scottsville 3209, South Africa.
E-mail: longwap4all@yahoo.com ; massfort@yahoo.fr, massamba@ukzn.ac.za