On the existence of the Yamabe problem on contact Riemannian manifolds

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Abstract. It was proved in [25] that for a contact Riemannian manifold with non-integrable almost complex structure, the Yamabe problem is subcritical in the sense that its Yamabe invariant is less than that of the Heisenberg group. In this paper we give a complete proof of the solvability of the contact Riemannian Yamabe problem in the subcritical case. These two results implies that the Yamabe problem on a contact Riemannian manifold is always solvable. By constructing normal coordinates on a contact Riemannian manifold, we can osculate the contact Riemannian structure at each point by the standard structure on the Heisenberg group. This osculation makes the machine of singular integral operators work on general contact Riemannian manifolds. We apply it to obtain the regularity of the SubLaplacians and the Yamabe equation, which allow us to solve the contact Riemannian Yamabe problem in the subcritical case by Jerison-Lee's approach in the CR case. We also clarify two claims in their proof.

M.S.C. 2010: 53D10, 53D35, 32V99.
Key words: contact Riemannian manifolds, the contact Riemannian Yamabe problem, the contact Riemannian Yamabe equation, normal coordinates, constrained variational problem, the machine of singular integral operators.

1 Introduction

Let M be a $(2n+1)$-dimensional *contact manifold*, i.e. it is equipped with a real 1-form θ, called a *contact form*, such that $\theta \wedge d\theta^n \neq 0$ everywhere on M. There exists a unique vector field T, called the *Reeb vector field*, such that $\theta(T) = 1$ and $i_T d\theta = 0$, where i is the interior product. It is known that given a contact manifold (M, θ), there exist a Riemannian metric h and a $(1,1)$-tensor field J on M, called an *almost complex structure*, such that

$$(1.1) \qquad h(X,T) = \theta(X), \qquad J^2 = -Id + \theta \otimes T, \qquad d\theta(X,Y) = h(X,JY),$$

Balkan Journal of Geometry and Its Applications, Vol.22, No.2, 2017, pp. 101-128.

for any vector field X, Y (cf. p. 278 in [4]). Given a contact form θ, h is uniquely determined once J is fixed and vise versa. (M, θ, h, J) is called a *contact Riemannian manifold*. $H(M) := Ker(\theta)$ is the *horizontal subbudle* of the tangent bundle TM. Tanno [20] constructed a canonical connection for a contact Riemannian manifold, called the *Tanaka-Webster-Tanno connection* now (or *TWT connection* briefly). In the CR case, this is exactly the Tanaka-Webster connection. Recently, people are interested in geometric analysis of contact Riemannian manifolds (cf. [3] [4] [9] [10] [24] [25] and references therein).

Under the conformal transformation

$$\widehat{\theta} = f\theta \tag{1.2}$$

for some positive function f, the contact Riemannian structure (θ, J, T, h) is changed to $(\widehat{\theta}, \widehat{J}, \widehat{T}, \widehat{h})$ (cf. (2.28)). The *contact Riemannian Yamabe problem* is to find $\widehat{\theta}$ conformal to θ such that its scalar curvature is constant. This problem was formulated by Tanno [20] as an analog of the CR Yamabe problem. Denote by s_θ and $s_{\widehat{\theta}}$ the scalar curvatures of the TWT connection for (M, θ, h, J) and $(M, \widehat{\theta}, \widehat{h}, \widehat{J})$, respectively. If we write the conformal transformation (1.2) with $f = u^{\frac{4}{Q-2}}$, the scalar curvatures transform as

$$b_n \triangle_\theta u + s_\theta u = s_{\widehat{\theta}} u^{2^*-1}, \qquad b_n = \frac{4(n+1)}{n}, \tag{1.3}$$

(cf. Proposition 2.2), where $2^* = \frac{2Q}{Q-2} = 2 + \frac{2}{n}$, i.e. $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{Q}$, is the *critical Sobolev exponent*, $Q = 2n+2$ is the *homogeneous dimension* of M and $\triangle_\theta$ is the SubLaplacian (2.18). (1.3) is the *contact Riemannian Yamabe equation.* When J is integrable, the manifold is CR. The CR Yamabe problem was solved by Jerison and Lee [17] [18] [19] for non-spherical CR manifolds with dimension > 3, while the remaining cases were solved by Gamara and Yacoub [13] [14].

Denote by ψ_θ be the volume element associated to the metric h, which is $\theta \wedge (d\theta)^n$ up to a constant. Let $d_b = \pi \circ d$, where π be the projection from T^*M to H^*. As in the Riemannian and CR cases, (1.3) is the Euler-Lagrangian equation for the constrained variational problem

$$\lambda(M) = \inf_\theta \{A_\theta(u); B_\theta(u) = 1\}, \tag{1.4}$$

where

$$A_\theta(u) = \int_M (b_n |d_b u|_\theta^2 + s_\theta u^2)\psi_\theta, \qquad B_\theta(u) = \int_M |u|^{2^*} \psi_\theta. \tag{1.5}$$

We prove the following result for the contact Riemannian Yamabe problem.

Theorem 1.1. *Suppose that (M, θ, h, J) is a compact contact Riemannian manifold with $\dim M = 2n+1$. Then,*

(1) $\lambda(M) \leq \lambda(\mathcal{H}^n)$, where $\mathcal{H}^n$ is the Heisenberg group with standard contact Riemannian structure;

(2) If $\lambda(M) < \lambda(\mathcal{H}^n)$, then the infimum of (1.4) is achieved by a positive C^∞ solution u to (1.3), i.e. the contact form $\widehat{\theta} = u^{\frac{4}{Q-2}}\theta$ has constant scalar curvature $s_{\widehat{\theta}} = \lambda(M)$.

We proved in [25] that for a contact Riemannian manifold (M, θ, h, J) with non-integrable almost complex structure J, we have $\lambda(M) < \lambda(\mathcal{H}^n)$. This result together with Theorem 1.1 implies that the contact Riemannian Yamabe problem is solvable. To make the solvability of the contact Riemannian Yamabe problem in [25] have a solid basis, we need a complete proof of Theorem 1.1. Note that in [1] the authors sketched a proof of Theorem 1.1 for a different connection on a contact Riemannian manifold. Although it is possible to use [1] to give a proof by proving several analytic assumptions, we here give a detailed proof along the approach of Jerison and Lee [17] in the CR case. Geometry and analysis involved in this proof might have applications to other problems on contact Riemannian manifolds. We also clarify two claims in their proof. The compactness of the imbedding of the Folland-Stein space $S_1^s(M)$ into $L^r(M)$ for $\frac{1}{r} > \frac{1}{s} - \frac{1}{2n+2}$ (cf. proposition 5.6 of [17]) plays a key role in solving the CR Yamabe equations with subcritical exponents. They used the calculus of pseudodifferential operators on the Heisenberg group to prove this compactness, but details have not been published so far. We give a simple proof of this compactness for the case $s = 2$ in Proposition 4.15, which is sufficient for our purpose. The other claim is the regularity of the Yamabe equation and Harnack inequality, which is uniform under the small perturbation of the pseudohermitian structures (details have not been published yet so far). We use Darboux's theorem to avoid the uniform estimates (cf. Remark 5.1).

In Section 2, we recall the standard contact Riemannian structure on the Heisenberg group and give the expression of the SubLaplacian, and show the contact Riemannian Yamabe equation (1.3) under a conformal transformation. In Section 3, we prove that the contact Riemannian structure at any given point ξ can be osculated by the standard structure of the Heisenberg group, and construct local normal coordinates, which depend on the point ξ smoothly. This kind of coordinates were originally constructed by Folland and Stein [12] for strictly pseudoconvex CR manifolds, and the smooth dependence is very important to make the machine of singular integral operators work. The method of harmonic analysis on CR manifolds can also be adapted to contact Riemannian manifolds, in particular the theory of singular integral operators. Many aspects of harmonic analysis has already been generalized to spaces equipped with smooth vector fields satisfying the Hörmander's condition (see, e.g. [6]), and the Harnack inequality and Poincaré-type inequality in such setting are known. We obtain the regularity of the SubLaplacians and the Yamabe equation on contact Riemannian manifolds in Section 4. The main theorem is proved in the last section.

2 Some basic facts

2.1 The standard contact Riemannian structure on the Heisenberg group

The multiplication of the *Heisenberg group* $\mathcal{H}^n$ can be written as

$$(x,t)\cdot(y,s) = \left(x+y, t+s+\sum_{a,b=1}^{2n} B_{ab}x_a y_b\right), \tag{2.1}$$

where $x = (x_1, \ldots, x_{2n}), y = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n}$, $t, s \in \mathbb{R}$, and $B = (B_{ab})$ is the antisymmetric matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{2.2}$$

Here I_n is the $n \times n$ identity matrix. The group of automorphisms of $\mathcal{H}^n$ is generated by the following transformations: (1) *dilations*: $D_\delta : (x,t) \longrightarrow (\delta x, \delta^2 t)$, for $(x,t) \in \mathcal{H}^n, \delta > 0$; (2) *translations*: $\tau_{(x,t)} : (y,s) \longrightarrow (x,t) \cdot (y,s)$, for $(x,t),(y,s) \in \mathcal{H}^n$; (3) *rotations*: $(x,t) \longrightarrow (Ux,t)$, for $U \in \mathrm{U}(n)$; (4) the *inversion* R, which is defined on $\mathcal{H}^n \setminus \{0\}$ by $R : (x,t) \longrightarrow \left(\frac{-x}{|x|^2-t}, \frac{-t}{|x|^4+|t|^2}\right)$. The following vector fields

$$Y_b = \frac{\partial}{\partial x_b} + \sum_{a=1}^{2n} B_{ab} x_a \frac{\partial}{\partial t}, \tag{2.3}$$

are left invariant on the Heisenberg group by the multiplication law (2.1) of the Heisenberg group, and

$$[Y_a, Y_b] = 2B_{ab} \frac{\partial}{\partial t}, \tag{2.4}$$

for $a, b = 1, \cdots, 2n$. Then the horizontal subspace $H_0 := \mathrm{span}_{\mathbb{R}}\{Y_1, \cdots, Y_{2n}\}$ generates the corresponding Lie algebra of the Heisenberg group. Denote

$$\theta_0 := dt - \sum_{a,b=1}^{2n} B_{ab} x_a dx_b. \tag{2.5}$$

It is obvious that $\theta_0(Y_a) = 0$ for $a = 1, \cdots, 2n$ and

$$d\theta_0 = -\sum_{a,b=1}^{2n} B_{ab} dx_a \wedge dx_b. \tag{2.6}$$

Define the standard Riemannian metric h_0 on $\mathcal{H}^n$ by

$$h_0(T,T) = 1, \qquad h_0(Y,T) = 0, \qquad h_0(Y_a, Y_b) = 2\delta_{ab}, \qquad a, b = 1, \cdots, 2n, \tag{2.7}$$

for any $Y \in H_0$, and the standard almost complex structure J_0 as a transformation given by

$$JT = 0, \qquad J_0 Y_b = \sum_{a=1}^{2n} B_{ba} Y_a, \tag{2.8}$$

for $b = 1, \ldots, 2n$. It is easy to see that J_0 satisfies the second identity in (1.1) and

$$d\theta_0(Y_a, Y_b) = -2B_{ab} = h_0(Y_a, J_0 Y_b), \tag{2.9}$$

by (2.6) and the definition of wedge products: for 1-forms ω and ω',

$$\omega \wedge \omega'(X,Y) = \omega(X)\omega'(Y) - \omega(Y)\omega'(X) \tag{2.10}$$

for any vector fields X and Y. $(\mathcal{H}, h_0, \theta_0, J_0)$ is the *standard contact Riemannian structure* on the Heisenberg group.

2.2 The SubLaplacian on a contact Riemannian manifold

On a contact Riemannian manifold (M, θ, h, J), there exists a unique linear connection ∇ such that

$$\begin{aligned} &\nabla\theta = 0, \qquad \nabla T = 0, \qquad \nabla h = 0, \\ &\tau(X, Y) = 2d\theta(X, Y)T, \quad X, Y \in \Gamma(HM), \\ &\tau(T, JX) = -J\tau(T, X), \quad X \in \Gamma(TM), \end{aligned} \tag{2.11}$$

(cf. (7)-(9) in [4]), where τ is the torsion of ∇, i.e. $\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ for $X, Y \in \Gamma(TM)$. This connection is called the *TWT connection*. The *Tanno tensor* is the $(1, 2)$-tensor field Q defined by $Q(X, Y) := (\nabla_Y J)X$, for $X, Y \in \Gamma(TM)$. Tanno prove that a contact Riemannian manifold is a CR manifold if and only if $Q \equiv 0$ (cf. proposition 2.1 of [20]). The curvature tensor of the TWT connection is $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$. The Ricci tensor is defined by $Ric(Y, Z) = tr\{X \longrightarrow R(X, Z)Y\}$ for any $X, Y, Z \in TM$. The *scalar curvature* is $s_\theta = tr(Ric)$.

Let ξ be an arbitrary point of a contact Riemannian manifold (M, θ, h, J) and U be a sufficiently small neighborhood of ξ. Now choose a local real horizontal vector field $\{X_1\}$ over U such that $h(X_1, X_1) = 2$ and set $X_{n+1} := JX_1$. Note that the metric h restricted to the horizontal subbundle is invariant under J, i.e. $h(JX, JY) = h(X, Y)$ for any horizontal vector fields X and Y by the third identity in (1.1). Then we get

$$\begin{aligned} h(X_1, X_{n+1}) &= h(X_1, JX_1) = d\theta(X_1, X_1) = 0, \\ h(X_{n+1}, X_{n+1}) &= h(JX_1, JX_1) = h(X_1, X_1) = 2, \end{aligned}$$

i.e., X_{n+1} is orthogonal to X_1. We can choose X_2 orthogonal to $span_{\mathbb{R}}\{X_1, JX_1\}$ with norm $\sqrt{2}$ and define $X_{n+2} := JX_2$. Repeating the procedure, we find a local orthogonal frame $\{X_1, \cdots, X_{2n}\}$ of the horizontal subspace $H|_U = \ker\theta|_U$ with norm $\sqrt{2}$ and

$$JX_\alpha = X_{\alpha+n}, \qquad JX_{\alpha+n} = -X_\alpha, \qquad \alpha = 1, \ldots, n. \tag{2.12}$$

Here we choose the local frame with norm $\sqrt{2}$, because the standard frame (2.7) on the Heisenberg group has the norm $\sqrt{2}$. Let $\{\theta^1, \ldots, \theta^{2n}, \theta\}$ be the local coframe dual to $\{X_1, \cdots, X_{2n}, T\}$. Namely $\theta^a(T) = 0, \theta(X_b) = 0, \theta^a(X_b) = \delta^a_b, a, b = 1, \ldots, 2n$. Then we have

$$d\theta(X_a, X_b) = h(X_a, JX_b) = -2B_{ab}, \tag{2.13}$$

by (2.12), where B is given by (2.2). (2.13) is equivalent to the *structure equation*:

$$d\theta = -\sum_{a,b=1}^{2n} B_{ab}\theta^a \wedge \theta^b, \qquad \mod \quad \theta, \tag{2.14}$$

by the definition (2.10) of wedge products. Applying the formula of exterior derivative: for a 1-form ω,

$$d\omega(X, Y) = X(\omega(Y)) - Y(X(\omega)) - \omega([X, Y]) \tag{2.15}$$

for any vector fields X and Y, we see that the dual of (2.13) is

$$[X_a, X_b] = 2B_{ab}T, \qquad \mod \quad H, \tag{2.16}$$

where T is the Reeb vector. Then the volume element ψ_θ associated to the metric h is locally as

$$\psi_\theta = 2^n \theta \wedge \theta^1 \wedge \cdots \wedge \theta^{2n}, \tag{2.17}$$

since $\{T, \frac{1}{\sqrt{2}}X_1, \ldots, \frac{1}{\sqrt{2}}X_{2n}\}$ is a local orthonormal frame on the Riemannian manifold (M, h). It is direct to see that ψ_θ is equal to $\theta \wedge (d\theta)^n$ up to a constant by the structure equation (2.14).

The metric h restricted to H induces a dual metric on H^*. We denote it by $\langle \cdot, \cdot \rangle_\theta$. Define a norm $|\omega|_\theta^2 = \langle \omega, \omega \rangle_\theta$ for $\omega \in H^*$. It induces an L^2 inner product on $\Gamma(H^*)$ by $\langle \omega, \omega' \rangle := \int_M \langle \omega, \omega' \rangle_\theta \psi_\theta$. We define the *SubLaplacian* as the differential operator $\triangle_\theta$ satisfying

$$\int_M \triangle_\theta f \cdot g \psi_\theta = \int_M \langle d_b f, d_b g \rangle_\theta \psi_\theta \tag{2.18}$$

for any $f, g \in C_0^\infty(M)$. This definition is convenient for handling the variational problem (1.4). The SubLaplacian $\triangle_\theta$ also depends on the almost complex structure J. But under a conformal transformation, the action of J on the horizontal subspace H is unchanged (cf. (2.28)). So we omit the subscript J for the notations of the SubLaplacian and the norm on H^*, etc..

Since the TWT connection ∇ preserves H by $\nabla \theta = 0$, there exist 1-forms ${\omega_a}^b \in \Gamma(H^*)$ such that $\nabla_Y X_a = {\omega_a}^b(Y) X_b$ for $Y \in \Gamma(H)$. Write ${\omega_a}^c(X_b) = {\Gamma_{ab}}^c$. Then $\nabla_{X_b} X_a = {\Gamma_{ab}}^c X_c$. The SubLaplacian $\triangle_\theta$ has the following expression.

Proposition 2.1. *On a contact Riemannian manifold M, let $\{X_a\}_{a=1}^{2n}$ be a local orthogonal frame of the horizontal subspace H with norm $\sqrt{2}$. Then for $f \in C^\infty(M)$, we have*

$$\triangle_\theta f = \frac{1}{2} \sum_{a=1}^{2n} \left(-X_a X_a f + \sum_{b=1}^{2n} {\Gamma_{bb}}^a X_a f \right). \tag{2.19}$$

Proof. Let $\{\theta^a\}_{a=1}^{2n}$ be a dual basis of $\{X_a\}_{a=1}^{2n}$ for H^* and $\theta^a(T) = 0$ for each a. Recall that for a 1-form $\omega \in \Omega^1(M)$, $(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y)$. Since the TWT connection ∇ preserves H^* by $\nabla T = 0$, we find that

$$\nabla_{X_b} \theta^a = -{\Gamma_{cb}}^a \theta^c. \tag{2.20}$$

Let X_a^* be the formal adjoint operator of the differential operator X_a, i.e. $\int_M X_a f \cdot g \psi_\theta = \int_M f X_a^* g \psi_\theta$ for any $f, g \in C_0^\infty(M)$. We claim that

$$X_a^* = -X_a + \sum_{b=1}^{2n} {\Gamma_{bb}}^a. \tag{2.21}$$

By definition $d_b f = \sum_{a=1}^{2n} X_a f \cdot \theta^a$ and $\langle \theta^a, \theta^a \rangle_\theta = \frac{1}{2}$. It follows from the definition (2.18) of the SubLaplacian that

$$\triangle_\theta = \frac{1}{2}\sum_{a=1}^{2n} X_a^* X_a = \frac{1}{2}\sum_{a=1}^{2n}\left(-X_a X_a u + \sum_{b=1}^{2n} \Gamma_{bb}{}^a X_a u\right). \tag{2.22}$$

It remains to prove the claim (2.21). Note that $Xf = i_X df$, where i_X is the interior operator. By Stokes' formula, we get

$$\begin{aligned}\int_M X_a f \cdot g\psi_\theta &= \int_M g df \wedge i_{X_a}\psi_\theta = -\int_M f dg \wedge i_{X_a}\psi_\theta - \int_M fgd\left(i_{X_a}\psi_\theta\right)\\ &= -\int_M f X_a g \psi_\theta - \int_M fgd(i_{X_a}\psi_\theta)\end{aligned} \tag{2.23}$$

with ψ_θ as in (2.17). By the standard exterior differentiation formula,

$$d\phi(X,Y) = (\nabla_X\phi)(Y) - (\nabla_Y\phi)(X) + \phi(\tau_{X,Y}) \tag{2.24}$$

for a 1-form $\phi \in \Omega^1(M)$, where the torsion $\tau_{X,Y} = 2d\theta(X,Y)T$ for $X, Y \in H$ by (2.11), and using (2.20), we find that $d\theta^b(X_c, X_d) = -\Gamma_{dc}{}^b + \Gamma_{cd}{}^b$. Thus we get

$$d\theta^b = \frac{1}{2}\sum_{c,d=1}^{2n}\left(-\Gamma_{dc}{}^b + \Gamma_{cd}{}^b\right)\theta^c\wedge\theta^d, \quad \mod \quad \theta. \tag{2.25}$$

Note that

$$d(i_{X_a}\psi_\theta) = (-1)^a 2^n d\theta\wedge\theta^1\wedge\cdots\wedge\widehat{\theta^a}\wedge\cdots\wedge\theta^{2n} + \sum_{b\neq a} d\theta^b\wedge i_{X_b} i_{X_a}\psi_\theta. \tag{2.26}$$

The first $(2n+1)$-form in the right hand side of (2.26) vanishes since it annihilates the Reeb vector T: $i_T d\theta = 0$. Inserting $d\theta^b = \sum_a\left(-\Gamma_{ba}{}^b + \Gamma_{ab}{}^b\right)\theta^a\wedge\theta^b + \cdots$ by (2.25) into the second sum in the right hand side of (2.26) and using antisymmetry $\Gamma_{bc}{}^a = -\Gamma_{ac}{}^b$, which follows from the fact that the TWT connection ∇ preserves the metric h, we find that

$$d(i_{X_a}\psi_\theta) = \sum_{b\neq a}\left(-\Gamma_{ba}{}^b + \Gamma_{ab}{}^b\right)\psi_\theta = -\sum_{b\neq a}\Gamma_{bb}{}^a\psi_\theta = -\sum_{b=1}^{2n}\Gamma_{bb}{}^a\psi_\theta, \tag{2.27}$$

Here $\Gamma_{ba}{}^b \equiv 0$ by antisymmetry. The claim (2.21) follows from (2.23) and (2.27). The proposition is proved. □

2.3 The Yamabe equation on contact Riemannian manifolds

Under the conformal transformation (1.2), the contact Riemannian structure (θ, J, T, h) is changed to $(\widehat{\theta}, \widehat{J}, \widehat{T}, \widehat{h})$ with

$$\begin{aligned}\widehat{T} &= \frac{1}{f}(T+\zeta),\\ \widehat{h} &= fh - f\big(\theta\otimes\omega + \omega\otimes\theta\big) + f(f-1+||\zeta||^2)\theta\otimes\theta,\\ \widehat{J} &= J + \frac{1}{2f}\theta\otimes\big(\nabla f - T(f)T\big),\end{aligned} \tag{2.28}$$

(cf. (12) in [4] or Lemma 9.1 in [20]), where $\zeta = \frac{1}{2f} J\nabla f$ and ω satisfies $\omega(X) = h(X, \zeta)$ for $X \in TM$.

Proposition 2.2. *Under the conformal transformation $\widehat{\theta} = u^{\frac{4}{Q-2}}\theta$, the scalar curvatures of the TWT connections transform as (1.3).*

Proof. Tanno considered (cf. [20], p. 363) the second order differential operator $\triangle_H$ given by $\triangle_H u := \triangle_h u - T(Tu)$, $u \in C^\infty(M)$, where $\triangle_h$ is the Beltrami-Laplacian of the Riemannian manifold (M, h). Tanno proved that the scalar curvature $s_{\widehat{\theta}}$ of the TWT connection of $(M, \widehat{\theta}, \widehat{h}, \widehat{J})$ satisfies the equation

$$-\frac{4(n+1)}{n}\triangle_H u + s_\theta = s_{\widehat{\theta}} u^{1+\frac{2}{n}} \tag{2.29}$$

(cf. (10.10) in [20]). See also [4], p. 336-337, for a simpler proof by a different technique. It remains to show $\triangle_\theta = -\triangle_H$.

It is known that the TWT connection can be expressed in terms of the Levi-Civita connection $\nabla^\dagger_X$ for the Riemannian metric h as

$$\nabla_X Y = \nabla^\dagger_X Y + \theta(X)JY - \theta(Y)\nabla^\dagger_X T + \left[\left(\nabla^\dagger_X \theta\right) Y\right] T, \tag{2.30}$$

(cf. (6) in [4]). Let $\{\frac{1}{\sqrt{2}}X_a\}_{a=1}^{2n}$ be a local orthonormal frame of the horizontal subspace H as before. Then $\{T, \frac{1}{\sqrt{2}}X_1, \ldots\}$ is a local orthonormal frame on the Riemannian manifold (M, h). Recall that the Beltrami-Laplacian of the Riemannian manifold (M, h) has the following form: $\triangle_h u = T^2 u + \frac{1}{2}\sum_{a=1}^{2n}\left(X_a X_a u - \nabla^\dagger_{X_a} X_a u\right)$, since

$$\nabla^\dagger_T T = 0 \tag{2.31}$$

by lemma 1.1 of [20]. This together with the relationship (2.30) between the TWT connection and the Levi-Civita connection for h implies that

$$\triangle_H u = \frac{1}{2}\sum_{a=1}^{2n}(X_a X_a u - \nabla_{X_a} X_a u) + \frac{1}{2}\sum_{a=1}^{2n}\left[\left(\nabla^\dagger_{X_a}\theta\right) X_a\right] Tu. \tag{2.32}$$

Note that $\nabla^\dagger_X \theta = h\left(\cdot, \nabla^\dagger_X T\right)$ by (1.1) and the Levi-Civita connection preserving the metric h. So we have

$$\sum_{a=1}^{2n}\left(\nabla^\dagger_{X_a}\theta\right)(X_a) = \sum_{a=1}^{2n} h\left(X_a, \nabla^\dagger_{X_a} T\right) = 2div(T) = 0,$$

by (2.31) again. Here the vanishing of the divergence of T with respect to the Riemannian metric h is given again by lemma 1.1 of [20]. Consequently, we get $\triangle_H u = -\triangle_\theta u$ by the expression of the SubLaplacian in Proposition 2.1. (1.3) follows from substituting this identity into (2.29). $\square$

3 Osculating the Riemannian contact structure by the Heisenberg group

3.1 Normal coordinates

The purpose of this subsection is to define normal coordinates to approximate the contact Riemannian structure at each point of a contact Riemannian manifold M by the standard structure on the Heisenberg group. Let $\{X_1, \cdots, X_{2n}\}$ be a local orthogonal frame of H with norm $\sqrt{2}$ under h constructed in section 2.2. As in the CR case in Folland-Stein [12], for each $\xi \in M$, we define *the exponential map* E_ξ at ξ based on the local frame $\{X_0 := T, X_1, \cdots, X_{2n}\}$. For $v = (v_0, v_1, \cdots, v_{2n}) \in \mathbb{R}^{2n+1}$, we define $E_\xi(v) \in M$ to be the endpoint of integral curve $\eta(s), 0 \leq s \leq 1$, of the vector field $\sum_{j=0}^{2n} v_j X_j$ with $\eta(0) = \xi$. Then E_ξ is a smooth mapping of a star shaped neighborhood U_ξ of $0 \in \mathbb{R}^{2n+1}$ into M. It is clear that $E_{\xi*}(\frac{\partial}{\partial v_j})|_0 = X_j|_\xi$. So E_ξ is a diffeomorphism of a smaller neighborhood of U_ξ of 0, denoted also by U_ξ, onto a neighborhood V_ξ of M. Let

$$\Omega = \{(\xi, \eta) \in M \times M; \eta \in V_\xi\} \tag{3.1}$$

be a neighborhood of the diagonal of $M \times M$. Denote by Θ_ξ the coordinate mapping $E_\xi^{-1} : V_\xi \longrightarrow \mathbb{R}^{2n+1}$ and $\Theta(\xi, \eta) := \Theta_\xi(\eta)$. We also write $(x(\eta; \xi), t(\eta; \xi))$ or $v(\eta; \xi)$ for the coordinates of $\Theta_\xi(\eta)$. Define a norm on $\mathbb{R}^{2n+1}$ by

$$\|v\| = \left(\left(v_1^2 + \cdots + v_{2n}^2 \right)^2 + v_0^2 \right)^{\frac{1}{4}} \qquad \text{and} \qquad \rho(\xi, \eta) := \|\Theta(\xi, \eta)\|. \tag{3.2}$$

A function f on V_ξ is said to be O^k if $f(\eta) = O(\rho(\xi, \eta)^k)$ as $\eta \longrightarrow \xi$.

Proposition 3.1. *In the coordinates $x(\cdot; \xi), t(\cdot; \xi)$, we have*

$$\begin{aligned} X_a &= \frac{\partial}{\partial x_a} + \sum_{b=1}^{2n} B_{ba} x_b \frac{\partial}{\partial t} + \sum_{c=1}^{2n} O^1 \frac{\partial}{\partial x_c} + O^2 \frac{\partial}{\partial t}, \qquad a = 1, \cdots, 2n, \\ T &= \frac{\partial}{\partial t} + \sum_{c=1}^{2n} O^1 \frac{\partial}{\partial x_c} + O^2 \frac{\partial}{\partial t}. \end{aligned} \tag{3.3}$$

In this proposition, we identify $\Theta_{\xi*} X_a$ with X_a. We need the following lemma (cf. Lemma 14.2 and Lemma 14.5 in [12], p. 472-474; see also Lemma 3.2-3.3 in the book [10]) to prove Proposition 3.1.

Lemma 3.2. *Suppose that vector fields $\{Z_\mu\}_{\mu=1}^N$ satisfy the condition that they span the tangential space at each point of a star shaped neighborhood U of $0 \in \mathbb{R}^N$. Write $Z_\mu = \sum_{\nu=1}^N F_{\mu\nu}(v) \frac{\partial}{\partial v_\nu}$. Let $(A_{\mu\nu})$ be the inverse transport matrix of $(F_{\mu\nu})$. Then,*

(1) If $v \in U$, $|s| < 1$, we have $\sum_{\nu=1}^N A_{\mu\nu}(sv) v_\nu = v_\mu$.

(2) Define real functions $c_{\mu\nu\kappa}$ on U by

$$[Z_\mu, Z_\nu] = \sum_{\kappa=1}^N c_{\mu\nu\kappa} Z_\kappa, \tag{3.4}$$

and matrices $D(s,v)=(sA_{\mu\nu}(sv))$ *and* $\Gamma(s,v)$ *with entries* $\Gamma_{\mu\nu}(s,v)=\sum_{\mu=1}^{N}c_{\kappa\nu\mu}(sv)v_\kappa$ *for* $v\in U$, $|s|<1$. *We have*

$$\frac{\partial D}{\partial s}=I-\Gamma D. \tag{3.5}$$

Proof of Proposition 3.1. Apply this lemma to vector fields $X_0:=T,\dots X_{2n}$ over a neighborhood U_ξ of $\xi\in M$. Write $X_a=\sum_{b=0}^{2n}F_{ab}(v)\frac{\partial}{\partial v_b}$, $a=0,1,\cdots,2n$. We have $F_{ab}(0)=\delta_{ab}$ by $E_{\xi*}(\frac{\partial}{\partial v_a})|_0=X_a|_\xi$. By Taylor's theorem, we have the expansion $F(sv)=F(0)+sF^{(1)}(v)+s^2F^{(2)}(v)+\cdots$, where $F(0)=I$ and $F^{(1)},F^{(2)},\cdots$ are certain matrices. Similarly, write

$$A(sv)=I+sA^{(1)}(v)+s^2A^{(2)}(v)+\cdots,\qquad D(s,v)=sI+s^2A^{(1)}(v)+\cdots.$$

Since $FA^t=I$, we have $F^{(1)t}=-A^{(1)}$. Write $\Gamma(s,v)=\Gamma^{(0)}(v)+s\Gamma^{(1)}(v)+\cdots$. The above equation (3.5) implies that $2F^{(1)}(v)^t=\Gamma^{(0)}(v)$. Since $\Gamma^{(0)}_{bc}(v)=\sum_{d=0}^{2n}c_{dcb}(0)v_d$ by definition and $v_0=t=O^2$, we find that

$$F^{(1)}_{a0}(v)=\frac{1}{2}\Gamma^{(0)t}_{a0}=\frac{1}{2}\sum_{d=0}^{2n}c_{da0}(0)v_d=\frac{1}{2}\sum_{b=1}^{2n}c_{ba0}(0)v_b+O^2. \tag{3.6}$$

To determine $c_{ba0}(0)$ for $a,b=1,\cdots,2n$, note that

$$\sum_{c=0}^{2n}c_{bac}X_c=[X_b,X_a]=2B_{ba}T\qquad\mathrm{mod}\quad H, \tag{3.7}$$

by the structure equation (2.16) and the definition of coefficients $c_{\dots}$ in (3.4). It follows that at the origin

$$c_{ba0}(0)=2B_{ba}, \tag{3.8}$$

for $a,b=1,\cdots,2n$. Now in the coordinate chart U_ξ, for $a=1,\dots,2n$, we have

$$\begin{aligned}X_a&=\sum_{b=0}^{2n}F_{ab}(v)\frac{\partial}{\partial v_b}=\frac{\partial}{\partial v_a}+F_{a0}(v)\frac{\partial}{\partial v_0}+\sum_{b=1}^{2n}O^1\frac{\partial}{\partial v_b}\\&=\frac{\partial}{\partial v_a}+\frac{1}{2}\sum_{b=1}^{2n}c_{ba0}(0)v_b\frac{\partial}{\partial v_0}+\sum_{b=1}^{2n}O^1\frac{\partial}{\partial v_b}+O^2\frac{\partial}{\partial v_0}\\&=\frac{\partial}{\partial v_a}+\sum_{b=1}^{2n}B_{ba}v_b\frac{\partial}{\partial v_0}+\sum_{b=1}^{2n}O^1\frac{\partial}{\partial v_b}+O^2\frac{\partial}{\partial v_0}\end{aligned} \tag{3.9}$$

by using (3.6) and (3.8).

For T, note that $T=\frac{\partial}{\partial v_0}+\sum_{a=0}^{2n}F^{(1)}_{0a}(v)\frac{\partial}{\partial v_a}+\sum_{b=0}^{2n}O^2\frac{\partial}{\partial v_b}$ and $F^{(1)}_{00}(v)=\frac{1}{2}\sum_{b=1}^{2n}c_{b00}(0)v_b+O^2$ by (3.6). On the other hand, we have $d\theta(X_b,T)=-i_Td\theta(X_b)=0$, which implies $[X_b,T]\in H$. It is equivalent to $c_{b00}(0)=0$ for $b=1,\dots,2n$. Then (3.3) for T follows. The lemma is proved.

3.2 Osculating the contact Riemannian structure by the Heisenberg group

Theorem 3.3. *In the normal coordinates constructed in section 3.1, we have*

(1) $\Theta_\xi(\eta) = -\Theta_\eta(\xi) \in \mathbb{R}^{2n+1}$*;*

(2) $\Theta : \Omega \longrightarrow \mathbb{R}^{2n+1}$ *is* C^∞*;*

(3) $\Theta_\xi^*(2^n dv_0 \cdots dv_{2n})|_\xi$ *is the volume element on* M *at* ξ*;*

(4) Suppose $(\xi,\eta),(\xi,\zeta),(\zeta,\eta) \in \Omega$ *and* $\rho(\xi,\eta) \le \varepsilon_0, \rho(\zeta,\eta) \le \varepsilon_0$ *for some sufficiently small constant* $\varepsilon_0 > 0$*. Then, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|\Theta(\xi,\eta) - \Theta(\zeta,\eta)\| \le C\left(\rho(\xi,\zeta) + \rho(\xi,\zeta)^{\frac{1}{2}}\rho(\xi,\eta)^{\frac{1}{2}}\right), \\ &\rho(\zeta,\eta) \le C\left(\rho(\xi,\zeta) + \rho(\xi,\eta)\right). \end{aligned} \tag{3.10}$$

Namely, $\rho(\cdot,\cdot)$ *is a local pseudodistance on* M*.*

Proof. (1) follows from the definition of exponential map E_ξ. (2) follows from the well known theorem on smooth dependence on parameters of solutions to an O.D.E.. (3) follows from $\Theta_{\xi*}$ mapping $X_a|_\xi$ to $\frac{\partial}{\partial v_a}$, $a = 0,1,\cdots,2n$. For (4), we regard ζ as a function of $\xi \in M$ and $v \in U_\xi$ by the equation $\zeta = E_\xi(v)$. We write $\Theta(\zeta,\eta) = f(\eta,\xi,v) \in \mathbb{R}^{2n+1}$ with $f(\eta,\xi,0) = \Theta(\xi,\eta)$. We expand f in Taylor's series of the variable v about 0 to get

$$\Theta_a(\zeta,\eta) = \Theta_a(\xi,\eta) + \sum_{b=0}^{2n} \Lambda_{ab}(\eta,\xi)v_b + O(|v|^2), \tag{3.11}$$

$a = 0,\cdots 2n$, where $v_a = \Theta_a(\xi,\zeta)$, $\Lambda_{ab}(\eta,\xi) = \frac{\partial f_a(\eta,\xi,0)}{\partial v_b}$. Using (3.11) for $\eta = \xi$, we get

$$v_a = \Theta_a(\xi,\zeta) = -\Theta_a(\zeta,\xi) = -\sum_{b=0}^{2n} \Lambda_{ab}(\xi,\xi)v_b + O(|v|^2).$$

Hence $\Lambda_{ab}(\xi,\xi) = -\delta_{ab}$ and $|\Lambda_{ab}(\eta,\xi)| = O(\rho(\eta,\xi))$ for $a \ne b$. It follows that

$$\begin{aligned} &|\Theta_0(\xi,\eta) - \Theta_0(\zeta,\eta)| \le C(\rho(\xi,\zeta)^2 + \rho(\eta,\xi)\rho(\xi,\zeta)), \\ &|\Theta_a(\xi,\eta) - \Theta_a(\zeta,\eta)| \le C\rho(\xi,\zeta), \qquad a = 1,\cdots 2n. \end{aligned} \tag{3.12}$$

Now the first inequality of (3.10) follows from (3.12). The second inequality of (3.10) easily follows from the first one and definition of $\rho(\cdot,\cdot)$. □

For the standard contact Riemannian structure on the Heisenberg group in section 2.1, its TWT connection coefficients vanish, and so does its curvature. For $f \in C^1(\mathcal{H}^n)$, $df = \sum_{a=1}^{2n} Y_a f \cdot \theta^a + \frac{\partial f}{\partial t} \cdot \theta_0$ and $d_b f = \sum_{a=1}^{2n} Y_a f \cdot \theta^a$, where $\theta^a = dx_a$. Recall that $\langle \theta^a, \theta^b \rangle_{\theta_0} = \frac{1}{2}\delta_{ab}$, $a,b = 1,\cdots,2n$. Hence,

$$\langle d_b f, d_b g \rangle_{\theta_0} = \frac{1}{2}\sum_{a=1}^{2n} Y_a f \cdot Y_a g \tag{3.13}$$

if f and g are real valued. $\psi_{\theta_0} = 2^n dt \wedge dx_1 \wedge \cdots \wedge dx_{2n}$ is the Lebegues' measure on $\mathbb{R}^{2n+1}$ up to a constant. The SubLaplacian on the Heisenberg group is

$$\triangle_0 = -\frac{1}{2}\sum_{a=1}^{2n} Y_a Y_a. \tag{3.14}$$

Corollary 3.4. *Let $\{X_1, \cdots, X_{2n}\}$ be a local orthogonal frame of H with norm $\sqrt{2}$ under h as before. Then in the normal coordinates constructed in section 3.1, we have*

(1) $\left(\Theta_\xi^{-1}\right)^* \theta = \theta_0 + O^1 dt + O^2 dx;$

(2) $\left(\Theta_\xi^{-1}\right)^* \psi_\theta = (1 + O^1)\psi_{\theta_0};$

(3) $\left(\Theta_\xi^{-1}\right)^* h = h_0 + O^1.$

(4) $\Theta_{\xi *}\triangle_\theta = \triangle_0 + \mathcal{E}(\partial_x) + O^1\mathcal{E}(\partial_t; \partial_x^2) + O^2\mathcal{E}(\partial_x \cdot \partial_t) + O^3\mathcal{E}(\partial_t^2)$. *Where $O^k\mathcal{E}$ indicates an operator involving combinations of the indicated derivatives with coefficients in O^k.*

Proof. (1) follows from the expansion of X_a in Proposition 3.1. (2) follows from (1). Since ${\Gamma_{ab}}^c$ bounded, ${\Gamma_{ab}}^c X_c = \mathcal{E}(\partial_x) + O^1\mathcal{E}(\partial_t)$ by expansion (3.3). (4) follows from the expression of the SubLaplacian $\triangle_\theta$ in (2.19) and the expansion of X_a in Proposition 3.1. □

4 The Regularity of $\triangle_\theta$ and the Yamabe equation

4.1 Singular integral operators on contact Riemannian manifolds and the regularity of $\triangle_\theta$

In this subsection, let U be a relatively compact open subset of a normal coordinate neighborhood of some point ξ in a contact Riemannian manifold M, and let $\{X_1, \cdots, X_{2n}\}$ be a local orthogonal frame of H with norm $\sqrt{2}$ under h as before. Define the *Folland-Stein norm* as

$$\|u\|_{S_k^s(U)} := \sum_{|L|\le k} \left\|X^L u\right\|_{L^s(U)} \tag{4.1}$$

for $u \in C^\infty(U)$, where $X^L = X_{a_1}\cdots X_{a_l}$ for a multiindex $L = (a_1, \cdots, a_l)$ with $a_1, \cdots, a_l \in \{1, \ldots, 2n\}$, $|L| = l$. The *Folland-Stein space* $S_k^s(U)$ is the completion of $C^\infty(U)$ with respect to this norm. Define

$$\|u\|_{\Gamma_\beta(U)} := \sup_{x\in U}|u(x)| + \sup_{x,y\in U}\frac{|u(x) - u(y)|}{\rho(x,y)^\beta}, \qquad \text{for } 0 < \beta < 1, \tag{4.2}$$

and $\Gamma_\beta(U)$ is the completion of $C^\infty(U)$ with respect to this norm. For $k < \beta < k+1$, $k \in \mathbb{Z}_+$,

$$\Gamma_\beta(U) = \left\{u \in C(U); X^L u \in \Gamma_{\beta-k} \text{ for any multiindex } L \text{ with } |L| \le k\right\} \tag{4.3}$$

with the norm

$$\|u\|_{\Gamma_\beta(U)} = \sup_{|L|=k} \left\|X^L u\right\|_{\Gamma_{\beta-k}(U)}. \tag{4.4}$$

Now fix an open covering $\{U_1, \cdots, U_\tau\}$ of M such that each U_j is a relatively compact open set of a normal coordinate neighborhood $\Omega_\xi \subset M$ for some $\xi \in M$. Let $\{\chi_j\}_{j=1}^{\tau}$ be a unit partition of M such that supp$\chi_j \subset U_j$. Define

$$S_k^s(M) = \{u; \chi_j u \in S_k^s(U_j) \text{ for all } j\}, \tag{4.5}$$

and the norm of u to be the sum of $\|\chi_j u\|_{S_k^s(U_j)}$.

Theorem 4.1. *Let U be a relatively compact open subset of a normal coordinate neighborhood of some point ξ in a contact Riemannian manifold M. For $1 < s < \infty$ and $k = 0, 1, \ldots$, there exists a constant $C > 0$ such that for $u \in C_0^\infty(U)$, we have*

$$\|u\|_{S_{k+2}^s(U)} \leq C(\|\triangle_\theta u\|_{S_k^s(U)} + \|u\|_{L^s(U)}).$$

For strictly pseudoconvex CR manifolds, such estimates for $\Box_b$ were proved by Folland and Stein in theorem 16.6 of [12]. We need the machine of singular integral operators to prove this theorem. A smooth function k defined on $\mathbb{R}^{2n+1} \setminus \{0\}$ is said to be *of type* λ ($\lambda > 0$) if $k(D_\delta(v)) = \delta^{-Q+\lambda} k(v)$, for any $v \in \mathbb{R}^{2n+1} \setminus \{0\}$, where D_δ is the dilation. k is said to be *of type* 0 if the above holds for $\lambda = 0$ and the *mean value* of k vanishes, i.e. $\int_{\alpha \leq \|v\| \leq \beta} k(v)\psi_{\theta_0}(v) = 0$ for any $\alpha, \beta > 0$. A function $K(\xi, \eta)$ on $M \times M$ is said to be a *kernel of type* λ if for each positive integer m, we can write

$$K(\xi, \eta) = \sum_{l=1}^{L} a_l(\xi) k_\eta^{(l)}(\Theta_\eta(\xi)) b_l(\eta) + E_m(\xi, \eta), \tag{4.6}$$

for some positive integer L and any (ξ, η) in the neighborhood Ω of the diagonal $M \times M$ in (3.1), with (1) $E_m \in C^m(M \times M)$, $a_l, b_l \in C_0^\infty(M)$; (2) for each l, functions $k_\eta^{(l)}(\cdot)$ defined on $\mathbb{R}^{2n+1} \setminus \{0\}$ is of type $\geq \lambda$ and depends smoothly on η. An operator T is said to be *of type* λ ($\lambda > 0$) if $Tf(\xi) = \int_M K(\xi, \eta) f(\eta) \psi_\theta(\eta)$ for some kernel K of type $\geq \lambda$, and an operator T is said to be *of type* 0 if

$$Tf(\lambda) = \lim_{\epsilon \to 0} \int_{\rho(\xi, \eta) > \epsilon} K(\xi, \eta) f(\eta) \psi_\theta(\eta) + a(\xi) f(\xi)$$

for some kernel K of type 0 and $a \in C_0^\infty(M)$. The kernel with $k_\eta^{(l)}(\cdot)$ replaced by $k_\xi^{(l)}(\cdot)$ in (4.6) is also a kernel of type λ by the following argument (4.9).

Proposition 4.2. *On a compact contact Riemannian manifold M,*

(1) If S is an operator of type $0 < \lambda < Q$, then S is bounded from $L^s(M)$ to $L^r(M)$ with $\frac{1}{r} = \frac{1}{s} - \frac{\lambda}{Q}$ and $1 < s < r < \infty$.

(2) If S is an operator of type $\lambda \geq 1$ and X be a horizontal vector field, then XS and SX are operators of type $\lambda - 1$.

(3) Given a normal coordinate neighborhood Ω_ξ for some $\xi \in M$, if S is an operators of type 0, there exist operators $S_0, S_1, \ldots, S_{2n}$ of type 1 such that

$$Sf = \sum_{a=1}^{2n} S_a X_a f + S_0 f \tag{4.7}$$

for $f \in S_1^s(M)$ and supported in Ω_ξ.

(4) The operator of type λ for $\lambda = 0, 1, 2, \cdots$ is bounded from $S_k^s(M)$ to $S_{k+\lambda}^s(M)$ for $k = 0, 1, \cdots, 1 < s < \infty$.

(5) The operator of type λ for $\lambda = 0, 1, 2, \cdots$ is bounded from $L^s(M)$ to $\Gamma_\beta(M)$ if $\beta = \lambda - \frac{2n+2}{s} > 0$ and $s \geq 1$.

For strictly pseudoconvex CR manifolds, these results are theorem 15.11, theorem 15.15, theorem 15.19 and theorem 15.20 of Folland-Stein [12], respectively. With the osculation by the Heisenberg group and the pseudodistance in Theorem 3.3, the machine of singular integral operators works on contact Riemannian manifolds without modification as on strictly pseudoconvex CR manifolds. This is because that harmonic analysis is a theory of real variables and the almost complex structure is not involved. The proof is exactly the same and so we omit the details.

It is well known that the fundamental solution of the SubLaplacian $\triangle_0$ on the Heisenberg group is given by $G_0(x,t) := \frac{C_Q}{\|(x,t)\|^{Q-2}}$ for some constant $C_Q > 0$. For fixed $\eta \in M$ and coordinates defined by $\Theta_\eta(\xi) = v$, we have $\triangle_\theta = \triangle_0^\eta + \mathcal{R}^\eta$ where $\triangle_0^\eta = -\frac{1}{2}\sum_{b=1}^{2n} Y_a Y_a$ by (3.14) with $Y_a = \frac{\partial}{\partial v_a} + \sum_{b=1}^{2n} B_{ba} v_b \frac{\partial}{\partial b_0}$, and

$$\mathcal{R}^\eta = \sum_{b=1}^{2n} O(1)\frac{\partial}{\partial v_b} + O^1\frac{\partial}{\partial v_0} + \sum_{a,b=1}^{2n} O^1 \frac{\partial^2}{\partial v_a \partial v_b} + \sum_{b=1}^{2n} O^2 \frac{\partial}{\partial v_b \partial v_0} + O^3 \frac{\partial^2}{\partial v_0^2}$$

by Corollary 3.4 (4). For $\eta \in M$ and $v = (x,t) \in \mathcal{H}^n$, we have $\triangle_0^\eta G_0(v) = \delta_0(v)$ by definition. Let $\psi(\xi,\eta)$ be a real C_0^∞ function supported in $\Omega \cap \{(\xi,\eta); \rho(\xi,\eta) \leq 1\}$ with $\psi(\xi,\eta) = \psi(\eta,\xi)$, and $\psi(\xi,\eta) = 1$ on a neighbourhood of the diagonal in $M \times M$. Now define an operator B by

$$B\phi(\xi) = \phi(\xi) - \triangle_\theta A\phi(\xi), \tag{4.8}$$

where A is the operator of type 2 given by $A\phi(\xi) = \int_M \psi(\xi,\eta) G_0(\Theta_\eta(\xi))\phi(\eta)\psi_\theta(\eta)$.

Proposition 4.3. *B is an operator of type 1.*

Since G_0 is of type 2, it is easy to see that $Y_a G_0$ is of type 1, TG_0 is of type 0, and so $\mathcal{R}^\eta G_0$ is a kernel of type 1. The proof is exactly the same as proposition 16.2 of Folland-Stein [12] for strictly pseudoconvex CR manifolds.

Proof of Theorem 4.1. For functions $k_\eta(u)$ of type $\lambda \geq 1$ smoothly depending on $\eta \in M$, by Taylor's expansion in terms of $v = \Theta_\eta(\xi)$, we have

$$k_\xi(v) = k_\eta(v) + \sum_{|\alpha|=1}^{L} k_\eta^{(\alpha)}(v)\frac{v^\alpha}{\alpha!} + R_m(v,\eta,\xi), \tag{4.9}$$

where $k_\eta^{(\alpha)}(v)$ denotes the appropriate derivatives of $k_\eta(v)$ with respect to η as we do in (3.11). If $k_\eta(\cdot)$ is of type λ, then so is $k_\eta^{(\alpha)}(\cdot)$. $k_\eta^{(\alpha)}(v)v^\alpha$ is therefore of type $\geq \lambda$ and for L sufficiently large, $R_m(v,\eta,\xi) \in C^m(\mathbb{R}^{2n+1} \times M \times M)$. Thus, $k_\xi(\Theta_\xi(\eta)) = k_\xi(-\Theta_\eta(\xi))$ is a kernel of type λ by the above argument. Note the adjoint T^* of the operator T of type λ with kernel $K(\xi,\eta)$ in (4.6) has the kernel

$$K^*(\xi,\eta) = K(\eta,\xi) = \sum_{l=1}^{L} a_l(\eta)\ k_\xi^{(l)}(\Theta_\xi(\eta))b_l(\xi) + E_m(\eta,\xi),$$

which is also a kernel of type λ by the above result. So T^* is also an operator of type λ. As the adjoint of (4.8), we get

$$I = A^* \triangle_\theta + B^* \tag{4.10}$$

with A^* and B^* operators of type 2 and 1, respectively. Set $A^{[p]} := \sum_{j=0}^{p-1}(B^*)^j A^*$. Then, $I - A^{[p]}\triangle_\theta = I - \sum_{j=0}^{p-1}(B^*)^j(I - B^*) = (B^*)^p$. Note that $\|B^*\phi\|_{S^s_{k+1}} \leq C_{s,k}\|\phi\|_{S^s_k}$, $\|A^*\phi\|_{S^s_{k+2}} \leq C_{s,k}\|\phi\|_{S^s_k}$ for some constant $C_{s,k} > 0$ for fixed positive integers k and s by Proposition 4.2 (4). Then we get

$$\|\phi\|_{S^s_{k+2}} \leq \|A^{[k+2]}\triangle_\theta\phi\|_{S^s_{k+2}} + \|(B^*)^{k+2}\phi\|_{S^s_{k+2}} \leq C\left(\|\triangle_\theta\phi\|_{S^s_k} + \|\phi\|_{L^s}\right).$$

The theorem is proved. □

Theorem 4.4. (The Sobolev embedding) *(1) The inclusion $S^s_k(U) \subset L^r(U)$ is continuous for $\frac{1}{r} = \frac{1}{s} - \frac{k}{Q}$ and $1 < s < r < \infty$.*

(2) Suppose that $\beta = k - \frac{2n+2}{s} > 0$. The inclusion $S^s_k(U) \subset \Gamma^\beta(U)$ is continuous if β is not an integer.

Proof. (1) For strictly pseudoconvex CR manifolds, this is theorem 5.5 of Jerison-Lee [17]. Now apply Proposition 4.2 (3) to the identity operator to get that given a normal coordinates neighborhood Ω_ξ for some $\xi \in M$, there exist operators $S_0, S_1, \ldots, S_{2n}$ of type 1 such that

$$f = \sum_{a=1}^{2n} S_a X_a f + S_0 f, \tag{4.11}$$

for $f \in S^s_1$ and supported in Ω_ξ. Since S_a is bounded from L^s to L^r with $\frac{1}{r} = \frac{1}{s} - \frac{1}{Q}$ by Proposition 4.2 (1), we get the result for $k = 1$. For higher k, we just substitute (4.11) into itself repeatedly as in Folland-Stein [12].

(2) The proof is exactly the same as that of theorem 21.1 of Folland-Stein [12] if we use the above representation formula repeatedly and Proposition 4.2 (5). □

The following regularity can be proved exactly as theorem 16.7 of Folland-Stein [12].

Theorem 4.5. *Suppose that $f, g \in L^1_{loc}(U)$ and $\triangle_\theta f = g$ in the sense of distributions on U. Then, for any $\eta \in C_0^\infty(U)$, we have*

(1) if $g \in L^s(U)$ with $n+1 < s \leq \infty$, then $\eta f \in \Gamma_\beta(U)$ with $\beta = 2 - \frac{2n+2}{s}$;

(2) if $g \in S^s_k(U)$ with $1 < s < \infty$ and $k = 0, 1, \ldots,$ then $\eta f \in S^s_{k+2}(U)$;

Harmonic analysis on strictly pseudoconvex CR manifolds has already generalized to general smooth vector fields $Z_1, \ldots, Z_m$ on $U \subset \mathbb{R}^N$ satisfying the *Hörmander's condition for hypoellipticity*: the rank of Lie $[Z_1, \ldots, Z_m] = N$, at every point $x \in U$. See e.g. [6] for the theory of singular integral operators in this general setting, in particular for results corresponding to our Theorem 4.1-4.4 and Proposition 4.2.

A piecewise C^1 curve $\gamma : [0, r] \to U$ is said to be *subunitary* if for every $\xi \in \mathbb{R}^N$ and $t \in (0, r)$, $(\gamma'(t) \cdot \xi)^2 \leq \sum_{j=1}^m (Z_j(\gamma(t)) \cdot \xi)^2$. A natural distance associated to the vector fields $Z_1, \ldots, Z_m$ is defined as follows. Given two points $x, y \in U$, the distance from x to y is defined by

$$d(x,y) := \inf\{r > 0; \text{there exists a subunitary } \gamma : [0,r] \to U, \text{ with } \gamma(0) = x, \gamma(r) = y\}. \tag{4.12}$$

For $x \in U$ and $R > 0$, let $B(x, R) := \{y \in U; d(x, y) < R\}$ be the ball of radius R respect to this distance.

Denote by $D_{\mathcal{L}} u = (Z_1 u, \ldots, Z_m u)$ the *subelliptic gradient* of u. Given an open set $U \subset \mathbb{R}^N$, denote by $S^s_k(U)$ the completion of $C^k(U)$ under the norm (4.1) with the vector fields X_j's replaced by Z_j's. We also need the following subelliptic estimates for operators more general than the SubLaplacian.

Theorem 4.6. (Theorem 0.1-0.2 in [6]) *Let $Z_1, \ldots, Z_m$ be smooth vector fields on $U \subset \mathbb{R}^N$ satisfying Hörmander's condition and $\mathcal{L} = \sum_{i,j=1}^m a_{ij}(x) Z_i Z_j$. The coefficients $a_{ij}(x)$ are real valued bounded measurable functions defined in U belonging to the class $VMO(U)$; the matrix $a_{ij}(x)$ (not necessarily symmetric) is uniformly elliptic:*

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2, \qquad for \quad any \qquad \xi \in \mathbb{R}^m, \ a.e. \ x \in U \tag{4.13}$$

for some positive constant μ. Then, for every $s \in (1, \infty)$, any $U' \subset\subset U$, there exists a constant c depending on the vector fields Z_j's, the numbers N, m, s, μ, the VMO moduli of the coefficients a_{ij}, U', U such that for every $u \in S^s_2(U)$,

$$\|u\|_{S^s_2(U')} \leq C(\|\mathcal{L}u\|_{L^s(U)} + \|u\|_{L^s(U)}). \tag{4.14}$$

Moreover, the following estimate holds: for every $u \in S^s_{k+2}(U)$,

$$\|u\|_{S^s_{k+2}(U')} \leq C(\|\mathcal{L}u\|_{S^s_k(U)} + \|u\|_{L^s(U)}) \tag{4.15}$$

for every positive integer k such that $a_{ij} \in S^\infty_k(U)$. The dependence on the VMO moduli of the coefficients a_{ij} is replaced by the $S^\infty_k(U)$ norm of a_{ij}.

4.2 Harnack inequality, Poincaré inequality and the regularity of the Yamabe equation

Let us recall the Harnack inequality and Poincaré-type inequality (See [7] [8] for example). For smooth vector fields $Z_1, \ldots, Z_m$ on $U \subset \mathbb{R}^N$ satisfying Hörmander's condition, consider the equation

$$\sum_{j=1}^m Z_j^\dagger A_j(x, u, Z_1 u, \ldots, Z_m u) = F(x, u, Z_1 u, \ldots, Z_m u), \tag{4.16}$$

where $Z_j^\dagger$ is the formal adjoint of Z_j with respect to the Lebegues' measure, with measurable functions $A = (A_1, \ldots, A_m) : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^1$. We assume that A_j's and F satisfy the following structural conditions: *There exist $p \in (1, \infty)$, $c_1 > 0$, and measurable functions $F_l, F_2, F_3, g_2, g_3, h_3$ on $\mathbb{R}^N$, such that for a.e. $x \in \mathbb{R}^N$, $u \in \mathbb{R}$ and $\zeta \in \mathbb{R}^m$, we have*

$$\begin{aligned} &|A(x,u,\zeta)| \le c_1|\zeta|^{p-1} + g_2(x)|u|^{p-1} + g_3(x), \\ &|F(x,u,\zeta)| \le F_1(x)|\zeta|^{p-1} + F_2(x)|u|^{p-1} + F_3(x), \\ &A(x,u,\zeta)\cdot\zeta \ge |\zeta|^p - F_2(x)|u|^{p-1} - h_3(x). \end{aligned} \tag{4.17}$$

Since the problem is local, we may assume A and F only defined over a fixed open subset U of $\mathbb{R}^N$. The integrability requirements on the functions $F_i, , g_i, h_i$ in the structural assumptions are

(i) $g_2, g_3 \in L^r_{loc}(U)$, with $r = \frac{Q}{p-1}$ if $p < Q$, and $r > \frac{Q}{Q-1}$ if $p = Q$;
(ii) $F_2, F_3, h_3 \in L^s_{loc}(U)$, with $s > \frac{Q}{p}$;
(iii) $F_1 \in L^t_{loc}(U)$, with $t > Q$.

Theorem 4.7. (Theorem 3.1 of Capogna-Danielli-Garofalo [7]) *Let $u \in S_1^p(U)$ be a nonnegative solution to the equation (4.16). Then there exist constant $C > 0$, only depending on U, p, $\|F_1\|_{L^t}$, $\|F_2\|_{L^s}$ and $\|g_2\|_{L^r}$, and $R_0 > 0$ such that for any $B_R = B(x,R)$ with $B(x,4R) \subset U$ and $R < R_0$, we have*

$$ess\ sup_{B_R} u \le C(ess\ inf_{B_R} u + K_R). \tag{4.18}$$

Here,

$$K_R := \left(|B_R|^{\frac{s}{Q}}\|F_3\|_{L^s(B_R)} + \|g_3\|_{L^r(B_R)}\right)^{\frac{1}{p-1}} + \left(|B_R|^{\frac{s}{Q}}\|h_3\|_{L^s(B_R)}\right)^{\frac{1}{p}} \tag{4.19}$$

with r, s as in (i), (ii) above.

See [7], p. 783, for the dependence of the constant C on $U, p, \ldots$. This makes the constant in the following Harnack inequality (4.20) not depend on u. It is better than the constant of the Harnack inequality in proposition 5.12 of Jerison-Lee [17], which depends on $\|u\|_{L^{2^*}(U)}$.

Proposition 4.8. (Harnack inequality) *Let U be a relatively compact open set of a normal coordinate neighborhood of a point in a contact Riemannian manifold M. Suppose $f \in L^\infty(U)$, $u \in L^{2^*}(U)$, $u \ge 0$ and $(\triangle_\theta + f)u = 0$ in the sense of distributions on U. Then, for any $K \subset\subset U$,*

$$ess\ sup_{x\in K} u(x) \le C ess\ inf_{x \in K} u(x), \tag{4.20}$$

where the constant C depends only on U, K, $\|f\|_{L^\infty(U)}$ and the choice of the frame.

Proof. Note that $\triangle_\theta u = -fu \in L^{2^*}(U) \subset L^2(U)$ by Hölder's inequality since U is bounded. Then $u \in S_1^2(U')$ for an open subset U' such that $U' \subset\subset U$ by the regularity Theorem 4.5. Note that if we write $\psi_\theta = V dv$ locally for some positive function V, where dv is the Lebegues' measure, then we have $X_a^* = X_a^\dagger - X_a V$, where X_a^* and $X_a^\dagger$ are the formal adjoint operators of X_a with respect to the measure ψ_θ and the

Lebegues' measure, respectively. So the equation $\triangle_\theta u = \frac{1}{2}\sum_{a=1}^{2n} X_a^* X_a u = -fu$ can be written as $\sum_{a=1}^{2n} X_a^\dagger X_a u = \sum_{a=1}^{2n} X_a V \cdot X_a u - 2fu$. Now we apply Theorem 4.7 to this equation with $Z_1 = X_1, \dots, Z_{2n} = X_{2n}$ and

$$p = 2, \qquad A(x,u,\zeta) = \zeta, \qquad F(x,u,\zeta) = \sum_{a=1}^{2n} X_a V \cdot \zeta_a - 2f(x)u,$$

in the equation (4.16). In particular, we have

$$F_1 = \sum_{a=1}^{2n} |X_a V| \in L^\infty, \qquad F_2 = 2|f| \in L^\infty, \qquad F_3 \equiv 0, \qquad c_1 \equiv 1, \qquad g_2 \equiv g_3 \equiv h_3 \equiv 0.$$

Then $K_R = 0$ by (4.19). The result follows from Theorem 4.7. □

Proposition 4.9. (Poincaré-type inequality) *Suppose that vector fields $Z_1, \dots, Z_m$ satisfy Hörmander's condition. Let $B_R \subset U$ be a ball of radius R with respect to the natural distance (4.12) associated to the vector fields. Then, for each u with $|D_{\mathcal{L}} u|^q \in L^1(U)$ with $1 < q < \infty$, we have*

$$\int_{B_R} |u - u_{B_R}|^q \leq C R^q \int_{B_R} |D_{\mathcal{L}} u|^q, \tag{4.21}$$

for some constant C only depending on U and the choice of the frame, where $u_{B_R} = \int_{B_R} u/|B_R|$, the average of u.

The inequality (4.21) was proved by Jerison [16] for all $f \in C^\infty(B_R)$. The general case is a consequence. The following interpolation inequality for the space S_1^s is a simple corollary of the above Poincaré-type inequality as in proposition 5.14 of Jerison-Lee [17].

Proposition 4.10. (Interpolation inequality for the space S_1^s) *If $u \in L^1(U)$ and $\sum_{a=1}^{2n} |X_a u|^s \in L^1(U)$ with $1 < r < \infty$, then $u \in S_1^s(U)$ and*

$$\|u\|_{S_1^s(U)} \leq C \left(\left\| \sum_{a=1}^{2n} |X_a u|^s \right\|_{L^1(U)} + \|u\|_{L^1(U)} \right), \tag{4.22}$$

for some constant C only depending on U and the choice of the frame.

Proposition 4.11. *Let U be a relatively compact open set of a normal coordinate neighborhood of a point in a contact Riemannian manifold M. Suppose $f \in L^{n+1}(U)$, $u \in L^{2^*}(U)$, $u \geq 0$ and*

$$(\triangle_\theta + f)u = 0$$

in the sense of distributions on U. Then, for any $\eta \in C_0^\infty(U)$, $\eta u \in L^s(U)$ for each $0 < s < \infty$.

This proposition is the contact Riemannian version of the *higher integrability* in proposition 5.10 of [17], a variant of results of Yamabe [26], Trudinger [22] and Brezis and Kato [5] for the original Yamabe equation. Its proof is the same as the CR case given in the Appendix of [17]. We need the following lemma A.1 of [17]: Let U' be an open set such that $U' \subset\subset U$, then with the hypotheses in this proposition we have $u \in S_1^2(U')$. The proof of this lemma is similar to that of lemma A.1 of [17] if we use the representation formula (4.8), (4.10) and the regularity of singular integral operators of type $0, 1, 2$ on contact Riemannian manifolds given in §4.1. The remaining part of proof is exactly the same as in the Appendix of [17].

Proposition 4.12. *With the hypotheses of Proposition 4.11 and the additional assumption $f \in L^s(U)$ with $s > n+1$, then $u \in \Gamma_\beta(U)$ for some $\beta > 0$ and for any $K \subset\subset U$,*

$$\|u\|_{\Gamma_\beta(K)} \leq C$$

for constant C depending only on K, $\|f\|_{L^s(U)}$, $\|u\|_{L^{2^*}(U)}$ and the choice of the frame.

It can be simply proved as proposition 5.10 of Jerison-Lee [17], without modification, by using the regularity of the SubLaplacian $\triangle_\theta$ in Theorem 4.5 and the Sobolev embedding Theorem 4.4.

Theorem 4.13. *Let U be a relatively compact open set of a normal coordinate neighborhood of of a point in a contact Riemannian manifold M. Suppose $f, g \in C^\infty(U)$, $u \geq 0$ on U, $u \in L^s(U)$, $s > 2^*$ and*

$$\triangle_\theta u + gu = fu^{q-1}$$

in the sense of distributions on U for some $2 \leq q \leq 2^*$. Then, $u \in C^\infty(U)$ and $u > 0$. If $K \subset\subset U$, then $\|u\|_{C^k(K)}$ depends only on K, $\|u\|_{L^s(K)}$, $\|f\|_{C^k(K)}$, $\|g\|_{C^k(K)}$, the choice of the frame, but not on q.

Proof. It is similar to theorem 5.15 in Jerison-Lee [17]. The equation can be written as $\triangle_\theta u = hu$ with $h = fu^{q-2} - g \in L^{\frac{s}{q-2}}(U)$. By Hölder's inequality, $h \in L^r(U)$ with $r = \frac{s}{2^*-2} > \frac{2^*}{2^*-2} = n+1$ and $\|h\|_r$ depends only on the stated bounds. Then choosing K' with $K \subset\subset K' \subset\subset U$, we see that $u \in \Gamma_\beta(K')$ for some $\beta > 0$ by Proposition 4.12. Note that h is bounded by the continuity of $\Gamma_\beta(K')$ functions. So we can use Harnack inequality in Proposition 4.8 to obtain that u is bounded away from zero by a constant depending on the stated bounds. Apply subelliptic regularity in Theorem 4.5 to $\triangle_\theta u = fu^{q-1} - gu \in L^r(U)$ to get $u \in S_2^r(K)$. Since u is bounded away from zero, $u^{q-1} \in S_2^r(K)$ with the norm depending only on the stated bounds. Thus, replacing K with a smaller set that we still denote K, we conclude from subelliptic regularity in Theorem 4.5 that $u \in S_4^r(K)$. Repeating this procedure, we get $u \in S_{2k+2}^r(K)$ for any positive integer k. Obviously $u \in \Gamma_{2k}(K)$ by Sobolev embedding Theorem 4.4 (2). We have $\Gamma_{2k}(K) \subset C^k(K)$ because $[X_a, X_b] = 2B_{ab}T \mod H$ locally. □

Corollary 4.14. *Under the assumptions of Theorem 4.13, but $s > 2^*$ replaced by $s = 2^*$, we still have $u > 0$ in U and $u \in C^\infty(U)$.*

Proof. The equation can be written as $\triangle_\theta u = hu$ with $h = fu^{q-2} - g \in L^{n+1}(U)$ as above. Proposition 4.11 grantees $u \in L^s(U')$ with $s > 2^*$ for any open subset $U' \subset\subset U$. □

Proposition 4.15. *On a compact contact Riemannian manifold, the unit ball in the space $S_1^2(M)$ is compact in $L^s(M)$ for $1 < s < 2^*$.*

To prove this proposition, we need the Sobolev space over the Euclidean space $L^s_\sigma(\mathbb{R}^{2n+1}) := \{f \in L^s(\mathbb{R}^{2n+1}); (1+|\xi|^2)^{\frac{\sigma}{2}}\widehat{f}(\xi) \in L^s(\mathbb{R}^{2n+1})\}$ for $0 \le \sigma < \infty$, where $\widehat{f}$ is the Fourier transformation of f. As before, we choose an open covering $\{U_1, \cdots, U_\tau\}$ of M such that each U_j is a relatively compact open set of a normal coordinate neighborhood $\Omega_\xi \subset M$ for some $\xi \in M$. Let $\{\chi_j\}_{j=1}^\tau$ be a unit partition of M such that $\mathrm{supp}\chi_j \subset U_j$. Define $L^s_\sigma(M) = \{u; \chi_j u \in L^s_\sigma(U_j)$ for all $j\}$, and the norm of u to be the sum of $\|\chi_j u\|_{L^s_\sigma(U_j)}$.

Proposition 4.16. *For a compact contact Riemannian manifold M, we have the continuous inclusion $S^2_1(M) \subset L^2_{\frac{1}{2}}(M)$.*

This is a special case of theorem 5.4.7 of Folland-Kohn [11] for general smooth vector fields satisfying Hörmander's condition. See also section 19 in Folland-Stein [12] for results over CR manifolds.

Proof of Proposition 4.15. See also proposition 4.10 of [23]. By Proposition 4.16, we have continuous inclusion $S^2_1(M) \subset L^2_{\frac{1}{2}}(M)$. Now the usual Sobolev imbedding theorem guarantees that the inclusion $L^2_{\sigma+\kappa}(M) \subset L^2_\sigma(M)$ is compact for $\kappa > 0$, $0 \le \sigma < \infty$ (cf. proposition 3.4 in the book [21]). In particular $L^2_{\frac{1}{2}}(M) \subset L^2(M)$ is compact. Recall that we have obviously continuous inclusion $L^2(M) \subset L^1(M)$ for a compact manifold M by Hölder's inequality. Consequently, the inclusion $S^2_1(M) \subset L^1(M)$ is compact by Proposition 4.16. Now suppose that $\{u_l\}$ is a bounded sequence of $S^2_1(M)$. Then, there exists a subsequence, which is still denoted by u_l, converging to u in $L^1(M)$. Note that for $1 < s < 2^*$,

(4.23)
$$\|u_l - u_{l'}\|_{L^s(M)} \le \|u_l - u_{l'}\|^a_{L^1(M)} \|u_l - u_{l'}\|^{1-a}_{L^{2^*}(M)}, \qquad \text{for} \quad 0 < a = \frac{\frac{1}{s} - \frac{1}{2^*}}{1 - \frac{1}{2^*}} < 1,$$

(cf. proposition 3.62 of [2]) by applying Hölder's inequality to $\|u_l - u_{l'}\|^s_{L^s(M)} = \int |u_l - u_{l'}|^{sa} |u_l - u_{l'}|^{s(1-a)} \psi_\theta$ with exponents $\frac{1}{sa}, \frac{2^*}{s(1-a)} > 1$, which satisfy $sa + \frac{s(1-a)}{2^*} = 1$. Note that $\|u_l - u_{l'}\|_{L^{2^*}(M)}$ is bounded by its $S^2_1(M)$ norm by the Sobolev embedding Theorem 4.4 with the critical exponent 2^*. We see that $\{u_l\}$ is also a Cauchy sequence in L^s. The compactness is proved. □

5 The existence of extremals

The extremal problem (1.4) on $\mathcal{H}^n$ is

$$\lambda(\mathcal{H}^n) = \inf \left\{ \frac{1}{2} \int_{\mathcal{H}^n} b_n \sum_{a=1}^{2n} |Y_a u|^2 \psi_{\theta_0}; \int_{\mathcal{H}^n} |u|^{2^*} \psi_{\theta_0} = 1 \right\}. \tag{5.1}$$

It is known that $0 < \lambda(\mathcal{H}^n) < \infty$.

Lemma 5.1. *$\lambda(M) \le \lambda(\mathcal{H}^n)$ for any compact contact Riemannian manifold M.*

Proof. Its proof is similar to that in [17]. The class of test functions defining $\lambda(\mathcal{H}^n)$ can be restricted to C^∞_0 functions. For each $\varepsilon > 0$, choose $u \in C^\infty_0(\mathcal{H}^n)$ such that $B_{\theta_0}(u) = 1$ and $A_{\theta_0}(u) < \lambda(\mathcal{H}^n) + \varepsilon$. Let

$$u_\delta(\zeta) := \delta^{-\frac{Q-2}{2}} u(D_{\delta^{-1}}\zeta), \quad \zeta \in \mathcal{H}^n, \qquad \text{and} \qquad f_\delta(\eta) := u_\delta(\Theta_\xi(\eta)),$$

for fixed $\xi \in M$. For δ sufficiently small, supp u_δ is contained in $\Theta_\xi(\Omega_\xi)$. So f_δ has compact support in Ω_ξ, and can be extended to a C^∞ function on M. It is easy to check that $B_{\theta_0}(u_\delta) = B_{\theta_0}(u) = 1$ and $A_{\theta_0}(u_\delta) = A_{\theta_0}(u) < \lambda(\mathcal{H}^n) + \varepsilon$ by rescaling. Also, $\int_{\mathcal{H}^n} |u_\delta|^2 \psi_{\theta_0} = \delta^2 \int_{\mathcal{H}^n} |u|^2 \psi_{\theta_0} \longrightarrow 0$ as $\delta \longrightarrow 0$. By Proposition 3.1 and Corollary 3.4, we have

(5.2)
$$\left(\Theta_\xi^{-1} \circ D_\delta\right)^* \theta = \delta^2 \theta_0 + \delta^3 (O^1 dt + O^2 dx), \qquad \left(\Theta_\xi^{-1} \circ D_\delta\right)^* h = \delta^2 h_0 + \delta^3 O^1,$$
$$\left(\Theta_\xi^{-1} \circ D_\delta\right)^* \psi_\theta = \delta^Q (1 + \delta O^1) \psi_{\theta_0},$$
$$\left(D_{\delta^{-1}} \circ \Theta_\xi\right)_* X_a = \delta^{-1}(Y_a + \delta O^1 \mathcal{E}(\partial_x) + \delta^1 O^2 \mathcal{E}(\partial_t)), \qquad a = 1, \cdots, 2n,$$

where $(Y_1, \cdots, Y_{2n})$ are left invariant vector fields on the Heisenberg group in (2.3). For any fixed $R > 0$, $D_{\delta^{-1}} \Theta_\xi(\Omega_\xi) \supset B_R$ for sufficiently small δ. Therefore,

$$\begin{aligned} B_\theta(f_\delta) &= \int_M |u_\delta(\Theta_\xi(\eta))|^{2^*} \psi_\theta = \int_{\mathcal{H}^n} \delta^{-Q} |u|^{2^*} \left(\Theta_\xi^{-1} \circ D_\delta\right)^* \psi_\theta \\ &= \int_{\mathcal{H}^n} |u|^{2^*} (1 + \delta O^1) \psi_{\theta_0} \longrightarrow B_{\theta_0}(u) = 1. \end{aligned} \tag{5.3}$$

Similarly, $A_\theta(f_\delta) \longrightarrow A_{\theta_0}(u) < \lambda(\mathcal{H}^n) + \varepsilon$. Since ε is arbitrary, the lemma follows. □

For each $2 \le q < 2^*$, consider the following variational problem,

(5.4)
$$\lambda_q(M) := \inf\{A_\theta(u); u \in S_1^2(M), B_{\theta,q}(u) = 1\}, \qquad \text{where} \quad B_{\theta,q}(u) = \int_M |u|^q \psi_\theta.$$

Theorem 5.2. *For each $2 \le q < 2^*$, there exists a positive C^∞ solution u_q to the equation*

$$b_n \triangle_\theta u_q + s_\theta u_q = \lambda_q(M) u_q^{q-1}, \tag{5.5}$$

satisfying $A_\theta(u_q) = \lambda_q(M)$ and $B_{\theta,q}(u_q) = 1$.

Proof. The proof of this theorem is the same as that of theorem 6.2 of [17]. Choosing a minimizing sequence u_j for (5.4), namely, $A_\theta(u_j) \to \lambda_q(M)$ and $B_{\theta,q}(u_j) = 1$. We may assume $u_j \ge 0$ after replacing u_j by $|u_j|$. Since $\{u_j\}$ is bounded in $S_1^2(M)$, there exists a subsequence converging weakly in $S_1^2(M)$ to $u \in S_1^2(M)$. By the compactness in Proposition 4.15, we can assume a subsequence converges in L^q, and so $B_{\theta,q}(u) = 1$. Applying Hölder's inequality we see that $\int s_\theta u_j^2 \to \int s_\theta u^2$, and so $A_\theta(u) \le \lambda_q(M)$. Consequently, $A_\theta(u) = \lambda_q(M)$ since $\lambda_q(M)$ is the minimum. By a standard variational argument, u satisfies the equation (5.5) in the sense of distributions. Moreover, $u \in L^{2^*}(M)$ by Sobolev embedding in Theorem 4.4. So u is strictly positive and smooth by Corollary 4.14. □

To prove Theorem 1.1, it is enough to prove the following theorem.

Theorem 5.3. *If $\lambda(M) < \lambda(\mathcal{H}^n)$, then there exists a sequence q_j tending to 2^* from below such that u_{q_j} converges in $C^m(M)$ for any m to a function $u \in C^\infty(M)$ such that $u > 0$ and*

$$b_n \triangle_\theta u + s_\theta u = \lambda(M) u^{2^*-1}, \tag{5.6}$$

with $A_\theta(u) = \lambda(M)$ and $B_{\theta,2^*}(u) = 1$.

The following behavior of λ_q can be proved exactly as lemma 6.4 of [17].

Proposition 5.4. *Suppose that θ is normalized, i.e. $\int_M \psi_\theta = 1$. Then*

(1) If $\lambda_q(M) < 0$ for some q, then $\lambda_q(M) < 0$ for all $q \geq 2$. λ_q is a nondecreasing function of q.

(2) If $\lambda_q(M) \geq 0$ for some (hence all) $q \geq 2$, then λ_q is nonincreasing of q and is continuous from left.

Proof of Theorem 5.3. We assume that θ is normalized.

Case 1. $\lambda(M) < 0$. For each $2 \leq q < 2^*$, let u_q be a positive C^∞ solution of equation (5.5) given by Theorem 5.2. For $\phi \in S_1^2(M)$, we have

$$\int_M \left(\langle d_b u_q, d_b\phi\rangle_\theta + s_\theta u_q \phi\right)\psi_\theta = \int_M \lambda_q u_q^{q-1}\phi\psi_\theta. \tag{5.7}$$

Let $\phi = u_q^{q-1}$. Since $\lambda_q(M) < 0$, we have

$$\int_M (q-1)u_q^{q-2}|d_b u_q|_\theta^2 \psi_\theta \leq \int_M |s_\theta u_q^q|\psi_\theta. \tag{5.8}$$

It follows that

$$\int_M |d_b w_q|^2\psi_\theta \leq C\int_M w_q^2\psi_\theta = C\int_M u_q^q\psi_\theta = C \tag{5.9}$$

for $w_q := u_q^{\frac{q}{2}}$ and some constant $C > 0$ independent of $2 \leq q < 2^*$. By Sobolev embedding Theorem 4.4 and $\|w_q\|_{S_1^2(M)}$ uniformly bounded, we get $\int_M w_q^{2^*}\psi_\theta \leq C'$ for some positive constant C' independent of q. Now choose $q_0 > 2$ and set $r = \frac{q_0}{2}2^* > 2^*$. Then for $q \geq q_0$, $\|u_q\|_{L^r(M)}$ is uniformly bounded. It follows from regularity result in Theorem 4.13 that $\|u_q\|_{C^k(M)}$ is uniformly bounded, and there exists a subsequence u_{q_j} converging in $C^k(M)$ to u for any k. The limit u satisfies $b_n\triangle_\theta u + s_\theta u = \lambda u^{2^*-1}$ with $\lambda = A_\theta(u) = \lim \lambda_{q_j}(M)$ and $B_{\theta,2^*}(u) = 1$. We also have $\lambda \leq \lambda(M)$ by proposition 5.4 (1). Consequently, $\lambda = \lambda(M)$ by the definition of $\lambda(M)$.

Case 2. $\lambda(M) \geq 0$.

Case 2i. For some sequence $q_j \longrightarrow 2^*$, $sup_M|d_b u_{q_j}|_\theta$ is uniformly bounded. Note that $u_{q_j} \in L^{q_j}(M) \subset L^1(M)$. By the interpolation inequality (4.22), u_{q_j} are uniformly bounded in $S_1^s(M)$ for any s. In particular, u_{q_j} are in $L^s(M)$ for any s. The result follows as in the case 1.

Case 2ii. $sup_M|d_b u_q|_\theta \longrightarrow \infty$ as $q \longrightarrow 2^*$. Choose $\xi_q \in M$ such that $|d_b u(\xi_q)|_\theta = \sup_M |d_b u_q|_\theta$. Let Θ_{ξ_q} be the normal coordinates constructed in section 3.1. If necessary by passing to a subsequence $\{q_j\}$ (we omit the subscript j), we can assume that there is a fixed neighborhood U of the origin of $\mathcal{H}^n$ contained in the image of Θ_{ξ_q} for all q, and $\xi_q \longrightarrow \xi \in M$. We will identify U with a neighborhood of ξ_q by $(x,t) = \Theta_{\xi_q}(\eta)$ for $\eta \in U$. Define $(\tilde{x},\tilde{t}) := D_{\delta_q^{-1}}(x,t)$. Then $\tilde{\theta}_0 := d\tilde{t} - \sum_{a,b=1}^{2n} B_{ab}\tilde{x}_a d\tilde{x}_b = \delta_q^{-2}(D_{\delta_q}^*\theta_0)$. Define

$$f_q(\tilde{x},\tilde{t}) := \delta_q^{\frac{2}{q-2}} u_q(\delta_q\tilde{x}, \delta_q^2\tilde{t}), \qquad \text{on the open set } U_{(q)} := D_{\delta_q{}^{-1}}U \subset \mathcal{H}^n, \tag{5.10}$$

where $\delta_q > 0$ is so chosen that $|d_b f_q(\mathbf{0})|_{\tilde{\theta}_0} = 1$, where $\mathbf{0}$ is the origin. Note that

$$1 = |d_b f_q(\mathbf{0})|_{\tilde{\theta}_0} = \delta_q^{1+\frac{2}{q-2}} |d_b u_q(\xi_q)|_\theta, \tag{5.11}$$

since $\theta|_{\mathbf{0}} = \theta_0|_{\mathbf{0}} = \widetilde{\theta}_0|_{\mathbf{0}}$ by (5.2). We have $\delta_q \longrightarrow 0$ as $q \longrightarrow 2^*$ and $D_{\delta_q^{-1}} U \longrightarrow \mathcal{H}^n$. Define

$$\theta_{(q)} := \delta_q^{-2} \cdot D_{\delta_q}^* \theta, \qquad h_{(q)} := \delta_q^{-2} \cdot D_{\delta_q}^* h, \qquad J_{(q)} Y := D_{\delta_q *}^{-1} [J(D_{\delta_q *} Y)] \qquad \text{on} \qquad U_{(q)}. \tag{5.12}$$

By (5.2) we see that

$$\theta_{(q)}\big|_{\mathbf{0}} = \theta_0|_{\mathbf{0}}, \qquad h_{(q)}\big|_{\mathbf{0}} = h_0|_{\mathbf{0}}, \qquad J_{(q)}\big|_{\mathbf{0}} = J_0|_{\mathbf{0}}, \tag{5.13}$$

where $\mathbf{0}$ is the origin. It is direct to check that $(U_{(q)}, \theta_{(q)}, h_{(q)}, J_{(q)})$ is a contact Riemannian manifold, i.e. (1.1) holds. For example, its Reeb vector field is $T_{(q)} = \delta_q^2 D_{\delta_q *}^{-1} T$ and

$$d\theta_{(q)}(X, Y) = h_{(q)}(X, J_{(q)} Y) \qquad \text{on} \qquad U_{(q)}.$$

Since the conformal factor in (5.12) is a constant, it is easy to show that the pull back connection $D_{\delta_q}^* \nabla$ defined by

$$\left(D_{\delta_q}^* \nabla \right)_X Y := D_{\delta_q *}^{-1} \left[\nabla_{D_{\delta_q *} X} \left(D_{\delta_q *} Y \right) \right],$$

is the TWT connection for $(U_{(q)}, \theta_{(q)}, h_{(q)}, J_{(q)})$ by the uniqueness of the TWT connection and checking (2.11) for it.

We claim that on the contact Riemannian manifold $(U_{(q)}, \theta_{(q)}, h_{(q)}, J_{(q)})$,

$$\triangle_{\theta_{(q)}} f_q(\tilde{x}, \tilde{t}) = \delta_q^{2+\frac{2}{q-2}} \left(\triangle_\theta u_q \right) \left(D_{\delta_q}(\tilde{x}, \tilde{t}) \right), \qquad \text{for } (\tilde{x}, \tilde{t}) \in U_{(q)}. \tag{5.14}$$

Let $(X_1^q, \cdots, X_{2n}^q)$ be the local orthogonal frame of the horizontal subspace $H|_U$ with norm $\sqrt{2}$, which is used to define Θ_{ξ_q}. We may assume it converging to $(X_1, \cdots, X_{4n})$ as $q \longrightarrow 2^*$. Let

$$Y_a^q := \delta_q D_{\delta_q *}^{-1} X_a^q, \qquad a = 1, \cdots, 2n. \tag{5.15}$$

Then $(Y_1^q, \cdots, Y_{2n}^q)$ is a horizontal orthogonal frame for $(U_{(q)}, \theta_{(q)}, h_{(q)}, J_{(q)})$ with norm $\sqrt{2}$, i.e. we have $h_{(q)}(Y_j^q, Y_k^q) = h(X_j, X_k) = 2\delta_{jk}$, and so

$$\triangle_{\theta_{(q)}} v = \frac{1}{2} \sum_{a=1}^{2n} \left(-Y_a^q Y_a^q v + \sum_{b=1}^{2n} \widetilde{\Gamma}_{bb}{}^a Y_a^q v \right) \tag{5.16}$$

by the expression (2.19) of the SubLaplacian in Proposition 2.1, where $\widetilde{\Gamma}_{cb}{}^a$ is the connection coefficients of $D_{\delta_q}^* \nabla$ with respect to this frame, i.e.

$$\widetilde{\Gamma}_{cb}{}^a Y_a^q = \left(D_{\delta_q}^* \nabla \right)_{Y_b^q} Y_c^q = D_{\delta_q *}^{-1} \left[\nabla_{D_{\delta_q *} Y_b^q} \left(D_{\delta_q *} Y_c^q \right) \right] = \delta_q^2 D_{\delta_q *}^{-1} \left[\nabla_{X_b^q} X_c^q \right].$$

Thus

$$\widetilde{\Gamma}_{cb}{}^a(\tilde{x}, \tilde{t}) = \delta_q \Gamma_{cb}{}^a(D_{\delta_q}(\tilde{x}, \tilde{t})). \tag{5.17}$$

Now applying (5.16) to $v = f_q = \delta_q^{\frac{2}{q-2}} D^*_{\delta_q} u_q$ and using (5.17), we get (5.14). Apply (5.14) to the equation (5.5) satisfied by u_q to get the equation for f_q:

$$b_n \triangle_{\theta_{(q)}} f_q + D^*_{\delta_q} s_\theta \cdot \delta_q^2 f_q = \lambda_q(M) f_q^{q-1} \text{ on } U_{(q)}. \tag{5.18}$$

By using (5.2), we see that the frame $\mathcal{Y}^q := \{Y_1^q, \cdots, Y_{2n}^q\}$ in (5.15) converge in $C^k(B_R)$ to the standard frame $\mathcal{Y} := \{Y_1, \cdots, Y_{2n}\}$ on the Heisenberg group for any fixed $k, R > 0$ as $\delta_q \longrightarrow 0$. Here and in the sequel B_R is the ball in the Heisenberg group, centered at the origin with radius R with respect to the norm (3.2). Obviously $U_{(q)} \supset B_R$ when δ_q sufficiently small. Similarly, $\triangle_{\theta_{(q)}}$ converges uniformly in $C^k(B_R)$ to $\triangle_0$ for each $k, R > 0$ by using (5.16) since the connection coefficients converges uniformly to zero by (5.17). $\theta_{(q)}$ and $h_{(q)}$ converge uniformly to θ_0 and h_0, respectively, in the same way by (5.2).

Noting that

$$|d_b f_q|_{\theta_{(q)}} = \delta_q^{1+\frac{2}{q-2}} |d_b u_q|_\theta , \tag{5.19}$$

we see that $|d_b f_q|_{\theta_{(q)}}$ bounded in B_R since it attains its maximum 1 at the origin by (5.11), and

$$\int_{|(\tilde{x},\tilde{t})|<R} |f_q(\tilde{x},\tilde{t})|^q \psi_{\theta_{(q)}} = \delta_q^{\frac{2q}{q-2}-Q} \int_{|(x,t)|<\delta_q R} |u_q(x,t)|^q \psi_\theta. \tag{5.20}$$

When $q < 2^*$, we have that $\frac{2q}{q-2} - Q > 0$ since $\frac{2q}{q-2}$ is decreasing in q for $q > 2$, and so the right side of (5.20) is uniformly bounded. Moreover, $\psi_{\theta_{(q)}} = (1 + \delta_q O^1)\psi_{\theta_0}$ on B_R by (5.2). We find that $f_q \in L^q(B_R)$ is uniformly bounded. Here and in the sequel the L^q norm is taken with respect to the standard volume on the Heisenberg group. Consequently, $f_q \in L^1(B_R)$ with a uniform bound. This fact together with $|d_b f_q|_{\theta_{(q)}}$ uniformly bounded by 1 implies $f_q \in S_1^s(B_R, \mathcal{Y}^q)$ for each $s < \infty$ by interpolation inequality (4.22). Here $S_k^s(B_R, \mathcal{Y}^q)$ denotes the Folland-Stein space with the norm defined by the frame $\mathcal{Y}^q$. Consequently, $\eta f_q \in L^s(M)$ for each $s \geq 1$ by the Sobolev embedding Theorem 4.4. We see that $\eta f_q \in C^\infty(M)$ by applying Theorem 4.13 to $\triangle_{\theta_{(q)}}$ for any fixed q, but we can not obtain a uniform C^k bound from this theorem directly. We claim that

(5.21)
when q is close to $2^*, \eta f_q$ is uniformly bounded in $C^{4/3}(B_R)$ for any fixed $R > 0$.

Now taking a subsequence $q_j \longrightarrow 2^*$ if necessary, we find a function f on $\mathcal{H}^n$ by first choosing a subsequence f_{q_j} convergent in $C^1(B_1)$; then choosing a subsequence of f_{q_j} convergent in $C^1(B_2)$, etc. Note $f \geq 0$, $f \in C^1(\mathcal{H}^n)$ and f is not zero since $|d_b f(\mathbf{0})|_{\theta_0} = 1$. Since $\theta_{(q_j)} \longrightarrow \theta_0$ and $\lambda_{q_j}(M) \longrightarrow \lambda(M)$ by the continuity of $\lambda_q(M)$ from left in q in Proposition 5.4, by letting $q_j \longrightarrow 2^*$ in (5.18), we get that for $\phi \in C_0^\infty(\mathcal{H}^n)$,

$$\int_{\mathcal{H}^n} \left(b_n \langle d_b f, d_b \phi \rangle_{\theta_0} - \lambda(M) f^{2^*-1} \phi \right) \psi_{\theta_0} = 0. \tag{5.22}$$

Since $\psi_{\theta_{(q_j)}} \longrightarrow \psi_{\theta_0}$ by (5.2) again, (5.20) implies $\int_{B_R} |f|^{2^*} \psi_{\theta_0} \leq 1$ for each $R > 0$. Hence,

$$\int_{\mathcal{H}^n} |f|^{2^*} \psi_{\theta_0} \leq 1. \tag{5.23}$$

On the other hand,

$$\int_{B_R} |d_b f|^2_{\theta_0}\, \psi_{\theta_0} = \lim_{j\longrightarrow\infty} \int_{B_R} \left|d_b f_{q_j}\right|^2_{\theta_{(q_j)}} \psi_{\theta_{(q_j)}} \le \lim_{j\longrightarrow\infty} \|f_{q_j}\|_{S^2_1(B_R,\mathcal{Y}^q)} < \infty \tag{5.24}$$

Thus $f \in S^2_1(\mathcal{H}^n)$.

By taking a subsequence $\phi_l \in C^\infty_0(\mathcal{H}^n)$ to approximate f in (5.22), we find that

$$b_n \int_{\mathcal{H}^n} |d_b f|^2_{\theta_0} \psi_{\theta_0} = \lambda(M) \int_{\mathcal{H}^n} |f|^{2^*} \psi_{\theta_0}. \tag{5.25}$$

Now taking $g = \frac{f}{\|f\|_{2^*}}$, we get

$$b_n \int_{\mathcal{H}^n} |d_b g|^2_{\theta_0} \psi_{\theta_0} = \lambda(M)\|f\|^{2^*-2}_{2^*} \le \lambda(M) < \lambda(\mathcal{H}^n), \qquad \|g\|_{2^*} = 1, \tag{5.26}$$

by (5.23), which contradicts the definition of $\lambda(\mathcal{H}^n)$. Thus case $2ii$ is impossible.

Let us prove the claim (5.21) now. By Darboux theorem, there exists a differential diffeomorphism $\Xi_q : B_R \to \mathcal{H}^n$ for each q such that

$$\Xi^*_q \theta_{(q)} = \theta_0. \tag{5.27}$$

Such diffeomorphisms can be constructed by using Moser's trick as follows (See section 2.5.1 in [15]). Suppose that θ is a contact form close to the standard 1-form θ_0 on $\mathcal{H}^n$ in the $C^k(B_R)$ norm for any fixed $k, R > 0$, and $\theta|_{\mathbf{0}} = \theta_0|_{\mathbf{0}}$, where $\mathbf{0}$ is the origin. Consider a family of contact 1-forms

$$\theta_u = (1-u)\theta_0 + u\theta, \qquad u \in [0,1].$$

We can construct a time dependent vector field X_u such that Ξ_u is the flow of X_u and $\Xi^*_u\theta_u = \theta_0$. The differentiation of $\Xi^*_u\theta_u = \theta_0$ with respect to u implies that

$$\dot{\theta}_u + d(\theta_u(X_u)) + i_{X_u} d\theta_u = 0, \tag{5.28}$$

where $\dot{\theta}_u$ is the derivative of θ_u with respect to u. Write $X_u = H_u R_u + Y_u$, where R_u is the Reeb vector field of θ_u, H_u is a function and $Y_u \in \ker\theta_u$. Then R_u is close to $\frac{\partial}{\partial t}$ if θ closes to θ_0 in the $C^k(B_R)$ norm. Inserting R_u into (5.28) gives

$$\dot{\theta}_u(R_u) + dH_u(R_u) = 0. \tag{5.29}$$

This an ODE of H_u: $R_u(H_u) = -\dot{\theta}_u(R_u)$. Note that the integral curves of the Reeb vector field R_u exist in B_R since R_u is close to $\frac{\partial}{\partial t}$ in the $C^k(B_R)$ norm. So we can integrate this ODE along the integral curves of R_u to get the solution $H_u(x,t)$ with $H_u(x,0) = 0$. We may require $H_u(\mathbf{0}) = 0$ and $dH_u|_{\mathbf{0}} = 0$ since $\dot{\theta}_u|_{\mathbf{0}} = 0$. Once H_u is chosen, Y_u is uniquely determined by (5.28), i.e. by

$$\dot{\theta}_u + dH_u + i_{Y_u} d\theta_u = 0, \tag{5.30}$$

which also have small $C^k(B_R)$ norm. Now we get the vector field X_u with small $C^k(B_R)$ norm and $X_u(\mathbf{0}) = 0$. Then we can integrate it to get diffeomorphisms Ξ_u close to the identity in $C^k(B_R)$ norm with $\Xi_u(\mathbf{0}) = \mathbf{0}$. Then Ξ_1 satisfying $\Xi^*_1\theta = \theta_0$.

Now applying this construction to $\theta_{(q)}$ we get a family of diffeomorphisms Ξ_q satisfying (5.27) such that for any fixed $k, R > 0$, Ξ_q is close to the identity mapping in the $C^k(B_R)$ norm uniformly for q sufficiently close to 2^*. Hence there exists small $\eta_0 >$ such that $B_{(1-\eta_0)R} \subset \Xi_q(B_R) \subset B_{(1+\eta_0)R}$. Thus

(5.31)

the frame $\widetilde{\mathcal{Y}}^q := \left\{\Xi_{q*}^{-1}Y_1^q, \ldots, \Xi_{q*}^{-1}Y_{2n}^q\right\}$ *is close to the standard frame* $\mathcal{Y}$ *on* $\mathcal{H}^n$

in the $C^k(B_R)$ norm uniformly for q. Then by pulling back, $(B_R, \theta_0, \Xi_q^* h_{(q)}, \Xi_q^* J_{(q)})$ is a contact Riemannian structure with $\Xi_q^* h_{(q)}$ and $\Xi_q^* J_{(q)}$ close to h_0 and J_0 uniformly in the $C^k(B_R)$ norms. It is easy to see that $\widetilde{\mathcal{Y}}^q$ is a frame for the horizontal subspace H_0 of the Heisenberg group, and $\Xi_q^* f \in S_1^s(B_R, \mathcal{Y})$ if $f \in S_1^s(B_{(1+\eta_0)R}, \mathcal{Y}^q)$. Note that $\Xi_{q*}^{-1}Y_j^q = Y_j + \sum_{k=1}^{2n} c_{jk}^q Y_k$ for some functions c_{jk}^q with small $C^k(B_R)$ norms uniformly, and the coefficients of TWT connection of $(B_R, \theta_0, \Xi_q^* h_{(q)}, \Xi_q^* J_{(q)})$ is obviously uniformly bounded. By the expression (5.16) of the SubLaplacian, we see that

$$\Xi_q^*(\triangle_{\theta_{(q)}} f_q) = \sum_{i,j=1}^{2n} a_{ij}^q Y_i Y_j \Xi_q^* f_q + \sum_{i,j=1}^{2n} b_j^q Y_j \Xi_q^* f_q \text{ on } B_R$$

for some a_{ij}^q, b_j^q with uniformly bounded $C^k(B_R)$ norm and a_{ij}^q satisfying uniformly elliptic condition (4.13) with some absolute constant $\mu > 0$, for q sufficiently close to 2^*.

Write $\widetilde{f}_q := \Xi_q^* f_q$, which is C^∞. Since $\widetilde{f}_q \in S_1^s(B_R)$ are uniformly bounded for any fixed $s \geq 1$, so are $\widetilde{f}_q^{q-1}$. Now pull back the equation (5.18) by Ξ_q to get

$$\mathcal{L}_q \widetilde{f}_q = \widetilde{g}_q, \qquad \text{where} \quad \mathcal{L}_q := \sum a_{ij}^q Y_i Y_j + \sum b_j^q Y_j, \tag{5.32}$$

with $\widetilde{g}_q := \lambda_q(M)\widetilde{f}_q^{q-1} - \Xi_q^*(D_{\delta_q}^* s_\theta)\delta_q^2 \widetilde{f}_q$. Obviously $\widetilde{g}_q \in S_1^s(B_R, \mathcal{Y})$ uniformly for q close to $2^* > 2$ for any fixed $s \geq 1$. Now applying the uniform estimate in Theorem 4.6 to the operator $\sum a_{ij}^q Y_i Y_j$ for smooth $\widetilde{f}_q$, we get

(5.33)

$$\begin{aligned}\|\widetilde{f}_q\|_{S_3^s(B_{R/2},\mathcal{Y})} &\leq C_1 \left(\left\| \sum a_{ij}^q Y_i Y_j \widetilde{f}_q \right\|_{S_1^s(B_{3R/4},\mathcal{Y})} + \|\widetilde{f}_q\|_{L^s(B_{3R/4})} \right) \\ &\leq C_1 \left(\left\| \mathcal{L}_q \widetilde{f}_q \right\|_{S_1^s(B_{3R/4},\mathcal{Y})} + \left\| \sum b_j^q Y_j \widetilde{f}_q \right\|_{S_1^s(B_{3R/4},\mathcal{Y})} + \left\| \widetilde{f}_q \right\|_{L^s(B_{3R/4})} \right)\end{aligned}$$

and by using the uniform estimate in Theorem 4.6 again,

(5.34)

$$\begin{aligned}\left\| \sum b_j^q Y_j \widetilde{f}_q \right\|_{S_1^s(B_{3R/4},\mathcal{Y})} &\leq C_2 \left\| \widetilde{f}_q \right\|_{S_2^s(B_{3R/4},\mathcal{Y})} \leq C_2 C_1' \left(\left\| \sum a_{ij}^q Y_i Y_j \widetilde{f}_q \right\|_{L^s(B_R,\mathcal{Y})} + \|\widetilde{f}_q\|_{L^s(B_R)} \right) \\ &\leq C_2 C_1' \left(\left\| \mathcal{L}_q \widetilde{f}_q \right\|_{L^s(B_R)} + C_2' \left\| \widetilde{f}_q \right\|_{S_1^s(B_R)} + \left\| \widetilde{f}_q \right\|_{L^s(B_R)} \right)\end{aligned}$$

Here C_1, C_1', C_2 and C_2' are all constants only depending on n, s, R, μ and the $C^3(B_R)$ norms of a_{ij}^q, b_j^q, but not on q. Consequently, we get

$$\left\| \widetilde{f}_q \right\|_{S_3^s(B_{R/2},\mathcal{Y})} \leq C \left(\|\widetilde{g}_q\|_{S_1^s(B_R,\mathcal{Y})} + \left\| \widetilde{f}_q \right\|_{S_1^s(B_R,\mathcal{Y})} \right). \tag{5.35}$$

So $\widetilde{f}_q \in S_3^s(B_{R/2}, \mathcal{Y})$ uniformly bounded for each $s \geq 1$. Then $\widetilde{f}_q \in \Gamma^{3-\epsilon_0}(B_{R/2}, \mathcal{Y})$ uniformly for any fixed $\epsilon_0 > 0$ by Sobolev embedding Theorem 4.4 (2). Consequently, we have $\widetilde{f}_q \in C^{\frac{4}{3}}(B_{R/2})$ uniformly bounded by the embedding $\Gamma_\beta(U) \hookrightarrow \Lambda_{\beta/2}(U)$ on the Heisenberg group by theorem 20.1 of Folland and Stein [12], where $\Lambda_{\beta/2}$ is the usual Lipschitzian space of order $\beta/2$. Hence $f_q \in C^{\frac{4}{3}}(B_{R/3})$. The claim is proved and so is the theorem.

Remark 5.1. We cannot apply the regularity Theorem 4.13 to the equation (5.18) directly to obtain a uniform bound of ηf_q in $C^k(B_R)$ for each k. This is because we have a family of contact Riemannian structures, while Theorem 4.13 can only be applied to a fixed contact Riemannian structure. Note that in general, horizontal subspaces $\ker \theta_{(q)}$ and the Folland-Stein spaces $S_1^s(B_R, \mathcal{Y}^q)$ may be different. This phenomenon does not happen in the Rienannian case. The advantage of the diffeomorphisms Ξ_q in (5.27) is that they transform $\theta_{(q)}$ to the standard one, and so do the horizontal subspaces and the Folland-Stein spaces.

Acknowledgements. Partially supported by National Nature Science Foundation (No. 11571306) in China.

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