A CHARACTERIZATION OF B-CONVEXITY AND J-CONVEXITY OF BANACH SPACES

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This paper is dedicated to Professor Themistocles M. Rassias.

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ABSTRACT. In [K.-I. Mitani and K.-S. Saito, J. Math. Anal. Appl. 327 (2007), 898–907] we characterized the strict convexity, uniform convexity and uniform non-squareness of Banach spaces using ψ-direct sums of two Banach spaces, where ψ is a continuous convex function with some appropriate conditions on [0, 1]. In this paper, we characterize the Bₙ-convexity and Jₙ-convexity of Banach spaces using ψ-direct sums of n Banach spaces, where ψ is a continuous convex function with some appropriate conditions on a certain convex subset of \( \mathbb{R}^n \).

1. INTRODUCTION AND PRELIMINARIES

A norm \( \| \cdot \| \) on \( \mathbb{C}^n \) is said to be absolute if

\[ \|(x_1, x_2, \ldots, x_n)\| = \|(|x_1|, |x_2|, \ldots, |x_n|)\| \]

for all \( x_1, x_2, \ldots, x_n \in \mathbb{C} \), and normalized if

\[ \|(1, 0, \ldots, 0)\| = \|(0, 1, 0, \ldots, 0)\| = \cdots = \|(0, \ldots, 0, 1)\| = 1. \]

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The $\ell_p$-norms $\| \cdot \|_p$ are such examples:

$$
\| (x_1, \ldots, x_n) \|_p = \begin{cases} 
(\sum_{j=1}^n |x_j|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max \{|x_1|, \ldots, |x_n|\} & \text{if } p = \infty.
\end{cases}
$$

Let $AN_n$ be the family of all absolute normalized norms on $\mathbb{C}^n$. The second author, Kato and Takahashi in [9] showed that for every absolute normalized norm $\| \cdot \|$ on $\mathbb{C}^n$, there corresponds a unique continuous convex function on $\Delta_n$ with some appropriate conditions, where

$$
\Delta_n = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : t_j \geq 0 \ (\forall j), \ \sum_{j=1}^n t_j = 1 \right\}.
$$

Indeed, for any $\| \cdot \| \in AN_n$, we define

$$
\psi(t_1, \ldots, t_n) = \| (t_1, \ldots, t_n) \|, \quad (t_1, \ldots, t_n) \in \Delta_n.
$$

Then $\psi$ is a continuous convex function on $\Delta_n$, and satisfies the following conditions:

$$
\begin{align*}
\psi(1, 0, \ldots, 0) &= \psi(0, 1, 0, \ldots, 0) = \cdots = \psi(0, 0, 1, \ldots, 0) = 1 \quad (A_0) \\
\psi(t_1, \ldots, t_n) &\geq (1 - t_1)\psi \left( 0, \frac{t_2}{1 - t_1}, \ldots, \frac{t_n}{1 - t_1} \right), \quad \text{if } t_1 \neq 1 \quad (A_1) \\
\psi(t_1, \ldots, t_n) &\geq (1 - t_2)\psi \left( \frac{t_1}{1 - t_2}, \frac{t_3 - 1}{1 - t_2}, \ldots, \frac{t_n}{1 - t_2} \right), \quad \text{if } t_2 \neq 1 \quad (A_2) \\
&\vdots
\end{align*}
$$

$$
\psi(t_1, \ldots, t_n) \geq (1 - t_n)\psi \left( \frac{t_1}{1 - t_n}, \ldots, \frac{t_{n-1}}{1 - t_n}, 0 \right), \quad \text{if } t_n \neq 1. \quad (A_n)
$$

Let $\Psi_n$ be the set of all continuous convex functions $\psi$ on $\Delta_n$ satisfying $(A_0) - (A_n)$. Conversely, for any $\psi \in \Psi_n$, we define

$$
\| (x_1, \ldots, x_n) \|_\psi = \begin{cases} 
\psi \left( \sum_{j=1}^n |x_j| \right) \psi \left( \sum_{j=1}^n |x_j|, \ldots, \sum_{j=1}^n |x_j| \right), & \text{if } (x_1, \ldots, x_n) \neq (0, \ldots, 0), \\
0, & \text{if } (x_1, \ldots, x_n) = (0, \ldots, 0).
\end{cases}
$$

Then $\| \cdot \|_\psi \in AN_n$ and satisfies (1.1) (cf. [9, Theorem 4.2]). Further, $AN_n$ and $\Psi_n$ are in a one-to-one correspondence. In particular, let $\psi_p$ be the function corresponding to $\ell_p$-norms on $\mathbb{C}^n$. According to Kato, the second author and Tamura in [6, 8], for any Banach spaces $X_1, X_2, \cdots, X_n$ and any $\psi \in \Psi_n$, we define the $\psi$-direct sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\psi$ to be their direct sum equipped with the norm

$$
\| (x_1, \ldots, x_n) \|_\psi := \|(\|x_1\|, \ldots, \|x_n\|)\|_\psi.
$$

Let $B_X = \{ x \in X : \| x \| \leq 1 \}$ be the closed unit ball of a Banach space $X$. A Banach space $X$ is said to be uniformed non-square if there exists a $\delta > 0$ such that $\|(x - y)/2\| > 1 - \delta$, $x, y \in B_X$ implies $\|(x + y)/2\| \leq 1 - \delta$ (cf. [5]). In [7], we characterized the uniform non-squareness of Banach spaces. That is, let
ϕ, ψ ∈ Ψ2. Assume that ϕ ≠ ψ∞ and ψ has a unique minimal point t0 in Δ2. Then a Banach space X is uniformly non-square if and only if

\[ \|A_2 : (X \oplus X)_\psi \rightarrow (X \oplus X)_\varphi\| < \frac{\|(1, 1)\|_\varphi}{\psi(t_0)} \]

holds, where A2 is the Littlewood matrix, that is, \( A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \).

In this paper, we give some characterizations of the B-convexity (or uniformly non-ℓ1-ness) and the J-convexity (which is related to superreflexivity) for Banach spaces using ψ-direct sums \( (X \oplus \cdots \oplus X)_\psi \). We note that these characterizations are an extension of some results in Takahashi and Kato [11].

We need some preliminaries. Recall the following properties concerning absolute norms on \( C^n \) (see [9]).

**Lemma 1.1** ([9]).

(i) Let \( \psi \in \Psi_n \). If \( |x_i| \leq |y_i| \) for all \( i \), then

\[ \|(x_1, x_2, \ldots, x_n)\|_\psi \leq \|(y_1, y_2, \ldots, y_n)\|_\psi. \]

(ii) Let \( \varphi, \psi \in \Psi_n \) with \( \varphi \leq \psi \). Put \( M = \max_{t \in \Delta_n} \psi(t)/\varphi(t) \). Then we have \( \| \cdot \|_\varphi \leq \| \cdot \|_\psi \leq M \| \cdot \|_\varphi \).

Let \( X \) be a Banach space. For any \( \psi \in \Psi_n \), we define the space \( \ell^n_\psi(X) \) by

\[ \ell^n_\psi(X) = (X \oplus \cdots \oplus X)_\psi. \]

In particular, for the case \( \psi = \psi_p \) (1 ≤ p ≤ ∞), let the space \( \ell^n_\psi(X) \) be \( \ell^n_p(X) = \ell^n_{\psi_p}(X) \).

## 2. B-CONVEXITY

The notion of B-convexity was introduced by Beck [2] in order to obtain a strong law of large numbers for certain vector-valued random variables. A Banach space \( X \) is called \( B_n \)-convex (or uniformly non-ℓ1) if there is a real number \( \delta > 0 \) such that for any \( x_1, \ldots, x_n \in B_X \)

\[ \min_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\| \leq n(1 - \delta) \]

holds. If \( X \) is \( B_n \)-convex for some \( n \), then \( X \) is called B-convex. For the fundamental properties of \( B \)-convexity, we refer to [11, 12, 13, 10, 11].

We consider the Rademacher matrices \( R_\mathbb{Z} \); that is,

\[ R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \begin{pmatrix} \vdots & R_n \\ 1 & \vdots \\ -1 \end{pmatrix} \quad (n = 1, 2, \cdots). \]
Proposition 2.1. Let $\varphi \in \Psi_{2^n}$ and $\psi \in \Psi_n$ where $n \geq 2$. Then for any Banach space $X$

$$
\|R_n : \ell^n(X) \to \ell_{\varphi}^{2^n}(X)\| \leq \frac{\|(1, \cdots, 1)\|_{\varphi}}{\min_{t \in \Delta_n} \psi(t)}.
$$

(2.1)

Proof. Put $R_n = (a_{ij})$. From Lemma 1.1 we have for any $x_1, \cdots, x_n \in X$,

$$
\left\| \left\{ \sum_{j=1}^{n} a_{ij} x_j \right\}^{2^n}_{i=1} \right\|_{\varphi} = \left\{ \left\| \sum_{j=1}^{n} a_{ij} x_j \right\|_{i=1}^{2^n} \right\}.
$$

$$
\leq \left\{ \sum_{j=1}^{n} \left\| x_j \right\|_{i=1}^{2^n} \right\}
$$

$$
= \left( \sum_{j=1}^{n} \left\| x_j \right\| \right) \|(1, \cdots, 1)\|_{\varphi}
$$

$$
= \|(x_1, \cdots, x_n)\|_1 \|(1, \cdots, 1)\|_{\varphi}
$$

$$
\leq \max_{t \in \Delta_n} \frac{\psi(t)}{\psi(t)} \|(x_1, \cdots, x_n)\|_1 \psi(1, \cdots, 1)\|_{\varphi}
$$

$$
= \frac{\|(1, \cdots, 1)\|_{\varphi}}{\min_{t \in \Delta_n} \psi(t)} \|(x_1, \cdots, x_n)\|_1 \psi,
$$

which implies (2.1). \hfill \square

The main theorem is the following.

Theorem 2.2. Let $\varphi \in \Psi_{2^n}$ and $\psi \in \Psi_n$ where $n \geq 2$. Assume that $\psi$ has a unique minimal point $t_0 = \{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all $j$ and $\varphi$ has the condition:

$$
\|(i, \cdots, 1, 0, 1, \cdots, 1)\|_{\varphi} < \|(1, \cdots, 1)\|_{\varphi}
$$

(2.2)

for every $i$. Then a Banach space $X$ is $B_n$-convex if and only if

$$
\|R_n : \ell^n(X) \to \ell_{\varphi}^{2^n}(X)\| < \frac{\|(1, \cdots, 1)\|_{\varphi}}{\psi(t_0)}
$$

holds.

We reformulate Theorem 2.2 as follows.

Theorem 2.3. Let $\varphi \in \Psi_{2^n}$ and $\psi \in \Psi_n$ where $n \geq 2$. Assume that $\psi$ has a unique minimal point $t_0 = \{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all $j$ and $\varphi$ has the condition:

$$
\|(i, \cdots, 1, 0, 1, \cdots, 1)\|_{\varphi} < \|(1, \cdots, 1)\|_{\varphi}
$$

(2.3)

for every $i$. Then for a Banach space $X$ the following are equivalent:

(i) $X$ is $B_n$-convex.
(ii) There exists a real number $\delta$ ($0 < \delta < 1$) such that for all $x_1, \cdots, x_n \in X$,

$$\left\| \left\{ \sum_{j=1}^{n} a_{ij}t_j x_j \right\}_{i=1}^{2^n} \right\| \leq (1 - \delta) \left\| (1, \cdots, 1) \right\| \phi \left\{ t_j x_j \right\}_{j=1}^{n} \psi \left( t_0 \right)$$

holds, where $R_n = (a_{ij})$.

We shall show Theorem 2.3. To do it, we need the following lemma about norm convexity.

**Lemma 2.4.** Let $X$ be a Banach space. Then the following are equivalent:

(i) There exists a real number $\delta$ ($0 < \delta < 1$) such that for all $x_1, \cdots, x_n \in B_X$,

$$\min_{1 \leq i \leq 2^n} \left\| \sum_{j=1}^{n} a_{ij} x_j \right\| \leq n(1 - \delta).$$

(ii) For any (resp. some) $\{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all $j = 1, \cdots, n$, there exists a real number $\delta_0$ ($0 < \delta_0 < 1$) such that for all $x_1, \cdots, x_n \in B_X$,

$$\min_{1 \leq i \leq 2^n} \left\| \sum_{j=1}^{n} a_{ij} t_j x_j \right\| \leq 1 - \delta_0.$$

**Proof.** (i)⇒(ii): Assume that the assertion (i) holds. We take $\{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all $j$, and fix $x_1, \cdots, x_n \in B_X$. Then there exists a number $i_0$ satisfying

$$\left\| \sum_{j=1}^{n} a_{i_0 j} x_j \right\| \leq n(1 - \delta).$$

We choose a number $k$ such that $t_k = \min\{t_1, \cdots, t_n\}$. Note that $t_k > 0$. Then we have

$$\left\| \sum_{j=1}^{n} a_{i_0 j} t_j x_j \right\| = \left\| \sum_{j=1}^{n} a_{i_0 j} t_k x_j + \sum_{j=1}^{n} a_{i_0 j} (t_j - t_k) x_j \right\|$$

$$\leq t_k \left\| \sum_{j=1}^{n} a_{i_0 j} x_j \right\| + \sum_{j=1}^{n} (t_j - t_k)$$

$$\leq nt_k (1 - \delta) + 1 - nt_k = 1 - nt_k \delta.$$

Put $\delta_0 = nt_k \delta$. Then the assertion (ii) holds.

(ii)⇒(i): Assume that the assertion (ii) holds. We take $\{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all $j$, and fix $x_1, \cdots, x_n \in B_X$. Then there exists a number $i_0$ such that

$$\left\| \sum_{j=1}^{n} a_{i_0 j} t_j x_j \right\| \leq 1 - \delta_0.$$
We choose a number \( \ell \) such that \( t_\ell = \max\{t_1, \cdots, t_n\} \). Then
\[
\left\| \sum_{j=1}^n a_{i_0j} x_j \right\| = \left\| \sum_{j=1}^n a_{i_0j} \frac{t_j}{t_\ell} x_j + \sum_{j=1}^n \left( 1 - \frac{t_j}{t_\ell} \right) a_{i_0j} x_j \right\|
\leq \frac{1}{t_\ell} \left\| \sum_{j=1}^n a_{i_0j} t_j x_j \right\| + \sum_{j=1}^n \left( 1 - \frac{t_j}{t_\ell} \right)
\leq \frac{1}{t_\ell} (1 - \delta_0) + n - \frac{1}{t_\ell} = n \left( 1 - \frac{1}{nt_\ell} \delta_0 \right).
\]

Put \( \delta = \frac{1}{nt_\ell} \delta_0 \). Then the assertion (i) holds. This completes the proof.

\(\square\)

**Proof of Theorem 2.3.** We put
\[
K = \frac{\| (1, \cdots, 1) \|_\varphi}{\psi(t_0)}.
\]

(i)\(\Rightarrow\)(ii): Assume that the assertion (ii) fails to hold. Then, for each positive number \( \ell \), there exist \( x_{\ell 1}, \cdots, x_{\ell n} \in X \) such that
\[
K \left( 1 - \frac{1}{t_\ell} \right) \left\| \{ t_j x_{\ell j} \} \right\|_\psi < \left\| \left\{ \sum_{j=1}^n a_{ij} t_j x_{\ell j} \right\} \right\|_\varphi.
\]

(2.4)

Since \( (x_{\ell 1}, \cdots, x_{\ell n}) \neq (0, \cdots, 0) \) for all \( \ell \), we may assume
\[
\max \{ \| x_{\ell 1} \|, \cdots, \| x_{\ell n} \| \} = 1
\]
for all \( \ell \). So we can take sequences \( \{ \ell(k) \}_{k=1}^\infty \), \( \{ \alpha_j \}_{j=1}^n \) and \( \{ \beta_i \}_{i=1}^2 \) such that for all \( i, j \),
\[
\| x_{\ell(k)j} \| \to \alpha_j \ (k \to \infty)
\]
and
\[
\left\| \sum_{j=1}^n a_{ij} t_j x_{\ell(k)j} \right\| \to \beta_i \ (k \to \infty).
\]

(2.5)

Note that \( 0 \leq \alpha_j \leq 1 \) for all \( j \), and \( 0 \leq \beta_i \leq 1 \) for all \( i \). By (2.4) and Lemma 1.1 we have
\[
K \left( 1 - \frac{1}{t_{\ell(k)}} \right) \left\| \{ t_j x_{\ell(k)j} \} \right\|_\psi < \left\| \left\{ \sum_{j=1}^n a_{ij} t_j x_{\ell(k)j} \right\} \right\|_\varphi
\leq \left\| \left\{ \sum_{j=1}^n t_j x_{\ell(k)j} \right\} \right\|_\varphi
= \sum_{j=1}^n t_j x_{\ell(k)j} \| (1, \cdots, 1) \|_\varphi.
\]

(2.6)

As \( k \to \infty \), we obtain
\[
K \| \{ t_j \alpha_j \} \|_\psi \leq \sum_{j=1}^n t_j \alpha_j \| (1, \cdots, 1) \|_\varphi.
\]
and so
\[ \psi \left( \frac{t_1 \alpha_1}{\sum_{j=1}^{n} t_j \alpha_j}, \ldots, \frac{t_n \alpha_n}{\sum_{j=1}^{n} t_j \alpha_j} \right) \leq \psi(t_0). \]
Since \( \psi(t) > \psi(t_0) \) for all \( t \in \Delta_n \) with \( t \neq t_0 \) by the assumption, we have
\[ t_0 = (t_1, \ldots, t_n) = \left( \frac{t_1 \alpha_1}{\sum_{j=1}^{n} t_j \alpha_j}, \ldots, \frac{t_n \alpha_n}{\sum_{j=1}^{n} t_j \alpha_j} \right), \]
which implies
\[ \alpha_1 = \alpha_2 = \cdots = \alpha_n = \sum_{j=1}^{n} t_j \alpha_j, \]
by \( t_j > 0 \) for all \( j \). Since \( \|x_{\ell(k)j}\| \to \alpha_j \) for all \( j \) and
\[ \max\{\|x_{\ell(k)1}\|, \ldots, \|x_{\ell(k)n}\|\} = 1, \]
we have \( \max\{\alpha_1, \ldots, \alpha_n\} = 1 \), and so \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = 1 \). We also have by (2.6),
\[ \left\| \left\{ \sum_{j=1}^{n} a_{ij} t_j x_{\ell(k)j} \right\}_{i=1}^{2^n} \right\| \to \|(1, \cdots, 1)\|_\varphi. \]
Hence we have by (2.5),
\[ \|(\beta_1, \beta_2, \cdots, \beta_{2^n})\|_\varphi = \|(1, 1, \cdots, 1)\|_\varphi. \]
If \( \beta_1 < 1 \), then we have by the assumption
\begin{align*}
\|(\beta_1, \beta_2, \cdots, \beta_n)\|_\varphi &\leq \|(\beta_1, 1, 1, \cdots, 1)\|_\varphi \\
&\leq (1 - \beta_1)\|(0, 1, 1, \cdots, 1)\|_\varphi + \beta_1\|(1, 1, \cdots, 1)\|_\varphi \\
&\leq (1 - \beta_1)\|(1, 1, \cdots, 1)\|_\varphi + \beta_1\|(1, 1, \cdots, 1)\|_\varphi \\
&= \|(1, 1, \cdots, 1)\|_\varphi,
\end{align*}
which is a contradiction. Hence \( \beta_1 = 1 \). We similarly have \( \beta_2 = \beta_3 = \cdots = \beta_n = 1 \). Namely, we obtain
\[ \left\| \sum_{j=1}^{n} a_{ij} t_j x_{\ell(k)j} \right\| \to 1 \quad (k \to \infty) \]
for all \( i \). Therefore it follows from Lemma 2.4 that (i) fails to hold.

(ii)\(\Rightarrow\)(i): Assume that the assertion (ii) holds. For any \( x_1, \cdots, x_n \in B_X \), we have
\begin{align*}
\min_{1 \leq i \leq 2^n} \left\| \sum_{j=1}^{n} a_{ij} t_j x_j \right\| \|(1, \cdots, 1)\|_\varphi &\leq \left\| \left\{ \left\| \sum_{j=1}^{n} a_{ij} t_j x_j \right\|_{i=1}^{2^n} \right\}_{i=1}^{2^n} \right\|_\varphi \\
&\leq K(1 - \delta)\|\{t_j \|x_j\|\}_{j=1}^{n}\|_\psi \\
&\leq K(1 - \delta)\|\{t_j\}_{j=1}^{n}\|_\psi \\
&= \|(1, \cdots, 1)\|_\varphi(1 - \delta)
\end{align*}
and so
\[ \min_{1 \leq i \leq 2^n} \left\| \sum_{j=1}^{n} a_{ij} t_j x_j \right\| \leq 1 - \delta. \]

Thus it follows from Lemma 2.4 that the assertion (i) holds. This completes the proof. \( \square \)

In Theorem 2.2, we suppose that \( \varphi \) is strictly convex on \( \Delta_n \). Then \( \varphi \) satisfies (2.2) for every \( i \). Therefore we have

Corollary 2.5. Let \( \varphi \in \Psi_{2^n} \) and \( \psi \in \Psi_n \) where \( n \geq 2 \). Assume that \( \psi \) has a unique minimal point \( t_0 = \{ t_j \}_{j=1}^{n} \in \Delta_n \) with \( t_j > 0 \) for all \( j \) and \( \varphi \) is strictly convex on \( \Delta_n \). Then a Banach space \( X \) is \( B_n \)-convex if and only if

\[ \| R_n : \ell_{\psi}^{n}(X) \rightarrow \ell_{\varphi}^{2^n}(X) \| < \frac{\| (1, \cdots, 1) \|_{\varphi}}{\psi(t_0)} \]

holds.

We remark that Theorem 2.2 is an extension of the following result in Takahashi and Kato [11].

Corollary 2.6 ([11]). Let \( 1 < r \leq \infty \) and \( 1 \leq s < \infty \). Then a Banach space \( X \) is \( B_n \)-convex if and only if \( \| R_n : \ell_{\psi}^{r}(X) \rightarrow \ell_{\varphi}^{2^n}(X) \| < 2^{n/s} n^{1/r'} \) holds, where \( 1/r + 1/r' = 1 \).

Proof. Note that \( \psi_r(t_1, \cdots, t_n) > \psi_r(\frac{1}{n}, \cdots, \frac{1}{n}) \) for all \( t = (t_1, \cdots, t_n) \in \Delta_n \) with \( t \neq (\frac{1}{n}, \cdots, \frac{1}{n}) \), and \( \psi_s \) satisfies (2.2), for all \( i \). Therefore we can apply Theorem 2.2 to \( \psi = \psi_r \) and \( \varphi = \psi_s \), and

\[ \frac{\| (1, \cdots, 1) \|_{\psi_s}}{\psi_r(\frac{1}{n}, \cdots, \frac{1}{n})} = \frac{(1^s + \cdots + 1^s)^{1/s}}{((\frac{1}{n})^r + \cdots + (\frac{1}{n})^r)^{1/r}} = 2^{n/s} n^{1/r'}. \]

Further we consider Theorem 2.2 for the case \( n = 2 \).

Corollary 2.7 ([7]). Let \( \psi \in \Psi_2 \) and \( \varphi \in \Psi_2 \). Assume that \( \psi \) has a unique minimal point \( t_0 \in \Delta_2 \) and \( \varphi \neq \psi_\infty \). Then a Banach space \( X \) is uniformly non-square if and only if

\[ \| A_2 : \ell_{\psi}^2(X) \rightarrow \ell_{\varphi}^2(X) \| < \frac{\| (1, 1) \|_{\varphi}}{\psi(t_0)} \]

holds.

Proof. If \( \varphi \in \Psi_2 \) satisfies \( \varphi \neq \psi_\infty \), then by the convexity of \( \varphi \), we have \( \varphi(\frac{1}{2}, \frac{1}{2}) > \frac{1}{2} \), which implies \( \| (0, 1) \|_{\varphi} < \| (1, 1) \|_{\varphi} \) and \( \| (1, 0) \|_{\varphi} < \| (1, 1) \|_{\varphi} \). Also, if \( \psi \in \Psi_2 \) has a unique minimal point \( t_0 = (t_1, t_2) \in \Delta_2 \), then it is obvious that \( t_1 > 0 \) and \( t_2 > 0 \). Thus we obtain this corollary. \( \square \)
3. J-convexity

A finite sequence of signs $\varepsilon_1, \cdots, \varepsilon_n$ will be called admissible if all $+$ signs are before all $-$ signs. A Banach space $X$ is called $J_n$-convex if there exists some $\delta > 0$ such that for every $x_1, x_2, \cdots, x_n \in B_X$, there is an admissible choice of signs $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$\left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \leq n(1 - \delta)$$

holds. If $X$ is $J_n$-convex for some $n$, then $X$ is called $J$-convex. Note that $X$ is uniformly non-square if and only if it is $J_2$-convex. It is well-known that $X$ is $J$-convex if and only if it is super-reflexive (see [1]). For the fundamental properties of $J$-convexity, we refer to [1], [5], [11] and so on.

The $n \times n$ matrices $A_n$ (called admissible matrices) are defined by

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 & \cdots & -1 \end{pmatrix}$$

$(n = 2, 3, \cdots)$.

As in Theorem 2.2 we can characterize $J_n$-convexity of Banach spaces using $\psi$-direct sums.

**Theorem 3.1.** Let $\psi \in \Psi_n$ and $\varphi \in \Psi_n$ where $n \geq 2$. Assume that $\psi$ has a unique minimal point $t_0 = \{t_j\} \in \Delta_n$ with $t_j > 0$ for all $j$ and $\varphi$ has the condition:

$$\| (1, \cdots, 1, (i), 1, 0, 1, \cdots, 1) \|_\varphi < \| (1, \cdots, 1) \|_\varphi$$

for every $i$. Then a Banach space $X$ is $J_n$-convex if and only if

$$\| A_n : \ell^n_\psi (X) \to \ell^n_\varphi (X) \| < \frac{\| (1, \cdots, 1) \|_\varphi}{\psi(t_0)}$$

holds.

In particular, we have

**Corollary 3.2 ([11]).** Let $1 < r \leq \infty$ and $1 \leq s < \infty$. Then a Banach space $X$ is $J_n$-convex if and only if $\| A_n : \ell^n_r (X) \to \ell^n_s (X) \| < n^{1/s+1/r'}$ holds, where $1/r + 1/r' = 1$.

**References**


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