A STUDY ON SOME NEW TYPES OF HARDY–HILBERT’S INTEGRAL INEQUALITIES

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Abstract. Some new kinds of Hardy–Hilbert’s integral inequality in which the weight function is homogeneous function are given. Other results are also obtained.

1. Introduction

Let \( f, g \geq 0 \) satisfy
\[
0 < \int_0^\infty f^2(t)dt < \infty \quad \text{and} \quad 0 < \int_0^\infty g^2(t)dt < \infty,
\]
then
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left( \int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{1/2}, \tag{1.1}
\]
where the constant factor \( \pi \) is the best possible (cf. Hardy et al. [2]). Inequality (1.1) is well known as Hilbert’s integral inequality. This inequality had been extended by Hardy [1] as follows: If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0 \) satisfy
\[
0 < \int_0^\infty f^p(t)dt < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(t)dt < \infty,
\]

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then
\[ \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(t) \, dt \right)^{1/p} \left( \int_0^\infty g^q(t) \, dt \right)^{1/q}, \quad (1.2) \]
where the constant factor \( \frac{\pi}{\sin(\pi/p)} \) is the best possible. Inequality (1.2) is called Hardy–Hilbert’s integral inequality and is important in analysis and applications (cf. Mitrinovic et al. [3]).

B. Yang gave the following extension of (1.2) as follows:

**Theorem [4].** If \( \lambda > 2 - \min\{p, q\} \) and \( f, g \geq 0 \) satisfy
\[ 0 < \int_0^\infty t^{1-\lambda} f^p(t) \, dt < \infty \quad \text{and} \quad 0 < \int_0^\infty t^{1-\lambda} g^q(t) \, dt < \infty, \]
then
\[ \int_0^\infty \int_0^\infty f(x)g(y) \frac{1}{(x+y)^\lambda} \, dx \, dy < k_\lambda(p) \left( \int_0^\infty t^{1-\lambda} f^p(t) \, dt \right)^{1/p} \left( \int_0^\infty t^{1-\lambda} g^q(t) \, dt \right)^{1/q}, \]
where the constant factor \( k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \) is the best possible \( B \) is the beta function. The function \( f(x, y) \) is said to be homogeneous of degree \( \lambda \), if
\[ f(tx, ty) = t^\lambda f(x, y) \quad (t > 0). \]

The object of this paper is that to give some new inequalities similar to that of Hardy–Hilbert’s integral inequality.

2. **Main result**

**Lemma 2.1.** Let \( K(t, 1), K(1, t) \) be positive increasing functions, \( 0 < \mu + 1 \leq \alpha \). Set for \( s \geq 1, \)
\[ f(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{K(1, t)} \, dt, \quad g(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{K(t, 1)} \, dt. \]

Then
\[ f(s) \leq f(1), g(s) \leq g(1). \]

**Proof.** We have
\[ f'(s) = s^{-\alpha} \frac{s^\mu}{K(1, s)} - \alpha s^{-\alpha-1} \int_0^s \frac{t^\mu}{K(1, t)} \, dt \]
\[ \leq \frac{s^{\mu-\alpha}}{K(1, s)} - \alpha s^{-\alpha-1} \int_0^s t^\mu \, dt \]
\[ = \frac{s^{\mu-\alpha}}{K(1, s)} \left( 1 - \frac{\alpha}{\mu + 1} \right) \leq 0. \]

This shows that \( f \) is nonincreasing and hence \( f(s) \leq f(1) \). The other part has a similar proof. \( \square \)

The following is our main result
Theorem 2.2. Let $f, g \geq 0$, $K(u, v)$ be positive, increasing, homogeneous function of degree $\lambda$, $0 < \lambda \leq \min\{(1 - b)/p, (1 - a)p/q\}$, $a, b > 0$, $p > 1$, $1/p + 1/q = 1$. Set

$$F(u) = \int_0^u f(t)dt, \quad G(v) = \int_0^v g(t)dt.$$  

Then

$$\int_0^T \int_0^T \frac{F(u)G(v)}{K(u, v)}du dv \leq T^{\alpha} \sqrt[p]{pK_1} \sqrt[q]{qK_2} \left( \int_0^T (T - t)F^{p-1}(t)f(t)dt \right)^{1/p} \times \left( \int_0^T (T - t)G^{q-1}(t)g(t)dt \right)^{1/q},$$

where

$$K_1 = \int_0^1 \frac{t^{a-1}}{K(1, t)}dt, \quad K_2 = \int_0^1 \frac{t^{b-1}}{K(t, 1)}dt.$$  

Proof.

$$\int_0^T \int_0^T \frac{F(u)G(v)}{K(u, v)}du dv = \int_0^T \int_0^T \frac{F(u)v^{a-1}}{u^{b-1}K^{1/p}(u,v)} \times \frac{G(v)u^{b-1}}{v^{a-1}/K^{1/q}(u,v)} du dv$$

$$\leq \left( \int_0^T \int_0^T \frac{F^p(u)v^{a-1}}{u(t-1)p/K(u,v)} du dv \right)^{1/p} \times \left( \int_0^T \int_0^T \frac{G^q(v)u^{b-1}}{v^{(a-1)/q}/K(u,v)} du dv \right)^{1/q}$$

$$= M^{1/p} N^{1/q}.$$ 

We first consider

$$M = \int_0^T u^{(1-b)p/q} F^p(u) du \int_0^T \frac{u^{a-1}}{K(u, v)} dv.$$ 

Observe that on putting $v = uy, dv = udv, 0 \leq y \leq t/u$, we have, in view of Lemma 2.1, by writing $\alpha = a + (1 - b)p/q - \lambda$,

$$\int_0^T \frac{u^{a-1}}{K(u, v)} dv = \int_0^{T/u}(uy)^{a-1}u \frac{dy}{K(u, uy)} = u^{a-\lambda} \int_0^{T/u} \frac{y^{a-1}}{K(1, y)} dy$$

$$= u^{a-\lambda} \left( \frac{y}{u} \right)^{a-\lambda} \frac{1}{K(1, y)} \int_0^{T/u} y^{a-1} dy$$

$$\leq T^{\alpha} u^{a-\lambda-\alpha} \int_0^1 \frac{y^{a-1}}{K(1, y)} dy = T^{\alpha} K_1 u^{a-\lambda-\alpha}.$$ 

By above we obtain

$$M \leq T^{\alpha} K_1 \int_0^T u^{a+(1-b)p/q-\lambda-\alpha} F^p(u) du$$

$$= T^{\alpha} K_1 \int_0^T F^p(u) du.$$ 

As

$$F^p(u) = \int_0^u (F^p(t))' dt = p \int_0^u F^{p-1}(t)f(t)dt,$$
we have
\[ M \leq pT^\alpha K_1 \int_0^T \int_0^u F^{p-1}(t)f(t)dtdu = pT^\alpha K_1 \int_0^T (T-t)F^{p-1}(t)f(t)dt. \]

Similarly, the other part follows by using Lemma 2.1, replacing \( \alpha \) by \( \beta \), where
\[ \beta = b + (1-a)q/p - \lambda \]
to obtain
\[ N \leq qT^\alpha K_2 \int_0^T (T-t)G^{q-1}(t)g(t)dt. \]

This completes the proof of the theorem. \( \square \)

3. Applications

Corollary 3.1. By an application of Theorem 2.2 for the special case \( a = b = \lambda/2 \), we have
\[
\int_0^T \int_0^T \frac{F(u)G(v)}{K(u,v)}dudv \leq T^\alpha \sqrt[1/p]{pK_1 \sqrt[K_3]{qK_3}} \left( \int_0^T (T-t)F^{p-1}(t)f(t)dt \right)^{1/p} \times \\
\left( \int_0^T (T-t)G^{q-1}(t)g(t)dt \right)^{1/q},
\]
where
\[ K_3 = \int_1^\infty \frac{t^{1/2-1}}{K(1,t)}dt, \]

Furthermore, when \( K(u,v) = (u+v)^\lambda \), we have
\[
\int_0^T \int_0^T \frac{F(u)G(v)}{(u+v)^\lambda}dudv \leq T^\alpha B\left( \frac{\lambda}{2}, \frac{\lambda}{2} \right) \sqrt[p]{p} \sqrt[q]{q} \left( \int_0^T (T-t)F^{p-1}(t)f(t)dt \right)^{1/p} \times \\
\left( \int_0^T (T-t)G^{q-1}(t)g(t)dt \right)^{1/q}.
\]

Proof. For \( a = b = \lambda/2 \), we have \( K_2 = K_3 \) as
\[
\int_0^1 \frac{t^{1/2-1}}{K(t,1)} = \int_0^1 \frac{t^{1/2-1}}{K(t,tt^{-1})}dt = \int_0^1 \frac{t^{-1/2-1}}{K(1,t^{-1})}dt = \int_1^\infty \frac{t^{1/2-1}}{K(1,t)}dt.
\]
The other part follows from the fact that for \( K(1,t) = (1+t)^\lambda \),
\[ K_1 = K_2 = K_3 = B\left( \frac{\lambda}{2}, \frac{\lambda}{2} \right). \]

\( \square \)
Corollary 3.2. By an application of Theorem 2.2 with \( K(u, v) = u^\lambda + v^\lambda \), we have

\[
\int_0^T \int_0^T \frac{F(u)G(v)}{u^\lambda + v^\lambda} \, du \, dv 
\leq T^\alpha qK_a q\sqrt{qK_b} \left( \int_0^T (T - t)F^{p-1}(t)f(t)dt \right)^{1/p} \times \left( \int_0^T (T - t)G^{q-1}(t)g(t)dt \right)^{1/q},
\]

where

\[
K_r = \int_0^1 \frac{t^{r-1}}{1 + t^\lambda} \, dt \quad (r \in \{a, b\}).
\]

References


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