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Generalization of Integral Inequalities for Functions whose Modulus of n^{th} Derivatives are Convex [1]

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

The aim of the present paper is to establish some new Ostrowski Grüss-Čebyšev type inequalities involving functions whose modulus of nth derivatives are convex. Our results are generalization of existing results in literature. Remarks given are important.

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1 Introduction

In 1938, A. M. Ostrowski [7] proved the following classical inequality:

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose first derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) i.e., $|f'(x)| \leq M < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)M, \tag{1.1}$$

for all $x \in [a,b]$, where M is a constant.

For two absolutely continuous functions $f, g : [a,b] \to \mathbb{R}$, consider the functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx \right) \left(\frac{1}{b-a}\int_a^b g(x)dx \right), \tag{1.2}$$

provided, the involved integrals exist.

In 1882, P. L. Čebyšev [11] proved that, if $f', g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{1.3}$$

In 1934, G. Grüss [11] showed that

$$T(f,g) \leq \frac{1}{4}(M-m)(N-n), \tag{1.4}$$

provided m, M, n and N are real numbers satisfying the condition $-\infty < m \leq f(x) \leq M < \infty$, $-\infty < n \leq g(x) \leq N < \infty$, for all $x \in [a,b]$.

During the past few years, many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see $[1-9]$, and the references cited therein. Motivated by the recent results given in $[1-3,\ 11]$ Ostrowski, Grüss, Čebyšev and Pachpatte, involving functions whose derivatives are bounded and whose modulus of derivatives are convex. The analysis used in the proofs is elementary and based on the use of integral identities proved in $[1-2]$.

2 Statement of Results

Let I be a suitable interval of the real line $\mathbb{R}$. A function $f: I \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$ (see [12]).

The following identities are proved in $[1-2]$ respectively

$$\frac{f(x)-f(t)}{x-t} = \frac{1}{x-t}\int_x^t f'(u)du = \int_0^1 f'\left((1-\lambda)x + \lambda t\right) d\lambda, \tag{2.1}$$

showing

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'\left((1-\lambda)x+\lambda t\right) d\lambda\right) dt, \tag{2.2}$$

and

$$\begin{aligned} \int_a^x (u-a)f'(u)du &= (x-a)^2\int_0^1 \lambda f'\left((1-\lambda)a+\lambda x\right) dt, \\ \int_x^b (u-b)f'(u)du &= (x-b)^2\int_0^1 \lambda f'\ \left((1-\lambda)b+\lambda x\right) dt, \end{aligned}$$

showing that

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{(x-a)^2}{b-a}\int_0^1 \lambda f'((1-\lambda)a+\lambda x)\,d\lambda$$

$$-\frac{(b-x)^2}{b-a}\int_0^1 \lambda f'(\lambda x+(1-\lambda)b)\,d\lambda, 2.3 \tag{1}$$

for all $x \in [a,b]$, where $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ and $\lambda \in [0,1]$.

We prove the following Lemmas.

Let f be absolutely continuous, then for any $x \in [a,b]$,

$$\frac{(x-a)^{n+1}}{n!}I_1 - \frac{(-1)^{n+1}(b-x)^{n+1}}{n!}I_2$$

$$= \int_a^x \frac{(u-a)^n}{n!}f^{(n)}(u)du - \int_x^b \frac{(u-b)^n}{n!}f^{(n)}(u)du, 2.4 \tag{2}$$

where

$$I_1 = \int_0^1 \lambda^n f^{(n)}((1-\lambda)a+\lambda x)\,d\lambda \text{ and } I_2 = \int_0^1 \lambda^n f^{(n)}((1-\lambda)b+\lambda x)\,d\lambda.$$

Proof. Consider

$$I_1 = \int_0^1 \lambda^n f^{(n)}((1-\lambda)a+\lambda x)\,d\lambda. \tag{2.5}$$

Let $u = (1-\lambda)a + \lambda x$. This gives $\frac{u-a}{x-a} = \lambda$, $\lambda \to 1$, $u \to x$, $\lambda \to 0, u \to a$ and $d\lambda = \frac{du}{x-a}$.

From (2.5) we have

$$\frac{(x-a)^{n+1}}{n!} I_1 = \int_a^x \frac{(u-a)^n}{n!} f^{(n)}(u) du,$$

and

$$\frac{(x-b)^{n+1}}{n!} I_2 = \int_b^x \frac{(u-b)^n}{n!} f^{(n)}(u) du.$$

Thus

$$\frac{(x-a)^{n+1}}{n!} I_1 - \frac{(-1)^{n+1}(b-x)^{n+1}}{n!} I_2 = \int_a^x \frac{(u-a)^n}{n!} f^{(n)}(u) du + \int_x^b \frac{(u-b)^n}{n!} f^{(n)}(u) du.$$

It completes the proof. ■

From (2.4), for $n = 1$, we get the identity (2.2), proved in [2].

Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous function on $[a, b]$ then for any $x \in [a, b]$,

$$\frac{(x-a)^{i+2} - (b-x)^{i+2}}{(i+2)} g^{(i+1)}(x) - \int_a^b \left[\int_t^x (i+1)(u-t)^i g^{(i+1)}(u) du \right] dt$$

$$(3)= \int_a^b \left[(x-t)^{i+2} \int_0^1 (1-\lambda)^{i+1} g^{(i+2)} \left((1-\lambda)x + \lambda t \right) d\lambda \right] dt.2.6$$

Proof. For any $x,\ t \in [a, b],\ x \neq t$, one has

$$\frac{f^{(i)}(x) - f^{(i)}(t)}{x-t} = \frac{1}{x-t} \int_t^x f^{(i+1)}(u) du,$$

for $i = 0, 1, ..., n-1$.

Let

$$f^{(i)}(x) = (x-t)^{i+1} g^{(i+1)}(x), \tag{2.7}$$

and

$$f^{(i)}(t) = 0.$$

Differentiating (2.7) w.r.t. x, we have

$$f^{(i+1)}(x) = (x-t)^{i+1} g^{(i+2)}(x) + (i+1)(x-t)^{i} g^{(i+1)}(x). \tag{2.8}$$

We have from $(2.7 - 2.8)$

(2.9)

$$(x-t)^{i+1} g^{(i+1)}(x) = \int_t^x \left[(u-t)^{i+1} g^{(i+2)}(u) + (i+1)(u-t)^{i} g^{(i+1)}(u) \right] du.$$

Integrating (2.9) w.r.t. t on $[a,b]$, we have:

$$\begin{aligned} & g^{(i+1)}(x) \int_a^b (x-t)^{i+1} dt \\ = & \int_a^b \left\{ \int_t^x [(u-t)^{i+1} g^{(i+2)}(u) + (i+1)(u-t)^{i} g^{(i+1)}(u)] du \right\} dt, \end{aligned}$$

by taking $u = (1-\lambda)x + \lambda t$, and $\frac{u-x}{x-t} = \lambda$, $\lambda \to 1$, $u \to t$, $\lambda \to 0$, $u \to x$, $(x-t)\, d\lambda = du$. We have

$$\begin{aligned} & \frac{(x-a)^{i+2} - (b-x)^{i+2}}{(i+2)} g^{(i+1)}(x) - \int_a^b \left[\int_t^x (i+1)(u-t)^{i} g^{(i+1)}(u) du \right] dt \\ (4)= & \int_a^b \left[(x-t)^{i+2} \int_0^1 (1-\lambda)^{i+1} g^{(i+2)} \left((1-\lambda)x + \lambda t \right) d\lambda \right] dt.2.10 \end{aligned}$$

■

From (2.9), for $i = 0$, we have the identity

$$(x-t)g'(x) = \int_t^x [(u-t)g''(u) + g'(u)]\,du,$$

implies

$$f(x) = f(t) + \int_t^x f'(u)du,$$

where

$$f(x) = (x-t)g'(x), \quad f(t) = 0 \text{ and } f'(x) = (x-t)g''(x) + g'(x),$$

which is the main identity proved in [1].

From (2.6), for $i = 0$, we have the identity for functions whose modules of second derivative may be convex

$$\begin{aligned} & g(x) - \frac{1}{b-a}\int_a^b g(t)dt + (x - \frac{a+b}{2})g'(x) \\ = \; & \frac{1}{b-a}\int_a^b (x-t)^2 \left[\int_0^1 (1-\lambda)g''\left((1-\lambda)x + \lambda t\right) d\lambda\right], \end{aligned}$$

where

$$u = (1-\lambda)x + \lambda t, \;\; \lambda \to 1, \quad u \to t \;\text{ and } \lambda \to 0, \;\; u \to x,$$

and

$$f(x) = (x-t)g'(x), \quad f(t) = 0 \text{ and } f'(x) = (x-t)g''(x) + g'(x).$$

The above identity is due to Pachpatte [11].

Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions on $[a,b]$. If $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$are convex on $[a,b]$, then

$$|S(f,g)|\leq\frac{1}{2}\left(|g(x)|\,|M(x)|+|f(x)|\,|N(x)|\right), \tag{2.11}$$

where

$$S(f,g)=f(x)g(x)+\frac{(-1)^n}{2\,(b-a)}\left(f(x)\int_a^b g(t)dt+g(x)\int_a^b f(t)dt\right),$$

$$\begin{aligned}|M(x)| &= \frac{1}{(n+2)!}\left\{\left(\frac{x-a}{b-a}\right)^{n+1}|f^{(n)}(a)|+\left(\frac{b-x}{b-a}\right)^{n+1}|f^{(n)}(b)|\right.\\ &\quad\left.+(n+1)\left[\left(\frac{x-a}{b-a}\right)^{n+1}+\left(\frac{b-x}{b-a}\right)^{n+1}\right]|f^{(n)}(x)|\right\}(b-a)^n\\ &\quad+\sum_{k=1}^{n-1}\frac{|G_k(x)|}{b-a},2.12\end{aligned} \tag{6}$$

$$\begin{aligned}|N(x)| &= \frac{1}{(n+2)!}\left\{\left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(a)|+\left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(b)|\right.\\ &\quad\left.+(n+1)\left[\left(\frac{x-a}{b-a}\right)^{n+1}+\left(\frac{b-x}{b-a}\right)^{n+1}\right]|g^{(n)}(x)|\right\}(b-a)^n\\ &\quad+\sum_{k=1}^{n-1}\frac{|F_k(x)|}{b-a},2.13\end{aligned} \tag{7}$$

$$G_k(x)=\frac{(b-x)^{k+1}+(-1)^k\,(x-a)^{k+1}}{(k+1)!}g^{(k)}(x),$$

and

$$F_k(x)=\frac{(b-x)^{k+1}+(-1)^k\,(x-a)^{k+1}}{(k+1)!}f^{(k)}(x).$$

Proof. Cerone, Dragomir and Roumeliotis in [3] proved the following identity for n times differentiable mappings:

$$\int_a^b K_n(x,t) f^{(n)}(t)dt$$

$$= (-1)^n \int_a^b f(t)dt + (-1)^{n+1} \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x)$$

$$(8) = (-1)^n \int_a^b f(t)dt + (-1)^{n+1} \sum_{k=0}^{n-1} F_k(x), 2.14$$

and empty sum is assumed to be zero and kernel $K_n(.,.) : [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a,x] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x,b], \end{cases}$$

for all $x \in [a,b]$ and $n \geq 1$ is a natural number.

From (2.4), we have:

(2.15)

$$\frac{(x-a)^{n+1}}{n!} I_1 - \frac{(-1)^{n+1}(b-x)^{n+1}}{n!} I_2 = (-1)^n \int_a^b f(t)dt + (-1)^{n+1} \sum_{k=0}^{n-1} F_k(x),$$

where

$$F_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x),$$

$$I_1 = \int_0^1 \lambda^n f^{(n)}\left((1-\lambda)a + \lambda x\right) d\lambda,$$

and

$$I_2 = \int_0^1 \lambda^n f^{(n)}\left((1-\lambda)b + \lambda x\right) d\lambda.$$

$$f(x) = \frac{(x-a)^{n+1}}{n!}\frac{I_1}{b-a} - \frac{(-1)^{n+1}(b-x)^{n+1}}{n!}\frac{I_2}{b-a}$$

(9) $$-\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(-1)^{n+1}}{b-a}\sum_{k=1}^{n-1} F_k(x), 2.16$$

and also

$$g(x) = \frac{(x-a)^{n+1}}{n!}\frac{I_3}{b-a} - \frac{(-1)^{n+1}(b-x)^{n+1}}{n!}\frac{I_4}{b-a}$$

(10) $$-\frac{(-1)^n}{b-a}\int_a^b g(t)dt - \frac{(-1)^{n+1}}{b-a}\sum_{k=1}^{n-1} G_k(x), 2.17$$

where

$$G_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} g^{(k)}(x),$$

$$I_3 = \int_0^1 \lambda^n g^{(n)}\left((1-\lambda)a + \lambda x\right) d\lambda,$$

and

$$I_4 = \int_0^1 \lambda^n g^{(n)}\left((1-\lambda)b + \lambda x\right) d\lambda.$$

Multiplying both sides of (2.16) and (2.17) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting, we have:

$$f(x)g(x) + \frac{(-1)^n}{2(b-a)}\left[g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt\right]$$

$$= \frac{g(x)}{2(b-a)n!}\left[(x-a)^{n+1}I_1 - (-1)^{n+1}(b-x)^{n+1}I_2\right]$$

$$+\frac{f(x)}{2(b-a)n!}\left[(x-a)^{n+1}I_3 - (x-b)^{n+1}I_4\right]$$

(11) $$-\frac{(-1)^{n+1}}{2(b-a)}\left[f(x)\sum_{k=1}^{n-1} G_k(x) + g(x)\sum_{k=1}^{n-1} F_k(x)\right].2.18$$

Since $|f^{(n)}|, |g^{(n)}|$ are convex on $[a, x]$ and $[x, b]$, from (2.18), we have

$$|S(f,g)| \leq \frac{1}{2}\left(g(x)\,|M(x)| + f(x)\,|N(x)|\right),$$

where

$$|M(x)| = \frac{1}{(n+2)!}\left\{\left(\frac{x-a}{b-a}\right)^{n+1}|f^{(n)}(a)| + \left(\frac{b-x}{b-a}\right)^{n+1}|f^{(n)}(b)| + (n+1)\left[\left(\frac{x-a}{b-a}\right)^{n+1} + \left(\frac{b-x}{b-a}\right)^{n+1}\right]|f^{(n)}(x)|\right\}(b-a)^n + \sum_{k=1}^{n-1}\frac{|G_k(x)|}{b-a}, 2.19 \quad (12)$$

$$|N(x)| = \frac{1}{(n+2)!}\left\{\left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(a)| + \left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(b)| + (n+1)\left[\left(\frac{x-a}{b-a}\right)^{n+1} + \left(\frac{b-x}{b-a}\right)^{n+1}\right]|g^{(n)}(x)|\right\}(b-a)^n + \sum_{k=1}^{n-1}\frac{|F_k(x)|}{b-a}, 2.19a \quad (13)$$

Let

$$M_1 = \frac{(x-a)^{n+1}I_1 - (-1)^{n+1}(b-x)^{n+1}I_2}{(b-a)n!} - (-1)^{n+1}\sum_{k=1}^{n-1}\frac{G_k(x)}{b-a}, \quad (2.20)$$

and

$$N_1 = \frac{(x-a)^{n+1}I_3 - (-1)^{n+1}(b-x)^{n+1}I_4}{(b-a)n!} - (-1)^{n+1}\sum_{k=1}^{n-1}\frac{F_k(x)}{b-a}. \quad (2.20a)$$

Using modulus and convex properties of $f^{(n)}$, we observe that

$$
\begin{aligned}
|I_1| &= \left| \int_0^1 \lambda^n f^{(n)}\left((1-\lambda)a + \lambda x\right) d\lambda \right| \\
&\le |f^{(n)}(a)| \int_0^1 \lambda^n (1-\lambda) d\lambda + |f^{(n)}(x)| \int_0^1 \lambda^{n+1} d\lambda \\
&= \frac{|f^{(n)}(a)|}{(n+1)(n+2)} + \frac{|f^{(n)}(x)|}{n+2}.2.21 \qquad (14)
\end{aligned}
$$

Similarly, we have

$$
|I_2| = \left| \int_0^1 \lambda^n f^{(n)}\left((1-\lambda)b + \lambda x\right) d\lambda \right| \le \frac{|f^{(n)}(b)|}{(n+1)(n+2)} + \frac{|f^{(n)}(x)|}{n+2}, \tag{2.22}
$$

$$
|I_3| = \left| \int_0^1 \lambda^n f^{(n)}\left((1-\lambda)a + \lambda x\right) d\lambda \right| \le \frac{|g^{(n)}(a)|}{(n+1)(n+2)} + \frac{|g^{(n)}(x)|}{n+2}, \tag{2.23}
$$

and

$$
|I_4| = \left| \int_0^1 \lambda^n g^{(n)}\left((1-\lambda)b + \lambda x\right) d\lambda \right| \le \frac{|g^{(n)}(b)|}{(n+1)(n+2)} + \frac{|g^{(n)}(x)|}{n+2}. \tag{2.24}
$$

From (2.20), (2.21) and (2.22), we have:

$$
\begin{aligned}
&|M_1| \\
\leq\ & \left(\frac{x-a}{b-a}\right)^{n+1}\left\{\frac{|f^{(n)}(a)|}{(n+1)(n+2)n!}+\frac{|f^{(n)}(x)|}{n!(n+2)}\right\}(b-a)^n \\
&+\left(\frac{b-x}{b-a}\right)^{n+1}\left\{\frac{|f^{(n)}(b)|}{(n+1)(n+2)n!}+\frac{|f^{(n)}(x)|}{n!(n+2)}\right\}(b-a)^n \\
&+\sum_{k=1}^{n-1}\frac{|G_k(x)|}{b-a} \\
=\ & \frac{1}{(n+2)!}\left\{\left(\frac{x-a}{b-a}\right)^{n+1}|f^{(n)}(a)|+\left(\frac{b-x}{b-a}\right)^{n+1}|f^{(n)}(b)|\right. \\
&\left.+(n+1)\left[\left(\frac{x-a}{b-a}\right)^{n+1}|f^{(n)}(x)|+\left(\frac{b-x}{b-a}\right)^{n+1}|f^{(n)}(x)|\right]\right\}(b-a)^n \\
(15)\quad &+\sum_{k=1}^{n-1}\frac{|G_k(x)|}{b-a}=|M(x)|\,.2.25
\end{aligned}
$$

Similarly, from (2.20a), (2.23) and (2.24), we have:

$$
\begin{aligned}
&|N_1| \\
\leq\ & \frac{1}{(n+2)!}\left\{\left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(a)|+\left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(b)|\right. \\
&\left.+(n+1)\left[\left(\frac{x-a}{b-a}\right)^{n+1}|g^{(n)}(x)|+\left(\frac{b-x}{b-a}\right)^{n+1}|g^{(n)}(b)|\right]\right\}(b-a)^n \\
(16)\quad &+\sum_{k=1}^{n-1}\frac{|F_k(x)|}{b-a}=|N(x)|\,.2.26
\end{aligned}
$$

Using (2.19), (2.19a), (2.25) and (2.26) we have the desired inequality ■

From (2.15), for $n=1$, we get the identity which is proved by Cerone and Dragomir [2].

From (2.18) , for $n = 1$, we have :

$$\begin{aligned} S(f,g) &= f(x)g(x) - \frac{1}{2(b-a)}\left(g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt \right) \\ &= \frac{g(x)}{2(b-a)}\Bigg[(x-a)^2 \int_0^1 \lambda f'((1-\lambda)a + \lambda x)\, d\lambda \\ &\quad - (x-b)^2 \int_0^1 \lambda f'((1-\lambda)b + \lambda x)\, d\lambda \Bigg] \\ &\quad + \frac{f(x)}{2(b-a)}\Bigg[(x-a)^2 \int_0^1 \lambda g'\left((1-\lambda)a + \lambda x\right) d\lambda \\ &\quad - (x-b)^2 \int_0^1 \lambda g'\left((1-\lambda)b + \lambda x\right) d\lambda \Bigg], 2.27 \end{aligned} \tag{17}$$

which is proved in [11].

Let $f, g = [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$.

If $|f^{(i)}|, |g^{(i)}|$ are convex on $[a,b]$, then

$$\begin{aligned} & |\widetilde{S}(f,g)| \\ \leq\ & \left[\frac{|f(x)|g^{(i+2)}(x)|}{i+3} + \frac{|f(x)|\|g^{(i+2)}\|_\infty}{(i+2)(i+3)} \right] \left[\frac{(x-a)^{i+3} + (b-x)^{i+3}}{i+3} \right] \\ & + \left[\frac{|g(x)|f^{(i+2)}(x)|}{i+3} + \frac{|g(x)|\|f^{(i+2)}\|_\infty}{(i+2)(i+3)} \right] \left[\frac{(x-a)^{i+3} + (b-x)^{i+3}}{i+3} \right], \\ & 2.28 \end{aligned} \tag{18}$$

where

$$\widetilde{S}(f,g) = f(x)\left\{\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}g^{(i+1)}(x)\right.$$

$$\left.-\int_a^b\left[\int_t^x (i+1)(u-t)^i g^{(x+1)}(u)du\right]dt\right\}$$

$$+g(x)\left\{\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}f^{(i+1)}(x)\right.$$

$$\left.-\int_a^b\left[\int_t^x (i+1)(u-t)^i f^{(i+1)}(u)du\right]dt\right\}, 2.29 \tag{19}$$

and $i = 1, 2, \ ..., \ n-1$.

Proof. From hypotheses of lemma 2 the following identities hold:

$$\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}g^{(i+1)}(x) - \int_a^b\left[\int_t^x (i+1)(u-t)^i g^{(i+1)}(u)du\right]dt$$

$$\int_a^b\left[(x-t)^{i+2}\int_0^1 (1-\lambda)^{i+1} g^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt, 2.30 \tag{20}$$

and

$$\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}f^{(i+1)}(x) - \int_a^b\left[\int_t^x (i+1)(u-t)^i f^{(i+1)}(u)du\right]dt$$

$$\int_a^b\left[(x-t)^{i+2}\int_0^1 (1-\lambda)^{i+1} f^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt, 2.31 \tag{21}$$

for all $x \in [a,b]$.

Multiplying both sides of (2.30) and (2.31) by $f(x)$ and $g(x)$ respectively adding the resulting identities and rewriting, we have:

$$\begin{aligned}
&\widetilde{S}(f,g) \\
= \ & f(x)\int_a^b \left[(x-t)^{i+2} + g^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt \\
&+g(x)\int_a^b \left[(x-t)^{i+2}\int_0^1 (1-\lambda)^{i+1} f^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt.
\end{aligned}$$

(22) 2.32

Since $|f^{(i)}|$ and $|g^{(i)}|$ are convex on $[a,b]$, from (2.32), we observe that

$$\begin{aligned}
&|\widetilde{S}(f,g)| \leq |f(x)|\int_a^b \left\{|x-t|^{i+2}\int_0^1 \left[(1-\lambda)^{i+2}|g^{(i+2)}(x)|\right.\right. \\
&\left.\left.+\ \lambda(1-\lambda)^{i+2}|g^{(i+2)}(t)|\right]d\lambda\right\}dt \\
&+|g(x)|\int_a^b \left\{|(x-t)|^{i+2}\int_0^1 \left[(1-\lambda)^{i+2}|f^{(i+2)}(x)|\right.\right. \\
&\left.\left.+\ \lambda(1-\lambda)^{i+1}|f^{(i+2)}(t)|\right]d\lambda\right\}dt.
\end{aligned}$$

Now

$$\int_0^1 (1-\lambda)^{i+2}d\lambda = \frac{1}{i+3} \text{ and } \int_0^1 \lambda(1-\lambda)^{i+1}d\lambda = \frac{1}{(i+2)(i+3)}.$$

Thus

$$
\begin{aligned}
&|\widetilde{S}(f,g)| \\
\leq\ & \frac{|f(x)||g^{(i+2)}(x)|}{i+3}\int_a^b |x-t|^{i+2}\,dt \\
&+\frac{|f(x)|}{(i+2)(i+3)}\int_a^b |x-t|^{i+2}\,|g^{(i+2)}(t)|dt \\
&+\frac{|g(x)||f^{(i+2)}(x)|}{i+3}\int_a^b |x-t|^{i+2}\,dt \\
&+\frac{|g(x)|}{(i+2)(i+3)}\int_a^b |x-t|^{i+2}\,|f^{(i+2)}(t)|dt \\
\leq\ & \left[\frac{|f(x)||g^{(i+2)}(x)|}{i+3}+\frac{|f(x)|}{(i+2)(i+3)}\|g^{(i+2)}\|_\infty\right]\int_a^b |x-t|^{i+2}\,dt \\
&+\left[\frac{|g(x)||f^{(i+2)}(x)|}{(i+3)}+\frac{|g(x)|}{(i+2)(i+3)}\|f^{(i+2)}\|_\infty\right]\int_a^b |x-t|^{i+2}\,dt \\
\leq\ & \left[\frac{|f(x)||g^{(i+2)}(x)|}{i+3}+\frac{|f(x)|\|g^{(i+2)}\|_\infty}{(i+2)(i+3)}\right]\frac{(x-a)^{i+3}(b-x)^{i+3}}{i+3} \\
&+\left[\frac{|g(x)||f^{(i+2)}(x)|}{i+3}+\frac{|g(x)|\|f^{(i+2)}\|_\infty}{(i+2)(i+3)}\right] \\
&\times\left[\frac{(x-a)^{i+3}(b-x)^{i+3}}{i+3}\right].2.33 \qquad (23)
\end{aligned}
$$

It completes the proof. ■

For $i = 0$, the L.H.S and R.H.S of (2.33) , are as follow

$$\begin{aligned} L.H.S &= |\widetilde{S}(f,g)| \\ &= \left| \frac{(x-a)^2-(b-x)^2}{2} g'(x)f(x) - f(x)\int_a^b (g(x)-g(t))\,dt \right. \\ &\quad \left. + \frac{(x-a)^2-(b-x)^2}{2} f'(x)g(x) - g(x)\int_a^b (f(x)-f(t))\,dt \right| \end{aligned}$$

$$\begin{aligned} &= \left| (b-a)(x-\frac{a+b}{2})| \left[f(x)g'(x)+g(x)f'(x) \right] + f(x)\int_a^b g(t)dt \right. \\ &\quad \left. + g(x)\int_a^b f(t)dt - 2f(x)g(x)(b-a) \right| \\ &= \left| 2(b-a)f(x)g(x) - f(x)\int_a^b g(t)dt \right. \\ &\quad \left. - g(x)\int_a^b f(t)dt - (b-a)(x-\frac{a+b}{2}) \left[f(x)g'(x)+g(x)f'(x) \right] \right| .2.34 \end{aligned} \tag{24}$$

and

$$
\begin{aligned}
R.H.S &= \left(\frac{|f(x)|\;||g''(x)|}{3} + \frac{|f(x)|\;||g''||_\infty}{6}\right)\frac{(x-a)^3+(b-x)^3}{3} \\
&\quad + \left(\frac{|g(x)|\;||f''(x)|}{3} + \frac{|g(x)|\;||f''||_\infty}{6}\right)\frac{(x-a)^3+(b-x)^3}{3} \\
&= \frac{b-a}{18}\left[|f(x)|(2|g''(x)| + ||g''||_\infty) + |g(x)|(2|f''(x)| + ||f''||_\infty)\right] \\
&\quad \times \left[\left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right] \\
&= \frac{(b-a)^3}{6}\left[|f(x)|(2|g''(x)| + ||g''||_\infty) + |g(x)|(2|f''(x)| + ||f''||_\infty)\right] \\
(25) &\quad \times \left[\frac{1}{12} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right].2.35
\end{aligned}
$$

From (2.34) and (2.35) we have

$$
\begin{aligned}
&\left| f(x)g(x) - \frac{1}{2}(x - \frac{a+b}{2})[f(x)g'(x) + g(x)f'(x)] - \frac{1}{2(b-a)}\left[f(x)\int_a^b g(t)dt \right.\right. \\
&\left.\left. + \; g(x)\int_a^b f(t)dt\right]\right| \\
\leq \; & \frac{(b-a)^2}{12}\left[|f(x)|(2|g''(x)| + ||g''||_\infty) + |g(x)|(2|f''(x)| + ||f''||_\infty)\right] \\
(26) \; & \times \left[\frac{1}{12} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right].2.35a
\end{aligned}
$$

We note that as a special case, if we take $f(x) = 1$ in the inequality (2.35a), we get Ostrowski inequality for functions whose, modulus of the second derivative is convex, i.e.,

$$\begin{aligned}&\left|g(x)-(x-\frac{a+b}{2})g'(x)-\frac{1}{b-a}\int_a^b g(t)dt\right|\\ = &\frac{(b-a)^2}{6}\left[2\left|g''(x)\right|+||g''||_\infty\right]\left[\frac{1}{12}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right].\end{aligned}$$

The rest of the inequalities in [1] can be generalized using our lemma 2.

Let $f,g=[a,b]\to\mathbb{R}$ be absolutely continuos functions on $[a,b]$. If $|f^{(i)}|,|g^{(i)}|$ one convex on $[a,b]$, then

$$\begin{aligned}&\left|\widetilde{T}(f,g)\right|\\ \leq &\frac{1}{b-a}\int_a^b\left\{|f(x)|\left[\frac{\left|g^{(i+2)}(x)\right|}{i+3}+\frac{\left\|g^{(i+2)}\right\|_\infty}{(i+2)(i+3)}\right]\right.\\ &\left.+|g(x)|\left[\frac{\left|f^{(i+2)}(x)\right|}{i+3}+\frac{\left\|f^{(i+2)}\right\|_\infty}{(i+2)(i+3)}\right]\right\}E(x)dx, 2.36 \qquad (27)\end{aligned}$$

where

$$E(x)=\frac{(x-a)^{i+3}+(b-x)^{i+3}}{i+3},$$

and $i=1,2,\ ...,\ n-1$.

Proof. From the hypothesis of Theorem 3, the following identity holds:

$$\begin{aligned}&\widetilde{S}(f,g)\\ = &f(x)\int_a^b\left[(x-t)^{i+2}\int_0^1(1-\lambda)^{i+1}g^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt\\ &+g(x)\int_a^b\left[(x-t)^{i+2}\int_0^1(1-\lambda)^{i+1}f^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt, 2.37 \qquad (28)\end{aligned}$$

where

$$\widetilde{S}(f,g) = f(x)\left[\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}g^{(i+1)}(x)\right.$$
$$\left. - \int_a^b \left(\int_t^x (i+1)u-t)^i g^{(i+1)}(u)du\right)dt\right]$$
$$+g(x)\left[\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}f^{(i+1)}(x)\right.$$
(29)
$$\left. - \int_a^b \left(\int_t^x (i+1)(u-t)^i f^{(i+1)}(u)du\right)dt\right].2.38$$

Integrating both sides of (2.37) w.r.t. x from a to b and rewriting, we have:

$$\frac{1}{b-a}\int_a^b \widetilde{S}(f,g)dx = \widetilde{T}(f,g).$$

$$\begin{aligned}
&\widetilde{T}(f,g) \\
= \ & \frac{1}{b-a}\int_a^b \left\{ f(x)\int_a^b \left[(x-t)^{i+2}\int_0^1 (1-\lambda)^{i+1}g^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt\right. \\
&\left. + g(x)\int_a^b \left[(x-t)^{i+2}\int_0^1 (1-\lambda)^{i+1}g^{(i+2)}\left((1-\lambda)x+\lambda t\right)d\lambda\right]dt\right\} dx,
\end{aligned}$$

where

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b \widetilde{S}(f,g)dx \\
=\ &\frac{1}{b-a}\int_a^b \left\{ f(x)\left[\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}g^{(i+1)}(x)\right.\right. \\
&\left.-\int_a^b\left(\int_t^x (i+1)(u-t)^i g^{(i+1)}(u)du\right)dt\right] \\
&+g(x)\left[\frac{(x-a)^{i+2}-(b-x)^{i+2}}{i+2}f^{(i+1)}(x)\right. \\
&\left.\left.-\int_a^b\left(\int_t^x (i+1)(u-t)^i f^{(i+1)}(u)du\right)dt\right]\right\} dx.2.39 \qquad (30)
\end{aligned}
$$

Since $|f^{(n)}|$ and $|g^{(n)}|$ are convex on $[a,b]$, we have: $i = 0, 1, ..., n-1$.

$$
\begin{aligned}
&|\widetilde{T}(f,g)| \\
\leq\ &\frac{1}{b-a}\int_a^b \left\{ |f(x)|\int_a^b \left[|(x-t)^{i+2}|\right.\right. \\
&\left.\times\int_0^1 (1-\lambda)^{i+2}|g^{(i+2)}(x)| + \lambda(1-\lambda)^{i+1}|g^{(i+2)}(t)|d\lambda\right] dt \\
&+|g(x)|\int_a^b \left[|(x-t)^{i+2}|\right. \\
&\left.\left.\times\int_0^1 (1-\lambda)^{i+2}|f^{(i+2)}(x)| + \lambda(1-\lambda)^{i+1}|f^{(i+2)}(t)|d\lambda\right] dt \right\} dx
\end{aligned}
$$

$$
\begin{aligned}
& |\widetilde{T}(f,g)| \\
\leq\ & \frac{1}{b-a}\int_a^b \left[|f(x)| \int_a^b |(x-t)^{i+2}| \left(\frac{|g^{(i+2)}(x)|}{i+3} + \frac{|g^{(i+2)}(t)|}{(i+2)(i+3)} \right) dt \right. \\
& \left. + |g(x)| \int_a^b |(x-t)^{i+2}| \left(\frac{|f^{(i+2)}(x)|}{i+3} + \frac{|f^{(i+2)}(t)|}{(i+2)(i+3)} \right) dt \right] dx \\
\leq\ & \frac{1}{b-a}\int_a^b \left\{ \left[|f(x)| \left(\frac{|g^{(i+2)}(x)|}{i+3} + \frac{\left\|g^{(i+2)}\right\|_\infty}{(i+2)(i+3)} \right) \right.\right. \\
& \left.\left. + |g(x)| \left(\frac{|f^{(i+2)}(x)|}{i+3} + \frac{\left\|f^{(i+2)}\right\|_\infty}{(i+2)(i+3)} \right) \right] \int_a^b |(x-t)^{i+2}| dt \right\} dx \\
=\ & \frac{1}{b-a}\int_a^b \left\{ \left[|f(x)| \left(\frac{|g^{(i+2)}(x)|}{i+3} + \frac{\left\|g^{(i+2)}\right\|_\infty}{(i+2)(i+3)} \right) \right.\right. \\
& \left.\left. + |g(x)| \left(\frac{|f^{(i+2)}(x)|}{i+3} + \frac{\left\|f^{(i+2)}\right\|_\infty}{(i+2)(i+3)} \right) \right] E(x) \right\} dx, 2.40 \qquad (31)
\end{aligned}
$$

where

$$E(x) = \int_a^b \left|(x-t)^{i+2}\right| dt = \frac{(x-a)^{i+3} + (b-x)^{i+3}}{i+3}.$$

■

From (2.40), for $i = 0$, we have:

$$
\begin{aligned}
|\widetilde{T}(f,g)| \leq\ & \frac{1}{b-a}\int_a^b \left[|f(x)| \left(\frac{|g''(x)|}{3} + \frac{\|g''\|_\infty}{6} \right) \right. \\
& \left. + |g(x)| \left(\frac{|f''(x)|}{3} + \frac{\|f''\|_\infty}{6} \right) \right] E(x) dx,
\end{aligned}
$$

where

$$E(x) = \frac{(x-a)^3 + (b-x)^3}{3} = (b-a)^3 \left[\frac{1}{12} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right],$$

implies

$$|\widetilde{T}(f,g)| \leq \frac{(b-a)^2}{6} \int_a^b [|f(x)|\,(2|g''(x)| + \|g''\|_\infty) + |g(x)|\,(2|f''(x)| + \|f''\|_\infty)] \times \left[\frac{1}{12} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] dx. 2.41 \tag{32}$$

Also, we have

$$\left|\frac{1}{b-a}\int_a^b \widetilde{S}(f,g)dx\right| = |\widetilde{T}(f,g)|$$

$$= \int_a^b \left|2f(x)g(x) - \frac{1}{b-a}\left(f(x)\int_a^b g(t)dt + g(x)\int_a^b f(t)dt\right) - (x - \frac{a+b}{2})[f(x)g'(x) + g(x)f'(x)]\right| dx. 2.42 \tag{33}$$

(2.41) and (2.42) are proved in [11].

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