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Rate of convergence on the mixed summation integral type operators [1]

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

Gupta and Erkus [3] introduced the mixed sequence of summation-integral type operators $S_n(f,x)$ and estimated some direct results in simultaneous approximation. We extend the study on these operators $S_n(f,x)$ and here we study the rate of convergence for functions having derivatives of bounded variation.

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1 Introduction

Very recently Gupta and Erkus [3] defined a mixed summation-integral type operators to approximate integrable functions on the interval $[0,\infty)$ .The operators introduced in [3] are defined as

(1) $$S_n(f,x) = \int_0^\infty W_n(x,t)f(t)dt$$

$$= (n-1)\sum_0^\infty s_{n,v}(x)\int_0^\infty b_{n,v-1}(t)f(t)dt + \exp(-nx)f(0), \qquad x \in [0,\infty)$$

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where

$$W_n(x,t) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x)b_{n,v-1}(t) + \exp^{-nx}\delta(t)$$

$\delta(t)$ being Dirac delta function, $s_{n,v}(x) = \frac{\exp(-nx)nx^v}{v!}$ and $b_{n,v}(t) = \binom{n+v-1}{v}t^v(1+t)^{-n-v}$ are respectively Szasz and Baskakov basis functions. Although these operators are similar to the generalized summation integral type operators recently introduced by Srivastava and Gupta [4], but the approximation properties of the operators (1) are different from those introduced in [4]. Here in summation and integration we are taking different basis functions.

We define

$$\beta_n(x,t) = \int_0^t W_n(x,s)ds$$

then in particular,

$$\beta_n(x,\infty) = \int_0^\infty W_n(x,s)ds = 1.$$

Let $DB_\gamma(0,\infty), \gamma \geq 0$ be the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying the growth condition $f(t) = O(t^\gamma), t \to \infty$ and having a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f \in BD_\gamma(0,\infty)$ posses for each $c > 0$ a representation

$$f(x) = f(c) + \int_c^x \psi(t)dt, \quad x \geq c.$$

We denote the auxiliary function f_x by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x; \\ 0, & t = x; \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

In [3] the authors studied some direct results in simultaneous approximation for the operators (1).The rates of convergence for functions having derivatives of bounded variation on Bernstein polynomials were studied in [1] and

[2]. This motivated us to study further on summation integral type operators and here we estimate the rate of convergence for the operator (1) with functions having derivatives of bounded variation.

2 Auxiliary Results

We shall use the following Lemmas to prove our main theorem.
Lemma 2.1. Let the function $\mu_{n,m}(x)$, $m \in \aleph^0$, be defined as

$$\mu_{n,m}(x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)(t-x)^m dt + (-x)^m \exp(-nx).$$

Then $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = \frac{2x}{n-2}$, $\mu_{n,2}(x) = \frac{nx(x+2)+6(x^2)}{(n-2)(n-3)}$, also we have the recurrence relation:

$$(n-m-2)\mu_{n,m+1}(x) = x[\mu^{(1)}_{n,m}(x) + m(x+2)\mu_{n,m-1}(x)$$
$$+ [m + 2x(m+1)]\mu_{n,m}(x); \qquad n > m+2.$$

Consequently for each $x \in [0,\infty)$ we from this recurrence relation that

$$\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$$

Remark 2.2.In particular given any number $\lambda > 1$ and $x \in (0,\infty)$,by lemma 2.1,we have for n sufficiently large

(2) $\qquad S_n((t-x)^2, x) \equiv \mu_{n,2} \leq \frac{\lambda x(x+2)}{n}$

Remark 2.3. From equation (2) it follows that

(3) $\qquad S_n(\mid t-x \mid, x) \leq [S_n((t-x)^2, x)]^{1/2} \leq \sqrt{\lambda x(x+2)/n}$

Lemma 2.4. Let $x \in (0,\infty)$ and $W_n(x,t)$are as in (1). then for $\lambda > 1$ and for n sufficiently large, we have

$$(i) \quad \beta_n(x,y) = \int_0^y W_n(x,t)dt \leq \frac{\lambda x(x+2)}{n(x-y)^2}, \quad 0 \leq y < x$$

$$(ii) \quad 1 - \beta_n(x,z) = \int_z^{\infty} W_n(x,t)dt \leq \frac{\lambda x(x+2)}{n(z-x)^2}, \quad x < z < \infty$$

Proof. First we prove (i), by (2), we have

$$\int_0^y W_n(x,t)dt \leq \int_0^y \frac{(x-t)^2}{(x-y)^2} W_n(x,t)dt$$

$$\leq (x,y)^{-2}\mu_{n,2}(x) \leq \frac{\lambda x(x+2)}{x(x-y)^2}.$$

The proof of (ii) is similar, we omit the details.

3 Main Result

In this section, we prove the following main theorem.

Theorem 3.1 Let $f \in DB_\gamma(0,\infty)$,$\gamma > 0$, and $x \in (0,\infty)$.Then for $\lambda > 2$ and for$n$ sufficiently large, we have

$$| S_n(f,x) - f(x) | \leq \frac{\lambda(x+2)}{n}(\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x))$$

$$+ \frac{\lambda(x+2)}{n}(| f(2x) - f(x) - xf'(x^+) | + | f(x) |)$$

$$+ \sqrt{\lambda x(x+2)/n}(M2^\gamma O(n^{\frac{-\gamma}{2}}) + | f(x^+) |)$$

$$+ 1/2\sqrt{\lambda x(x+2)/n} \mid f('x^+) | - | f'(x^-) |$$

$$+ \frac{x}{n-2} | f'(x^+) | + | f'(x^-) |,$$

where$\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$.

Proof. We have

$$S_n(f,x) - f(x) = \int_0^\infty W_n(x,t)(f(t) - f(x))dt$$

$$= \int_0^\infty (\int_x^t W_n(x,t)(f'(u)du)dt)$$

Using the identity

$$f'(u) = 1/2[f'(x^+) + f'(x^-)] + (f')_x(u) + 1/2[f'(x^+) - f'(x^-)]sgn(u-x)$$

$$+[f'(x)-1/2[f'(x^+)+f'(x^-)]]\chi_x(u),$$

it is easily verified that

$$\int_0^\infty(\int_x^t f'(x)-1/2[f'(x^+)+f'(x^-)]\chi_x(u)du))W_n(x,t)dt=0$$

Also

$$\int_0^\infty(\int_x^t 1/2[f'(x^+)-f'(x^-)]sgn(u-x)du)W_n(x,t)dt$$

$$=1/2[f'(x^+)-f'(x^-)]S_n(\mid t-x\mid,x)$$

and

$$\int_0^\infty(\int_x^t \frac{1}{2}[f'(x^+)+f'(x^-)]du)W_n(x,t)dt$$

$$=\frac{1}{2}[f'(x^+)+f'(x^-)]S_n((t-x),x).$$

Thus we have

$$\mid S_n(f,x)-f(x)\mid \tag{4}$$

$$\leq\mid\int_x^\infty(\int_x^t(f')_x(u)du)W_n(x,t)dt-\int_0^x(\int_x^t(f')_x(u)du)W_n(x,t)dt\mid$$

$$+\frac{1}{2}\mid f'(x^+)-f'(x^-)\mid S_n(\mid t-x\mid,x)$$

$$+\frac{1}{2}\mid f'(x^+)+f'(x^-)\mid S_n((t-x),x)$$

$$=\mid A_n(f,x)+B_n(f,x)\mid+\frac{1}{2}\mid f'(x^+)-f'(x^-)\mid S_n(\mid t-x\mid,x)$$

$$+\frac{1}{2}\mid f'(x^+)+f'(x^-)\mid S_n((t-x),x).$$

To complete the proof of the theorem it is sufficient to estimate the terms $A_n(f,x)$ and $B_n(f,x)$. Applying integration by parts, using Lemma 2.4 and taking $y=x-x/\sqrt{n}$,we have

$$\mid B_n(f,x)\mid=\mid\int_0^x(\int_x^t(f')_x(u)du)dt\beta_n(x,t)dt\mid$$

$$\int_0^x \beta_n(x,t)(f')_x(t)dt \leq (\int_0^y + \int_y^x) | (f')_x(t) || \beta_n(x,t) | dt$$

$$\leq \frac{\lambda x(x+2)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \int_y^x \bigvee_t^x ((f')_x) dt$$

$$\leq \frac{\lambda x(x+2)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x).$$

Let$u = \frac{x}{x-t}$.Then we have

$$\frac{\lambda x(x+2)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t^2)} dt = \frac{\lambda x(x+2)}{n} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x ((f')_x) du$$

$$\leq \frac{\lambda x(x+2)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^x ((f')_x).$$

Thus

(5) $$| \beta_n(f,x) | \leq \tfrac{\lambda x(x+2)}{n} \textstyle\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^x ((f')_x) + \tfrac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x).$$

On the other hand, we have

(6) $$| A_n(f,x) |=| \textstyle\int_x^\infty (\int_x^t (f')_x(u)du) W_n(x,t)dt |$$

$$=| \int_{2x}^\infty (\int_x^t (f')_x(u)du) W_n(x,t)dt + \int_x^{2x} (\int_x^t (f')_x(u)du) dt (1-\beta_n(x,t)) | dt$$

$$\leq | \int_{2x}^\infty (f(t)-f(x)) W_n(x,t)dt | +| f'(x^+) || \int_{2x}^\infty (t-x) W_n(x,t)dt |$$

$$+| \int_x^{2x} (f')_x(u)du) || (1-\beta_n(x,2x) | + \int_x^{2x} | (f')_x(t) | | (1-\beta_n(x,t) | dt$$

$$\leq \frac{M}{x} \int_{2x}^\infty W_n(x,t) t^\gamma | t-x | dt + \frac{| f(x) |}{x^2} \int_{2x}^\infty W_n(x,t)(t-x)^2 dt$$

$$+| f'(x^+) | \int_{2x}^{\infty} W_n(x,t)| (t-x) |dt + \frac{\lambda(x+2)}{nx}(| f(2x) - f(x) - xf'(x^+) |$$

$$+\frac{\lambda(x+2)}{n}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+\frac{x}{\sqrt{n}}} ((f')_x).$$

Next applying Holderş inequality, and Lemma 2.1, we proceed as follows for the estimation of the first two terms in the right hand side of (6):

$$\frac{M}{x}\int_{2x}^{\infty} W_n(x,t)t^{\gamma}| t-x |dt + \frac{| f(x) |}{x^2}\int_{2x}^{\infty} W_n(x,t)(t-x)^2 dt \tag{7}$$

$$\leq \frac{M}{x}(\int_{2x}^{\infty} W_n(x,t)t^{2\gamma}dt)^{\frac{1}{2}} + (\int_{0}^{\infty} W_n(x,t)(t-x)^2 dt)^{\frac{1}{2}}$$

$$+\frac{| f(x) |}{x^2}(\int_{2x}^{\infty} W_n(x,t)(t-x)^2 dt)$$

$$\leq M2^{\gamma}O(n^{-\gamma/2})\frac{\sqrt{\lambda x(x+2)}}{\sqrt{n}} + | f(x) | \frac{\lambda(x+2)}{nx}$$

Also the third term of the right side of (6) is estimated as

$$| f'(x^+) | \int_{2x}^{\infty} W_n(x,t)| t-x |dt$$

$$\leq | f'(x^+) | \int_{0}^{\infty} W_n(x,t)| t-x |dt$$

$$\leq | f'(x^+) | (\int_{0}^{\infty} W_n(x,t)(t-x)^2 dt)^{\frac{1}{2}} (\int_{0}^{\infty} W_n(x,t)dt)^{\frac{1}{2}}$$

$$=| f'(x^+) | \frac{\sqrt{\lambda x(x+2)}}{\sqrt{n}}$$

Combining the estimates (4)-(7), we get the desired result.

This completes the proof of Theorem 3.1.

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