General Mathematics Vol. 15, Nr. 2–3(2007), 184-189

Some preserving properties of a integral operator

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Abstract

In this paper we will give some preserving properties of some subclasses of functions with negative coefficients by using a integral operator.

2000 Mathematical Subject Classification: 30C45

Key words and phrases: Function with negative coefficients, integral operator, Sălăgean operator

1 Introduction and Preliminaries

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

We denote with T the subset of the functions $f \in S$, which have the form

(1) $$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \, , \; a_j \geq 0 \, , \; j \geq 2 \, , \; z \in U$$

and with $T^* = T \bigcap S^*$, $T^*(\alpha) = T \bigcap S^*(\alpha)$, $T^c = T \bigcap S^c$ and $T^c(\alpha) = T \bigcap S^c(\alpha)$, where $0 \leq \alpha < 1$.

Theorem 1. *[6] For a function f having the form (1) the following assertions are equivalents:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$.

Regarding the classes $T^*(\alpha)$ and $T^c(\alpha)$ with $0 \leq \alpha < 1$, we recall here the following result:

Theorem 2. *[6] A function f having the form (1) is in the class $T^*(\alpha)$ if and only if:*

(2) $$\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} a_j \leq 1 \, ,$$

and is in the class $T^c(\alpha)$ if and only if:

(3) $$\sum_{j=2}^{\infty} \frac{j(j-\alpha)}{1-\alpha} a_j \leq 1 \, .$$

Definition 1. *[2] Let $S^*(\alpha, \beta)$ denote the class of functions having the form (1) which are starlike and satisfy*

(4) $$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1-2\alpha)} \right| < \beta$$

for $0 \leq \alpha < 1$ *and* $0 < \beta \leq 1$*. And let* $C^*(\alpha, \beta)$ *denote the class of functions such that* $zf'(z)$ *is in the class* $S^*(\alpha, \beta)$*.*

Theorem 3. *[2] A function* f *having the form (1) is in the class* $S^*(\alpha, \beta)$ *if and only if:*

$$\sum_{j=2}^{\infty} \{(j-1) + \beta(j+1-2\alpha)\} \, a_j \leq 2\beta(1-\alpha)\,, \tag{5}$$

and is in the class $C^*(\alpha, \beta)$ *if and only if:*

$$\sum_{j=2}^{\infty} j\,\{(j-1) + \beta(j+1-2\alpha)\} \, a_j \leq 2\beta(1-\alpha)\,. \tag{6}$$

Let D^n be the Sălăgean differential operator (see [3]) defined as:

$$D^n : A \to A\,, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)\,, \quad D^n f(z) = D(D^{n-1} f(z)).$$

In [5] the author define the class $T_n(\alpha, \beta)$, from which by choosing different values for the parameters we obtain variously subclasses of analytic functions with negative coefficients (for example $T_n(\alpha, 1) = T_n(\alpha)$ which is the class of n-starlike of order α functions with negative coefficients and $T_0(\alpha, \beta) = S^*(\alpha, \beta)$ is the class defined by (4)).

Definition 2. *[5] Let* $\alpha \in [0, 1)$*,* $\beta \in (0, 1]$ *and* $n \in \mathbb{N}$*. We define the class* $S_n(\alpha, \beta)$ *of the* n*-starlike of order* α *and type* β *through*

$$S_n(\alpha, \beta) = \{f \in A\,;\, |J(f, n, \alpha; z)| < \beta\}$$

where $J(f,n,\alpha;z) = \dfrac{D^{n+1}f(z) - D^n f(z)}{D^{n+1}f(z) + (1-2\alpha)D^n f(z)}$, $z \in U$. *Consequently* $T_n(\alpha,\beta) = S_n(\alpha,\beta) \bigcap T$.

Theorem 4. *[5] Let* f *be a function having the form (1). Then* $f \in T_n(\alpha,\beta)$ *if and only if*

$$\sum_{j=2}^{\infty} j^n \left[j - 1 + \beta(j + 1 - 2\alpha)\right] a_j \leq 2\beta(1-\alpha)\,. \tag{7}$$

2 Main results

In [1] the authors consider the integral operator $I_{c+\delta} : A \to A$, $0 < u \leq 1$, $1 \leq \delta < \infty$, $0 < c < \infty$, defined by

$$f(z) = I_{c+\delta}(F(z)) = (c+\delta)\int_0^1 u^{c+\delta-2} F(uz)du. \tag{8}$$

Remark 1. *For* $F(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, *from (8) we obtain*

$$f(z) = z + \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} a_j z^j.$$

Also, we notice that $0 < \dfrac{c+\delta}{c+j+\delta-1} < 1$, where $0 < c < \infty$, $j \geq 2$, $1 \leq \delta < \infty$.

Remark 2. *It is easy to prove that for* $F(z) \in T$ *and* $f(z) = I_{c+\delta}(F(z))$, *we have* $f(z) \in T$, *where* $I_{c+\delta}$ *is the integral operator defined by (8).*

Theorem 5. *Let $F(z)$ be in the class $T^*(\alpha)$, $\alpha \in [0,1)$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = I_{c+\delta}(F(z)) \in T^*(\alpha)$, where $I_{c+\delta}$ is the integral operator defined by (8).*

Proof. From Remark 2 we obtain $f(z) = I_{c+\delta}(F(z)) \in T$.

We have $\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} a_j \leq 1$ and $f(z) = z - \sum_{j=2}^{\infty} b_j z^j$, where $b_j = \frac{c+\delta}{c+j+\delta-1} a_j$. By using the fact that $0 < \frac{c+\delta}{c+j+\delta-1} < 1$, where $0 < c < \infty$, $j \geq 2$, $1 \leq \delta < \infty$, we obtain $\frac{j-\alpha}{1-\alpha} b_j < \frac{j-\alpha}{1-\alpha} a_j$ and thus $\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} b_j \leq 1$. This mean (see Theorem 2) that $f(z) = I_{c+\delta}(F(z)) \in T^*(\alpha)$.

Similarly (by using Theorems 2 , 3 and 4) we obtain:

Theorem 6. *Let $F(z)$ be in the class $T^c(\alpha)$, $\alpha \in [0,1)$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = I_{c+\delta}(F(z)) \in T^c(\alpha)$, where $I_{c+\delta}$ is the integral operator defined by (8).*

Theorem 7. *Let $F(z)$ be in the class $C^*(\alpha, \beta)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = I_{c+\delta}(F(z)) \in C^*(\alpha, \beta)$, where $I_{c+\delta}$ is the integral operator defined by (8).*

Theorem 8. *Let $F(z)$ be in the class $T_n(\alpha, \beta)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $n \in \mathbb{N}$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = I_{c+\delta}(F(z)) \in T_n(\alpha, \beta)$, where $I_{c+\delta}$ is the integral operator defined by (8).*

Remark 3. *By choosing $\beta = 1$, respectively $n = 0$, in the above theorem, we obtain the similarly result for the classes $T_n(\alpha)$ and $S^*(\alpha, \beta)$.*

References

[1] M. Acu and all, *On some starlike functions with negative coefficients*, (to appear).

[2] V.P. Gupta, P.K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., 14(1976), 409-416.

[3] G.S. Sălăgean, *Geometria planului complex*, Ed. Promedia Plus, Cluj-Napoca, 1997.

[4] G.S. Sălăgean, *Subclasses of univalent functions*, Complex Analysis. Fifth Roumanian-Finnish Seminar, Lectures Notes in Mathematics, 1013, Springer-Verlag, 1983, 362-372.

[5] G.S. Sălăgean, *Analytic functions with negative coefficients*, Mathematica, 36(59), 2(1994), 219-224.

[6] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., 51(1975), 109-116.

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