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Relation between Greek means and various means [1]

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Abstract

In this paper, we obtain some inequalities between Greek means and various means. Further, we deduced the best possible values of various means with $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$. Also we studied the partial derivatives of important means and the value of α of second order partial derivatives.

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1 Introduction

In ([1]), Ten Greek means are defined on the basis of proportions of which six means are named and four means are unnamed and some distinguished results are obtained. In ([5], [6]) authors defined Oscillatory, r^{th} Oscillatory means and its duals and obtained some new inequalities and the best possible values with Logarithmic mean, Identric mean and Power mean. In ([7]) authors defined $Gn_{\mu,r}(a,b)$, $gn_{\mu,r}(a,b)$ deduced some important results and also shown applications to Ky-Fan inequalities. Here we find the best possible values of the parameters μ, r for which F_4, F_5 and F_6 are satisfied by the inequalities (15) to (22). Further in ([1]), the partial derivatives of means and some related results are given, using which we obtained parameter α for various means.

Let a, $b > 0$, then

(1) $$A(a,b) = F_1(a,b) = \frac{a+b}{2}$$

(2) $$G(a,b) = F_2(a,b) = \sqrt{ab}$$

(3) $$F_3(a,b) = \frac{2ab}{a+b}$$

(4) $$C(a,b) = F_4(a,b) = \frac{a^2+b^2}{a+b}$$

(5) $$F_5(a,b) = \frac{a-b+\sqrt{(a-b)^2+4b^2}}{2}$$

(6) $$F_6(a,b) = \frac{b-a+\sqrt{(a-b)^2+4a^2}}{2}$$

are respestively called Arithmetic mean, Geometric mean, Harmonic mean, contra Harmonic mean, first contra Geometric mean, second contra Geometric mean. Above are called named six Greek means.

$$L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b} & a \neq b \\ a & a = b \end{cases} \tag{7}$$

$$I(a,b) = \begin{cases} e^{\left(\frac{a\ln a - b\ln b}{a-b} - 1\right)} & a \neq b \\ a & a = b \end{cases} \tag{8}$$

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}} & r \neq 0 \\ \sqrt{ab} & r = 0 \end{cases} \tag{9}$$

$$H(a,b) = \frac{a+\sqrt{ab}+b}{3} \tag{10}$$

are respectively called Logarithmic mean, Identric mean and Power mean and Heron mean.

Definition 1 *([7]) For positive numbers a and b , r be a positive real number and $\mu \in (-2,\infty)$. Then $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ are defined as*

$$Gn_{\mu,r}(a,b) = \begin{cases} \frac{2}{\mu+2}A(a,b) + \frac{\mu}{\mu+2}M_r(a,b) & r \neq 0 \\ \frac{2}{\mu+2}A(a,b) + \frac{\mu}{\mu+2}G(a,b) & r = 0 \end{cases} \tag{11}$$

and

$$gn_{\mu,r}(a,b) = \begin{cases} M_r^{\frac{\mu}{\mu+2}}(a,b)A^{\frac{\mu}{\mu+2}}(a,b) & r \neq 0 \\ G^{\frac{\mu}{\mu+2}}(a,b)A^{\frac{\mu}{\mu+2}}(a,b) & r = 0 \end{cases}. \tag{12}$$

Definition 2 *([6]) Let $\alpha \in [0,1]$ and $r \geq 0$, then r^{th} Oscillatory mean and its dual are defined by*

$$O = O(a,b;\alpha,r) = \alpha M_r(a,b) + (1-\alpha)\, A(a,b) \tag{13}$$

and

$$o = o(a,b;\alpha,r) = M_r^{\alpha}(a,b) A^{1-\alpha}(a,b). \tag{14}$$

Let us conclude the introduction by a brief description of the contents of the paper. Section 2 contains new inequalities involving Greek means and other means and its proof are given. Also, we present table1 contain the best possible value of important means with $Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$, power mean, Oscillatory mean and r^{th} Oscillatory mean.Finally, Section 3 contains partial derivatives and consequences of symmetric mean, α values for important means are tabulated in Table 2 and two remarks.

2 Some Inequalities

Theorem 1 *For μ_1, $\mu_2 \neq -2$, $r \neq 0, 3$ and if $\mu_1 \leq \frac{4}{r-3} \leq \mu_2$, then*

$$(i)\ gn_{\mu_2,r}(a,b) \leq F_4(a,b) \leq Gn_{\mu_1,r}(a,b). \tag{15}$$

Furthermore $\mu_1 = \mu_2 = -\frac{4}{r-3}$ is the best possible for (15).

$$(ii)\ gn_{\mu_2,0}(a,b) \leq F_4(a,b) \leq Gn_{\mu_1,0}(a,b). \tag{16}$$

Furthermore $\mu_1 = \mu_2 = -\frac{4}{3}$ is the best possible for (16).

Proof. Applying Taylor's theorem and by setting $a = x = t+1$ and $b = 1$, we have

$$F_4(x,1) = F_4(t+1,1) = 1 + \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} - ...$$

$$Gn_{\mu_1,r}(x,1) = Gn_{\mu_1,r}(t+1,1) = 1 + \frac{t}{2} - \frac{(1-r)\,\mu_1}{(\mu_1+2)\,8}t^2 + ...$$

$$gn_{\mu_2,r}(x,1) = gn_{\mu_2,r}(t+1,1) = 1 + \frac{t}{2} - \frac{(1-r)\,\mu_2}{(\mu_2+2)\,8}t^2 + ...$$

Consider $gn_{\mu_2,r}(a,b) \leq F_4(a,b) \leq Gn_{\mu_1,r}(a,b) - \frac{(1-r)\mu_2}{(\mu_2+2)8} \leq \frac{1}{4} \leq \frac{(1-r)\mu_1}{(\mu_1+2)8}$

with simple manipulation we have $\mu_1 \leq \frac{4}{r-3} \leq \mu_2$, Hence the proof of (15) and (16).

Theorem 2 *For $\mu_1,\ \mu_2 \neq -2$, $r \neq 0,2$ and if $\mu_1 \leq \frac{2}{r-2} \leq \mu_2$, then*

(17) *(i)* $gn_{\mu_2,r}(a,b) \leq F_5(a,b) \leq Gn_{\mu_1,r}(a,b)$.

Furthermore $\mu_1 = \mu_2 = \frac{2}{r-2}$ is the best possible for (17)

(18) *(ii)* $gn_{\mu_2,0}(a,b) \leq F_5(a,b) \leq Gn_{\mu_1,0}(a,b)$.

Furthermore $\mu_1 = \mu_2 = -1$ is the best possible for (18).

Corollary 1 *For $\mu_1,\ \mu_2 \neq -2$, $r \neq 0,2$ and if $\mu_1 \leq \frac{2}{r-2} \leq \mu_2$, then*

(19) *(i)* $gn_{\mu_2,r}(a,b) \leq F_6(a,b) = F_5(a,b) \leq Gn_{\mu_1,r}(a,b)$.

Furthermore $\mu_1 = \mu_2 = \frac{2}{r-2}$ is the best possible for (19).

(20) *(ii)* $gn_{\mu_2,0}(a,b) \leq F_6(a,b) = F_5(a,b) \leq Gn_{\mu_1,0}(a,b)$.

Furthermore $\mu_1 = \mu_2 = -1$ is the best possible for (20).

Theorem 3 *Let $\alpha_1,\ \alpha_2 \in [0,1]$, $r \neq 0,1$, if $\alpha_1 \leq \frac{2}{r-1} \leq \alpha_2$, then*

$$O(a,b;\alpha_1,r) \geq F_4(a,b) \geq o(a,b;\alpha_2,r). \tag{21}$$

Furthermore $\alpha_1 = \alpha_2 = \frac{2}{r-1}$ is the best possible for (21).

Proof. Applying Taylor's theorem and by setting $a = x = t+1$ and $b = 1$, we have

$$F_4(x,1) = F_4(t+1,1) = 1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} - ...$$
$$O(a,b;\alpha,r) = 1 + \frac{t}{2} - \frac{\alpha\,(1-r)}{8}t^2 + ...$$
$$o(a,b;\alpha,r) = 1 + \frac{t}{2} - \frac{\alpha\,(1-r)}{8}t^2 + ...$$

Consider $\alpha_1 \leq \frac{2}{r-1} \leq \alpha_2$. With simple manipulations we get

$$\begin{aligned} -\frac{\alpha_1\,(1-r)}{8} &\geq \frac{1}{4} \geq -\frac{\alpha_2\,(1-r)}{8} \\ 1 + \frac{t}{2} - \frac{\alpha_1\,(1-r)}{8}t^2 + ... &\geq 1 + \frac{t}{2} + \frac{1}{4}t^2 + ... \geq 1 + \frac{t}{2} - \frac{\alpha_2\,(1-r)}{8}t^2 + ... \\ O(a,b;\alpha_1,r) &\geq F_4(a,b) \geq o(a,b;\alpha_2,r). \end{aligned}$$

Furthermore $\alpha_1 = \alpha_2 = \frac{2}{r-1}$ is the best possible (for 21).

Theorem 4 *Let $\alpha_1,\ \alpha_2 \in [0,1]$, $r \neq 0,1$, if $\alpha_1 \leq \frac{1}{r-1} \leq \alpha_2$, then*

$$O(a,b;\alpha_1,r) \geq F_5(a,b) \geq o(a,b;\alpha_2,r). \tag{22}$$

Furthermore $\alpha_1 = \alpha_2 = \frac{1}{r-1}$ is the best possible for (22).

Corollary 2 *Let $\alpha_1,\ \alpha_2 \in [0,1]$, $r \neq 0,1$, if $\alpha_1 \leq \frac{1}{r-1} \leq \alpha_2$, then*

$$O(a,b;\alpha_1,r) \geq F_5(a,b) = F_6(a,b) \geq o(a,b;\alpha_2,r). \tag{23}$$

Furthermore $\alpha_1 = \alpha_2 = \frac{1}{r-1}$ is the best possible for (23).

The proofs of the following remarks are obvious.

Remark 1

$$F_3 \leq F_2 \leq L \leq M_{1/3} \leq M_{1/2} \leq H \leq M_{2/3} \leq F_1 \leq F_6 \leq F_5 \leq F_4.$$

Remark 2

$$F_3 \leq F_2 \leq L \leq I \leq F_1 \leq F_6 \leq F_5 \leq F_4.$$

The following table gives the best possible value of important means with $Gn_{\mu,r}(a,b)$ and $gn_{\mu_2,r}(a,b)$ power mean, oscillatory mean and r^{th} oscillatory mean.

Table 1

Important means	$Gn_{\mu,0}(a,b)$	$Gn_{\mu,r}(a,b)$	$O(a,b;\alpha,r)$	$O(a,b;\alpha)$	$M_r(a,b)$
Arithmetic mean	0	0	0	0	1
Geometric mean	∞	$\frac{-2}{r}$	$\frac{1}{1-r}$	1	0
Contra Harmonic mean	$\frac{-4}{3}$	$\frac{4}{r-3}$	$\frac{2}{r-1}$	-2	3
I Contra Geometric mean	-1	$\frac{2}{r-2}$	$\frac{1}{1-r}$	-1	2
II Contra Geometric mean	-1	$\frac{2}{r-2}$	$\frac{1}{1-r}$	-1	2
Logarithmic mean	4	$\frac{4}{1-3r}$	$\frac{2}{3(1-r)}$	$\frac{2}{3}$	$\frac{1}{3}$
Identric Mean	1	$\frac{2}{2-3r}$	$\frac{1}{3(1-r)}$	$\frac{1}{3}$	$\frac{2}{3}$
Heron Mean	1	$\frac{2}{2-3r}$	$\frac{1}{3(1-r)}$	$\frac{1}{3}$	$\frac{2}{3}$
Power Mean	-1	$\frac{2(1-r)}{r}$	1	$1-r$	$---.$

3 Partial Derivatives and Consequences

For a symmetric mean $M(a,b)$ the partial derivatives are exist, then we have

$$M_a(c,c) + M_b(c,c) = 1 \tag{24}$$

$$M_a(c,c) \geq 0 \text{ and } M_b(c,c) \geq 0 \tag{25}$$

$$0 \leq M_a(c,c) \leq 1 \text{ and } 0 \leq M_b(c,c) \leq 1 \tag{26}$$

(26) property does not hold for arbitrary point.

$$M_a(c,c) = M_b(c,c) = \frac{1}{2} \tag{27}$$

$$M_{aa}(c,c) + 2M_{ab}(c,c) + M_{bb}(c,c) = 0 \tag{28}$$

$$M_{aa}(c,c) = -M_{ab}(c,c) = M_{bb}(c,c) \tag{29}$$

$$M_{aa}(c,c) = \frac{\alpha}{c} \text{ where } \alpha \text{ in } \mathbb{R}. \tag{30}$$

The proofs of the above results are obtained by simple direct computations.

Table 2

Important means	Notation	The value of 'α'
Arithmetic Mean	$F_1(a,b)$	0
Geometric mean	$F_2(a,b)$	$\frac{-1}{4}$
Harmonic Mean	$F_3(a,b)$	$\frac{-1}{2}$
Logarithmic Mean	$L(a,b)$	$\frac{-1}{6}$
Heron Mean	$h(a,b)$	$\frac{-1}{12}$
Identric Mean	I(a, b)	$\frac{-1}{12}$
Power Mean	$M_r(a,b)$	$\frac{r-1}{4}$
Contra Harmonic mean	$F_4(a,b)$	$\frac{1}{2}$
First Contra Geometric mean	$F_5(a,b)$	$\frac{1}{4}$
Second Contra Geometric mean	$F_6(a,b)$	$\frac{1}{4}$
Oscillatory mean	$o(a,b;\alpha)$	$\frac{-\alpha}{4}$
r^{th} Oscillatory mean	$o(a,b;r,\alpha)$	$\frac{-\alpha(1-r)}{4}$
Definition1.	$Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ $(r=0)$	$\frac{-\mu(1-r)}{4(\mu+2)}$
Definition1.	$Gn_{\mu,r}(a,b)$ and $gn_{\mu,r}(a,b)$ $(r \neq 0)$	$\frac{-\mu}{4(\mu+2)}$

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