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Quenching for semidiscretizations of a parabolic equation with a nonlinear boundary condition [1]

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Abstract

This paper concerns the study of the numerical approximation for the following initial-boundary value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \quad u_x(1,t) = (1 - u(1,t))^{-p}, \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \end{cases}$$

where $p > 0$. We obtain some conditions under which the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time. Finally, we give some numerical results to illustrate our analysis.

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1 Introduction

In this paper, we are interested in the numerical approximation for the following initial-boundary value problem

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 0, \quad u_x(1,t) = (1-u(1,t))^{-p}, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) \geq 0, \quad 0 \leq x \leq 1, \tag{3}$$

where $p > 0$, $u_0 \in C^2([0,1])$, $u_0'(x) > 0$, $u_0''(x) > 0$, $x \in (0,1)$, $u_0(0) = 0$, $u_0'(1) = (1-u_0(1))^{-p}$.

The particularity of this kind of problem is that the flux on the boundary admits a singularity at the point 1 and the solution u may reach this value in a finite time T. In this case, we say that u quenches in a finite time and the time T is called the quenching time of u. The solutions which quench in a finite time have been the subject of investigations of many authors (see [2], [4]–[7], [10], [11], [13]–[15], [20], [21] and the references cited therein). Under the conditions given on the initial data, the authors have proved that the solution u of (1)–(3) quenches in a finite time and given some estimations of the quenching time (see, for instance [15]). The condition $u_0''(x) > 0$, $x \in (0,1)$, allows the solution u of (1)–(3) to increase with respect to the second variable and the hypothesis $u_0'(x) > 0$, $x \in (0,1)$ permits the solution u to quench at the point $x = 1$.

In this paper, we are interested in the numerical study of the phenomenon of quenching. We start by the construction of a semidiscrete scheme as follows. Let I be a positive integer, and define the grid $x_i = ih$, $0 \leq i \leq I$, where $h = 1/I$. We approximate the solution u of (1)–(3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t), \quad 1 \leq i \leq I-1, \quad t \in (0, T_q^h), \tag{4}$$

$$U_0(t) = 0, \quad \frac{dU_I(t)}{dt} = \delta^2 U_I(t) + \frac{2}{h}(1 - U_I(t))^{-p}, \quad t \in (0, T_q^h), \tag{5}$$

$$U_i(0) = \varphi_i, \quad 0 \leq i \leq I, \tag{6}$$

where $\varphi_0 = 0, \quad \varphi_i > 0, \quad 1 \leq i \leq I,$

$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}, \quad \delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}.$$

Here, $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$ where $\|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|$. If T_q^h is finite, then we say that $U_h(t)$ quenches in a finite time, and the time T_q^h is called the semidiscrete quenching time of $U_h(t)$.

In this paper, we give some conditions under which the solution of (4)–(6) quenches in a finite time and estimate its semidiscrete quenching time.

We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. Nabongo and Boni have obtained in [19] similar results considering (1)–(3) in the case where the first boundary condition in (2) is replaced by $u_x(0, t) = 0$. Thus, the results found in the present paper generalize those obtained in [19], but let us notice that this is not a simple generalization. In fact, because of the condition $u(0, t) = 0$ in (2), the methods used in [19] can not be applied directly. So we utilize other methods. Our work was also motived by the papers in [1] and [3] where the authors have proved similar results about the blow-up phenomenon considering a semilinear parabolic equation with Dirichlet boundary conditions (we say that a solution blows up in a finite time if it takes an infinite value in a finite time). Also, previously in [3] the phenomenon of extinction is studied by numerical methods where a semilinear parabolic equation with Dirichlet boundary conditions is considered (we say that a solution extincts in a finite time it reaches the value zero in a finite time).

This paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle. In the third section, under some conditions, we prove that the solution of (4)–(6) quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we study the convergence of the semidiscrete quenching time. Finally, in the last section, we give some numerical results to illustrate our analysis.

2 Properties of the semidiscrete problem

In this section, we give some lemmas which will be used later.

The following lemma reveals a property of the operator δ^2.

Lemma 1 *Let V_h, $U_h \in \mathbb{R}^{I+1}$. If $\delta^-(U_I)\delta^-(V_I) \geq 0$,*

$$\delta^+(U_i)\delta^+(V_i) \geq 0 \quad \text{and} \quad \delta^-(U_i)\delta^-(V_i) \geq 0, \quad 1 \leq i \leq I-1,$$

then

$$\delta^2(U_iV_i) \geq U_i\delta^2 V_i + V_i\delta^2 U_i, \quad 1 \leq i \leq I,$$

where $\delta^+(U_i) = \frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i) = \frac{U_{i-1}-U_i}{h}$.

Proof. A straightforward computation yields

$$\delta^2(U_iV_i) = \delta^+(U_i)\delta^+(V_i) + \delta^-(U_i)\delta^-(V_i) + U_i\delta^2 V_i + V_i\delta^2 U_i, \ 1 \leq i \leq I-1,$$

$$\delta^2(U_IV_I) = 2\delta^-(U_I)\delta^-(V_I) + U_I\delta^2 V_I + V_I\delta^2 U_I.$$

Using the assumptions of the lemma, we obtain the desired result.

The result below reveals a property of the semidiscrete solution.

Lemma 2 *Let $U_h(t)$ be the solution of (4)–(6). Then, we have $U_i(t) > 0$, $1 \leq i \leq I$, $t \in (0, T_q^h)$.*

Proof. Let t_0 be the first $t > 0$ such that $U_i(t) > 0$ for $t \in [0, t_0)$, $1 \leq i \leq I$, but $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, ..., I\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the equality. We have

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} > 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$
$$\delta^2 U_{i_0}(t_0) = \frac{2U_{I-1}(t_0) - 2U_I(t_0)}{h^2} > 0 \quad \text{if} \quad i_0 = I.$$

We deduce that

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) < 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$
$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) - \frac{2}{h}(1 - U_{i_0}(t_0))^{-p} < 0 \quad \text{if} \quad i_0 = I,$$

which contradicts (4)–(5). This ends the proof.

The following lemma shows another property of the semidiscrete solution.

Lemma 3 *Let $U_h(t)$ be the solution of (4)–(6) such that the initial data at (6) satisfies $\delta^+\varphi_i > 0$, $0 \leq i \leq I-1$. Then, we have $\delta^+ U_i(t) > 0$, $0 \leq i \leq I-1$, $t \in (0, T_q^h)$.*

Proof. Let t_0 be the first $t > 0$ such that $\delta^+ U_i(t) > 0$, $0 \leq i \leq I-1$, $t \in [0, t_0)$ but $\delta^+ U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, ..., I\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the equality. If $i_0 = 0$, then we have $U_1(t_0) = U_0(t_0) = 0$, which contradicts Lemma 2. Put $Z_{i_0}(t) = U_{i_0+1}(t) - U_{i_0}(t)$. We have

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0 \quad \text{if} \quad 1 \leq i_0 \leq I-2,$$
$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0-1}(t_0) - 3Z_{i_0}(t_0)}{h^2} > 0 \quad \text{if} \quad i_0 = I-1,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) < 0 \quad \text{if} \quad 1 \leq i_0 \leq I-2,$$

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) - \frac{2}{h}(1 - U_{i_0}(t_0))^{-p} < 0 \quad \text{if} \quad i_0 = I - 1.$$

Therefore, we have a contradiction because of (4)–(5) and the proof is complete.

The above lemma reveals that if the initial data of the semidiscrete solution is increasing in space, then the semidiscrete solution is also increasing in space. This property will be used later to show that the semidiscrete solution attains its maximum at the last node.
The result below shows another property of the operator δ^2.

Lemma 4 *Let $U_h \in \mathbb{R}^{I+1}$ such that $\|U_h\|_\infty < 1$. Then, we have*

$$\delta^2(1 - U_i)^{-p} \geq p(1 - U_i)^{-p-1}\delta^2 U_i, \quad 1 \leq i \leq I.$$

Proof. Apply Taylor's expansion to obtain

$$\delta^2(1 - U_i)^{-p} = p(1 - U_i)^{-p-1}\delta^2 U_i + (U_{i+1} - U_i)^2\frac{p(p+1)}{2h^2}\theta_i^{-p-2}$$

$$+(U_{i-1} - U_i)^2\frac{p(p+1)}{2h^2}\eta_i^{-p-2} \quad \text{if} \quad 1 \leq i \leq I-1,$$

$$\delta^2(1 - U_I)^{-p} = p(1 - U_I)^{-p-1}\delta^2 U_I + (U_{I-1} - U_I)^2\frac{p(p+1)}{h^2}\eta_I^{-p-2},$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i the one between U_i and U_{i-1}. Use the fact that $\|U_h\|_\infty < 1$ to complete the rest of the proof.

The following lemma is a semidiscrete version of the maximum principle.

Lemma 5 *Let $a_h(t) \in C^0([0,T), \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T), \mathbb{R}^{I+1})$ such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + a_i(t)V_i(t) \geq 0, \quad 1 \leq i \leq I, \quad t \in (0,T), \tag{7}$$

$$V_0(t) \geq 0, \quad t \in (0,T), \tag{8}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{9}$$

Then, we have $V_i(t) \geq 0$ for $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t}V_h(t)$, where λ is such that $a_i(t) - \lambda > 0$ for $t \in [0,T_0]$, $0 \leq i \leq I$. Let $m = \min_{0\leq i\leq I, 0\leq t\leq T_0} Z_i(t)$. Since for $i \in \{0,...,I\}$, $Z_i(t)$ is a continuous function, there exists $t_0 \in [0,T_0]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in \{0,...,I\}$. It is not hard to see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \tag{10}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = I, \tag{11}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1. \tag{12}$$

Using (7), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I. \tag{13}$$

Due to (10)–(13), we arrive at $(a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$, $1 \leq i_0 \leq I$. Taking into account (8) and the fact that $a_{i_0}(t_0) - \lambda > 0$, we deduce that $V_h(t) \geq 0$ for $t \in [0,T_0]$, which leads us to the desired result.

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 6 *Let $V_h(t), U_h(t) \in C^1([0,T), \mathbb{R}^{I+1})$ and $f \in C^0(\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that for $t \in (0,T)$,*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t),t) < \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_i(t),t),\ 1 \le i \le I, \tag{14}$$

$$V_0(t) < U_0(t), \tag{15}$$

$$V_i(0) < U_i(0), \quad 0 \le i \le I. \tag{16}$$

Then, we have $V_i(t) < U_i(t)$, $0 \le i \le I$, $t \in (0,T)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t \in (0,T)$ such that $Z_h(t) > 0$ for $t \in [0,t_0)$, but

$$Z_{i_0}(t_0) = 0 \quad \text{for a certain } i_0 \in \{0,...,I\}.$$

We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0 \quad \text{if} \quad 1 \le i_0 \le I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 \quad \text{if} \quad i_0 = I,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0),t_0) - f(V_{i_0}(t_0),t_0) \le 0 \quad \text{if} \quad 1 \le i_0 \le I.$$

But, this inequality contradicts (14). If $i_0 = 0$, then we have a contradiction because of (15), and the proof is complete.

3 Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (4)–(6) quenches in a finite time and estimate its semidiscrete quenching time.

Our result about the quenching time is the following.

Theorem 1 *Let U_h be the solution of (4)–(6). Assume that there exists a constant $A > 0$ such that the initial data at (6) satisfies*

$$\delta^2\varphi_i + (1-\varphi_i)^{-p} \geq A\sin(ih\frac{\pi}{2})(1-\varphi_i)^{-p}, \quad 0 \leq i \leq I, \tag{17}$$

and

$$1 - \frac{\pi^2}{2A(p+1)}(1-\|\varphi_h\|_\infty)^{p+1} > 0. \tag{18}$$

Then, under the hypothesis of Lemma 3, the solution $U_h(t)$ quenches in a finite time T_q^h and the following estimate holds

$$T_q^h < -\frac{2}{\pi^2}\ln(1 - \frac{\pi^2}{2A(p+1)}(1-\|\varphi_h\|_\infty)^{p+1}). \tag{19}$$

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ such that

$$J_i(t) = \frac{dU_i(t)}{dt} - C_i(t)(1-U_i(t))^{-p}, \quad 0 \leq i \leq I,$$

where $C_i(t) = Ae^{-\lambda_h t}\sin(ih\frac{\pi}{2})$ with $\lambda_h = \frac{2-2\cos(\frac{\pi}{2}h)}{h^2}$. A straightforward computation gives

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) = \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) - pC_i(t)(1-U_i)^{-p-1}\frac{dU_i(t)}{dt} - \frac{dC_i(t)}{dt}(1-U_i)^{-p}$$

$$+\delta^2(C_i(t)(1-U_i)^{-p}), \quad 1 \leq i \leq I.$$

Obviously $\delta^+(C_i) > 0$, $0 \leq i \leq I-1$. From Lemmas 1, 3 and 4, we get

$$\delta^2(C_i(1-U_i)^{-p}) \geq pC_i(1-U_i)^{-p-1}\delta^2 U_i + (1-U_i)^{-p}\delta^2 C_i, \quad 1 \leq i \leq I.$$

Hence, we deduce that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \leq \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) - pC_i(t)(1-U_i)^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t))$$

$$-(1-U_i)^{-p}(\frac{dC_i(t)}{dt} - \delta^2 C_i(t)), \quad 1 \leq i \leq I.$$

It is not hard to see that $C_0(t) = 0$, $\frac{dC_i(t)}{dt} - \delta^2 C_i(t) = 0$, $1 \leq i \leq I$. Therefore using (4)–(6), we arrive at

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \leq 0, \quad 1 \leq i \leq I-1,$$

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \leq p(1-U_I(t))^{-p-1} J_I(t).$$

Obviously $J_0(t) = 0$ and from (17), $J_h(0) \geq 0$. We deduce from Lemma 5 that $J_h(t) \geq 0$ for $(0, T_q^h)$. We observe that $\lambda_h \leq \frac{\pi^2}{2}$ for h small enough. Therefore, we obtain

$$(1-U_I)^p dU_I \geq Ae^{-\frac{\pi^2}{2}t}dt \quad \text{for} \quad (0, T_q^h). \tag{20}$$

From Lemma 3, $\|U_h(t)\|_\infty = U_I(t)$. Integrating (20) over $(0, T_q^h)$ and using the fact that $\|U_h(0)\|_\infty = \|\varphi_h\|_\infty$, we arrive at

$$T_q^h < -\frac{2}{\pi^2}\ln(1 - \frac{\pi^2}{2A(p+1)}(1-\|\varphi_h\|_\infty)^{p+1}).$$

Taking into account (18), we see that T_q^h is finite and the proof is complete.

Remark 1 *Suppose that there exists a time $t_0 \in (0, T_q^h)$ such that*

$$1 - \frac{\pi^2}{2A(p+1)} e^{\frac{\pi^2}{2} t_0} (1 - \|U_h(t_0)\|_\infty)^{p+1} > 0.$$

Integrating the inequality (20) over (t_0, T_q^h) and using the fact that $\|U_h(t_0)\|_\infty = U_I(t_0)$, we find that

$$T_q^h - t_0 < -\frac{2}{\pi^2} \ln(1 - \frac{\pi^2}{2A(p+1)} e^{\frac{\pi^2}{2} t_0} (1 - \|U_h(t_0)\|_\infty)^{p+1}).$$

4 Convergence of the semidiscrete quenching time

In this section, under some assumptions, we show that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. We denote $u_h(t) = (u(x_0, t), ..., u(x_I, t))^T$.

We need the following result about the convergence of our scheme.

Theorem 2 *Assume that the problem (1)–(3) has a solution $u \in C^{4,1}([0,1] \times [0,T])$ such that $\sup_{t \in [0,T]} \|u(\cdot, t)\|_\infty = \alpha < 1$ and*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{21}$$

Then, for h sufficiently small, the problem (4)–(6) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that

$$\max_{0 \le t \le T} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad as \quad h \to 0. \tag{22}$$

Proof. We take $\rho > 0$ such that $\rho + \alpha < 1$. Let L be such that

$$2p(1 - \rho - \alpha)^{-p-1} \le L. \tag{23}$$

The problem (4)–(6) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \rho \quad \text{for} \quad t \in (0, t(h)). \tag{24}$$

The relation (21) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \leq \|u(\cdot, t)\|_\infty + \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0, t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leq \rho + \alpha < 1 \quad \text{for} \quad t \in (0, t^*(h)). \tag{25}$$

Apply Taylor's expansion to obtain

$$\delta^2 u(x_i, t) = u_{xx}(x_i, t) + \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t), \quad 1 \leq i \leq I - 1,$$

$$\delta^2 u(x_I, t) = -\frac{2}{h}(1 - u(x_I, t))^{-p} + u_{xx}(x_I, t) - \frac{h}{3} u_{xxx}(\widetilde{x}_I, t),$$

which implies that

$$u_t(x_i, t) - \delta^2 u(x_i, t) = -\frac{h^2}{12} u_{xxxx}(x_i, t), \quad 1 \leq i \leq I - 1,$$

$$u_t(x_I, t) - \delta^2 u(x_I, t) = \frac{2}{h}(1 - u(x_I, t))^{-p} + \frac{h}{3} u_{xxx}(\widetilde{x}_I, t).$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using the mean value theorem, we have for $t \in (0, t^*(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t), \quad 1 \leq i \leq I - 1,$$

$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) = \frac{2}{h} p(1 - \theta_I(t))^{-p-1} e_I(t) - \frac{h}{3} u_{xxx}(\widetilde{x}_I, t),$$

where $\theta_I(t)$ is an intermediate value between $U_I(t)$ and $u(x_I,t)$. Since $u \in C^{4,1}$, using (23) and (25), there exists a positive constant K such that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq Kh^2, \quad 1 \leq i \leq I-1, \tag{26}$$

$$\frac{de_I(t)}{dt} - \delta^2 e_I(t) \leq \frac{L|e_I(t)|}{h} + hK. \tag{27}$$

Now, consider the function $z(x,t)$ defined as follows

$$z(x,t) = e^{((M+1)t+Cx^2)}(\|\varphi_h - u_h(0)\|_\infty + Qh) \quad \text{in} \quad [0,1]\times[0,T],$$

where M, C, Q are positive constants which will be determined later. We observe that

$$z_t = (M+1)z,$$

$$z_x = 2Cxz,$$

$$z_{xx} = (2C + 4C^2x^2)z,$$

$$z_{xxx} = (12C^2x + 8C^3x^3)z,$$

$$z_{xxxx} = (12C^2 + 48C^3x^2 + 16C^4x^4)z.$$

A direct calculation reveals that

$$z_t(x_i,t) - z_{xx}(x_i,t) = (M+1-2C-4C^2x_i^2)z(x_i,t), \quad 1 \leq i \leq I.$$

On the other hand, use Taylor's expansion to obtain

$$\delta^2 z(x_i,t) = z_{xx}(x_i,t) + \frac{h^2}{12}z_{xxxx}(\widetilde{x}_i,t), \quad 1 \leq i \leq I-1,$$

$$\delta^2 z(x_I,t) = -\frac{4C}{h}z(x_I,t) + z_{xx}(x_I,t) - \frac{h}{3}z_{xxx}(\widetilde{x}_I,t),$$

which implies that

$$z_t(x_i,t)-\delta^2 z(x_i,t) = (M+1-2C-4C^2x_i^2)z(x_i,t)-\frac{h^2}{12}z_{xxxx}(\widetilde{x}_i,t),\ 1\le i\le I-1,$$

$$z(x_0,t)>0,$$

$$z_t(x_I,t)-\delta^2 z(x_I,t) = (M+1-2C-4C^2)z(x_I,t)+\frac{4C}{h}z(x_I,t)+\frac{h}{3}z_{xxx}(\widetilde{x}_I,t).$$

Since $z(x,t)\ge Qh$ for $(x,t)\in[0,1]\times[0,T]$, we may choose M, C, Q such that

$$\frac{dz(x_i,t)}{dt} > \delta^2 z(x_i,t)+Kh^2,\quad 1\le i\le I-1,\quad t\in(0,t^*(h)), \tag{28}$$

$$\frac{dz(x_I,t)}{dt} > \delta^2 z(x_I,t)+\frac{L}{h}|z(x_I,t)|+Kh,\quad t\in(0,t^*(h)), \tag{29}$$

$$z(x_0,t)>e_0(t),\quad t\in(0,t^*(h)), \tag{30}$$

$$z(x_i,0)>e_i(0),\quad 0\le i\le I. \tag{31}$$

Applying Comparison Lemma 6, we arrive at

$$z(x_i,t)>e_i(t)\quad\text{for}\quad t\in(0,t^*(h)),\quad 0\le i\le I.$$

In the same way, we also prove that

$$z(x_i,t)>-e_i(t)\quad\text{for}\quad t\in(0,t^*(h)),\quad 0\le i\le I,$$

which implies that

$$\|U_h(t)-u_h(t)\|_\infty \le e^{(Mt+C)}(\|\varphi_h-u_h(0)\|_\infty+Qh),\quad t\in(0,t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (24), we obtain

$$\frac{\varrho}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(MT+C)}(\|\varphi_h - u_h(0)\|_\infty + Qh). \tag{32}$$

Since the term in the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\varrho}{2} \leq 0$, which is impossible. Consequently $t^*(h) = T$, and the proof is complete.

Now, we are able to prove the following.

Theorem 3 *Suppose that the problem (1)–(3) has a solution u which quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0,T_q))$. Assume that the initial data at (6) satisfies the condition (21). Under the assumptions of Theorem 1, the problem (4)–(6) admits a unique solution U_h which quenches in a finite time T_q^h and we have $\lim_{h\to 0} T_q^h = T_q$.*

Proof. Let $\varepsilon \in (0, T_q/2)$. There exists a constant $\rho \in (0,1)$ such that

$$-\frac{1}{2\pi^2}\ln(1 - \frac{4\pi^2}{A(p+1)}e^{2\pi^2 T_q}(1-y)^{p+1}) < \frac{\varepsilon}{2} \quad \text{for} \quad y \in [1-\rho, 1). \tag{33}$$

Since u quenches at the time T_q, there exists $T_1 \in (T_q - \frac{\varepsilon}{2}, T_q)$ such that $1 > \|u(\cdot,t)\|_\infty \geq 1 - \frac{\varrho}{2}$ for $t \in [T_1, T_q)$. From Theorem 2, we know that the problem (4)–(6) admits a unique solution $U_h(t)$ such that the following estimate holds

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad t \in [0, T_2]$$

where $T_2 = \frac{T_1+T_q}{2}$. Using the triangle inequality, we get

$$\|U_h(t)\|_\infty \geq \|u_h(t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty \geq 1 - \frac{\rho}{2} - \frac{\rho}{2} \geq 1 - \rho \quad \text{for} \quad t \in [0, T_2],$$

which implies that $\|U_h(T_2)\|_\infty \geq 1-\rho$. Due to (33), it is not hard to see that

$$-\frac{1}{2\pi^2}\ln(1-\frac{4\pi^2}{A(p+1)}e^{2\pi^2 T_2}(1-\|U_h(T_2)\|_\infty)^{p+1}) < \frac{\varepsilon}{2}. \tag{34}$$

From Theorem 1, $U_h(t)$ quenches at the time T_q^h. Using (34) and Remark 1, we arrive at

$$|T_q^h - T_q| \quad \leq \quad |T_q^h - T_2| + |T_2 - T_q| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,$$

which leads us to the desired result.

5 Numerical experiments

In this section, we give some computational results about the approximation of the real quenching time. We consider the problem (1)–(3) in the case where $p = 1$ and $u_0(x) = \frac{1}{2}x^4$. For our numerical experiments, we propose some adaptive schemes as follows. Firstly, we approximate the solution u of (1)–(3) by the solution $U_h^{(n)}$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \leq i \leq I-1,$$

$$U_0^{(n)} = 0, \quad \frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + \frac{2}{h}(1-U_I^{(n)})^{-p},$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,$$

where $n \geq 0$. In order to permit the discrete solution to reproduce the property of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \{\frac{h^2}{2}, h^2(1-\|U_h^{(n)}\|_\infty)^{p+1}\}.$$

We also approximate the solution u of (1)–(3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2}, \quad 1 \le i \le I-1,$$

$$U_0^{(n)} = 0, \quad \frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + \frac{2}{h}(1 - U_I^{(n)})^{-p},$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I,$$

where $n \ge 0$. As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}.$$

In both cases, we take $\varphi_i = \frac{1}{2}(ih)^4$.
We need the following definition.

Definition 1 *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to+\infty} \|U_h^{(n)}\|_\infty = 1$, but the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical quenching time.*

In the tables 1 and 2, in rows, we present the numerical quenching times, numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders (s) of the approximations obtained with the explicit scheme

I	T^n	n	$CPUtime$	s
16	0.025538	163	-	-
32	0.023834	422	-	-
64	0.023270	1236	1	1.60
128	0.023093	4086	17	1.67
256	0.023039	14734	506	1.71

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders (s) of the approximations obtained with the implicit scheme

I	T^n	n	$CPUtime$	s
16	0.026126	164	-	-
32	0.024009	423	-	-
64	0.023317	1238	2	1.62
128	0.023105	4089	35	1.71
256	0.023043	14737	1140	1.77

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