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Subordination by p-valent convex functions [1]

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Abstract

We obtain an interesting subordination relation for analytic p-valent functions by using subordinating factor sequence $(b_k)_1^\infty$.

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1 Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=2}^{\infty} a_k z^{p+k-1}, \tag{1}$$

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which are analytic in the open unit disk $D = \{z : z \in C; |z| < 1\}$ and $p \in N$. Consider also its subclasses $C(p), S^*(p)$ consisting of $p-$valent convex and starlike functions respectivly, where $C(1) \equiv C$, $S^*(1) \equiv S^*$, the classes of univalent convex and starlike functions .

For $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=2}^{\infty} g_k z^{p+k-1}, \tag{2}$$

the convolution (or Hadamard product) of f and g, denoted by $f * g$, is defined by

$$(f * g)(z) = z^p + \sum_{k=2}^{\infty} a_k g_k z^{p+k-1}.$$

Let $g(z)$ given by (2) be a fixed function, with $g_k \geq g_2 > 0$ $(k \geq 2), \gamma < 1$, and let

$$T_g(\gamma) := \{f(z) \in A(p) : \sum_{k=2}^{\infty} |a_k g_k| < p(p-\gamma)\}.$$

Definition 1 *A sequence $(b_k)_1^\infty$ of complex numbers is said to be a subordinating factor sequence if for every convex p-valent function $f(z)$ given by (1)*

$$\sum_{k=2}^{\infty} a_k b_k z^{p+k-1} \prec f(z) \qquad (\prec \textit{ means subordinate}).$$

Theorem 1 *A sequence $(b_k)_1^\infty$ of complex numbers is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=2}^{\infty} b_k z^{k+p-1}\right\} > 0. \tag{3}$$

Proof. Assume that the sequence $(b_k)_1^\infty$ of complex numbers is a subordinating factor sequence. Then

$$\sum_{k=1}^{\infty} b_k z^{p+k-1} \prec \sum_{k=1}^{\infty} z^{p+k-1} = z^p \frac{z}{(1-z)},$$

then

$$\operatorname{Re}\left(\sum_{k=1}^{\infty} b_k z^{p+k-1}\right) \geq \operatorname{Re}\left(\frac{z^p}{(1-z)}\right),$$

and since

$$\operatorname{Re}\left(\frac{z^p}{(1-z)}\right) \geq (-\frac{\cos p\pi}{2}), -1 \leq \cos p\pi \leq 1$$

which is equivalent to

$$\operatorname{Re}\left\{\sum_{k=2}^{\infty} b_k z^{p+k-1}\right\} > -\frac{1}{2}, (|z| < 1).$$

2 Subordination with convex functions

We begin with the following subordination result.

Theorem 2 *If $f(z) \in T_g(\gamma)$ and $h(z) \in C(p)$, then*

(4) $$\frac{g_2}{2(g_2 + p(p-\gamma))}(f * h)(z) \prec h(z),$$

(5) $$\operatorname{Re}(f(z)) > -\frac{(g_2 + p(p-\gamma))}{g_2}, z \in D.$$

The constant factor

$$\frac{g_2}{2(g_2 + p(p-\gamma))}$$

in the subordination result (4) cannot be replaced by a larger number.

Proof. Let $G(z) = z + \sum_{k=2}^{\infty} g_2 z^{p+k-1}$. Since $T_g(\gamma) \subseteq T_G(\gamma)$, our result follows if we prove the result for the class $T_G(\gamma)$. Let $f(z) \in T_G(\gamma)$ and suppose that

$$h(z) = z^p + \sum_{k=2}^{\infty} c_k z^{p+k-1} \in C(p).$$

In this case,

$$\frac{g_2}{2(g_2 + p(p-\gamma))}(f * h)(z) = \frac{g_2}{2(g_2 + p(p-\gamma))}\left(z^p + \sum_{k=2}^{\infty} a_k c_k z^{p+k-1}\right).$$

Observe that the subordination result (4) holds true if

$$\left(\frac{g_2}{2(g_2 + p(p-\gamma))} a_k\right)_1^{\infty}$$

is a subordinating factor sequence (with $a_1 = 1$). In view of Theorem 2 , this is equivalent to the condition that

$$\operatorname{Re}\left(1 + \sum_{k=1}^{\infty} \frac{g_2}{(g_2 + p(p-\gamma))} a_k z^{p+k-1}\right) > 0. \tag{6}$$

Since $g_k \geq g_2 > 0$ for $k \geq 2$, we have

$$\operatorname{Re}\left(1 + \sum_{k=1}^{\infty} \frac{g_2}{(g_2 + p(p-\gamma))} a_k z^{p+k-1}\right) =$$

$$\operatorname{Re}\left(1 + \frac{g_2}{g_2 + p(p-\gamma)} z^p + \frac{1}{g_2 + p(p-\gamma)} \sum_{k=2}^{\infty} g_2 a_k z^{p+k-1}\right)$$

$$\geq 1 - \left(\frac{g_2}{g_2 + p(p-\gamma)} r^p + \frac{1}{g_2 + p(p-\gamma)} \sum_{k=2}^{\infty} |g_2 a_k|\, r^{p+k-1}\right)$$

$$> 1 - \left(\frac{g_2}{g_2 + p(p-\gamma)} r^p + \frac{p(p-\gamma)}{g_2 + p(p-\gamma)} r^p\right) > 0, (|z| = r < 1).$$

Thus(6) holds true in D, and proves (4). The inequality (5) follows by taking

$$h(z) = p\int_0^z \frac{t^{p-1}}{(1-t)^{2p}}dt = z^p + 2\sum_{k=2}^{\infty} B_k z^{p+k-1}, B_k = \frac{(2p)_{k-p}}{k(k-p)!}$$

in (4).

Now consider the function

$$F(z) = z^p - \frac{p(p-\gamma)}{g_2}z^2, (\gamma < 1).$$

Clearly, $F(z) \in T_g(\gamma)$. For this function $F(z)$, (4) becomes

$$\frac{g_2}{2(g_2 + p(p-\gamma))}F(z) \prec h(z) = p\int_0^z \frac{t^{p-1}}{(1-t)^{2p}}dt,$$

It is easily verified that

$$min\left\{\mathrm{Re}\left(\frac{g_2}{2(g_2 + p(p-\gamma))}F(z)\right)\right\} = \frac{-1}{2}, z \in D.$$

Therefore,

$$\frac{g_2}{2(g_2 + p(p-\gamma))}$$

cannot be replaced by any larger constant.

Corollary 1 *If* $f(z) \in T_p(j, \lambda, \alpha, n)$, *(Alkharsani and Alhajry [1]); and* $h(z) \in C(p)$, *then*

(7)
$$\frac{2^n(2p-\alpha)(\lambda(2p-1)+1)}{2(2^n(2p-\alpha)(\lambda(2p-1)+1) + (p-\alpha)(\lambda(p-1)+1))}(f * h)(z) \prec h(z),$$

$$\mathrm{Re}(f(z)) > -\frac{\left(2^n\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)+\left(p-\alpha\right)\left(\lambda\left(p-1\right)+1\right)\right)}{2^n\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)}, z \in D.$$

The constant factor

$$\frac{2^n\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)}{2\left(2^n\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)+\left(p-\alpha\right)\left(\lambda\left(p-1\right)+1\right)\right)}$$

in the subordination result (7) cannot be replaced by a larger number.

Corollary 2 *If $f(z) \in T(j,p,\lambda,\alpha)$ (Altintas et al. [2]); and $h(z) \in C(p)$, then*

(8)
$$\frac{\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)}{2\left(\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)+\left(p-\alpha\right)\left(\lambda\left(p-1\right)+1\right)\right)}(f*h)(z) \prec h(z),$$

$$\mathrm{Re}(f(z)) > -\frac{\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)+\left(p-\alpha\right)\left(\lambda\left(p-1\right)+1\right)}{\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)}, z \in D.$$

The constant factor

$$\frac{\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)}{2\left(\left(2p-\alpha\right)\left(\lambda\left(2p-1\right)+1\right)+\left(p-\alpha\right)\left(\lambda\left(p-1\right)+1\right)\right)}$$

in the subordination result (8) cannot be replaced by a larger number.

Remark 1 *The case $\lambda = 1$ and $p = 1$ in Corollary 2 was obtained by Rosihan et al. [6]*

Corollary 3 *If $f(z) \in T^*(p,j,\alpha)$ (Owa [5]); and $h(z) \in C(p)$, then*

(9)
$$\frac{\left(2p-\alpha\right)}{2\left(3p-2\alpha\right)}(f*h)(z) \prec h(z),$$

$$\mathrm{Re}(f(z)) > -\frac{\left(3p-2\alpha\right)}{\left(2p-\alpha\right)}, z \in D.$$

The constant factor

$$\frac{\left(2p-\alpha\right)}{2\left(3p-2\alpha\right)}$$

in the subordination result (9) cannot be replaced by a larger number.

Remark 2 *The case $\alpha = 0$ and $p = 1$ in Corollary 3 was obtained by Singh [7].*

Corollary 4 *If $f(z) \in p(j, \lambda, \alpha, n)$, (Aouf and Srivastava [3]); and $h(z) \in C$, then*

$$\frac{2^{n-1}(2-\alpha)(\lambda+1)}{(2^n(2-\alpha)(\lambda+1)+(1-\alpha))}(f*h)(z) \prec h(z), \tag{10}$$

$$\mathrm{Re}(f(z)) > -\frac{(2^n(2-\alpha)(\lambda+1)+(1-\alpha))}{2^{n-1}(2-\alpha)(\lambda+1)}, z \in D.$$

The constant factor

$$\frac{2^{n-1}(2-\alpha)(\lambda+1)}{(2^n(2-\alpha)(\lambda+1)+(1-\alpha))}$$

in the subordination result (10) cannot be replaced by a larger number.

Remark 3 *The case $\lambda = 0$ in Corollary 4 was obtained by Eker et al. [4].*

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