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On sufficient conditions for starlikeness of p-valently Bazilevič functions of the type β and order γ [1]

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Abstract

The aim of this paper is to establish certain sufficient conditions for the class of starlikeness p-valently Bazilevič functions of the type β and order γ. Our result is of general nature and capable of yielding a number of unknown new and interesting results.

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1 Introduction and definitions

Let $\mathcal{A}_n(p)$ be the class of normalized functions of the form

$$(1) \qquad f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \qquad (n \in N := \{1, 2, 3, ...\}),$$

which are *analytic* in the unit disk $\Delta := \{z : |z| < 1\}$. A function $f \in \mathcal{A}_n(p)$ is said to be in the class $\mathcal{S}_n^*(p, \alpha)$, if it satisfies

$$(2) \qquad \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (0 \leq \alpha < p;\ z \in \Delta).$$

A function in the class $\mathcal{S}_n^*(\alpha)$ is starlike of order α in Δ.

Let $\mathcal{K}_n(\alpha)$ be subclass of $\mathcal{A}_n(p)$ consisting of functions $f(z)$ which satisfies

$$(3) \qquad \text{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha \qquad (0 \leq \alpha < p;\ z \in \Delta),$$

where $g(z) \in \mathcal{S}_n(p, 0) := \mathcal{S}_n(p)$.

Also a function $f \in \mathcal{A}_n$ is said to be p-valently Bazilevič function of type β $(\beta \geq 0)$ and order γ $(0 \leq \gamma < p)$, if there exists a function $g(z) \in \mathcal{S}_n^*(p)$, such that

$$(4) \qquad \text{Re}\left\{\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right\} > \gamma \qquad (0 \leq \gamma < p; z \in \Delta).$$

We denote the class of all such functions by $\mathcal{B}_n(\beta, \gamma)$. In particular, when $\beta = 1$, a function $f \in \mathcal{K}_n(\gamma) := \mathcal{B}_n(1, \gamma)$ is p-valently close-to-convex of order γ in Δ. Moreover if $\beta = 0$, then $\mathcal{B}_n(0, \gamma) := \mathcal{S}_n^*(\gamma)$.

In order to prove our main results , we shall require the following lemma.

Lemma 1 *([3]) Let Ω be a set in the complex plane C and suppose that ϕ is a mapping from $C^2 \times \triangle$ to C which satisfies $\phi(ix; y; z) \notin \Omega$ for $z \in \triangle$ and for all real x and y such that $y \geq -n(1+x^2)/2$. If the function $P(z) = 1 + c_n z^n + ...$, is analytic in $\triangle$ and $\phi(P(z); zp(z); z) \in \Omega$ all $z \in \Delta$ then Re $P(z) > 0$.*

2 Main Results

Applying Lemma 1, we now derive the following

Theorem 1 *Let $f(z) \in \mathcal{A}_n(p)$, satisfies If*

$$\mathrm{Re}\left\{\frac{\alpha z f'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}} + \frac{\alpha z f''(z)}{f'(z)} - \alpha(1-\mu)\frac{zf''(z)}{f'(z)} - \alpha\mu\frac{zg'(z)}{g(z)} + 1\right\}\frac{zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}$$

$$> \alpha\beta\left(\beta\delta + \frac{n}{2} - 1\right) + \left(\beta - \frac{np\alpha}{2}\right)(z \in \triangle, 0 \leq \alpha, \beta < p), \tag{5}$$

then $f(z) \in \mathcal{B}_n(p, \mu, \beta)$.

Proof. Define P(z) by

$$\frac{zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}} = (p-\beta)P(z) + \beta, \tag{6}$$

then $P(z) = 1 + c_n z^n + ...$ is analytic in $\triangle$. Differentiate logarithmically (6) with respect to z and making a little simplification, we get

$$\frac{\delta z f'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}} + \frac{zf''(z)}{f'(z)} - (1-\mu)\frac{zf'(z)}{f(z)} - \mu\frac{zg'(z)}{g(z)}$$

$$= \frac{(p-\beta)zP'(z) + \delta\left[(p-\beta)P(z) + \beta\right]^2}{(p-\beta)P(z) + \beta} \tag{7}$$

$$\left[\frac{\alpha\delta zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}+\alpha\frac{zf''(z)}{f'(z)}-\alpha(1-\mu)\frac{zf'(z)}{f(z)}-\alpha\mu\frac{zg'(z)}{g(z)}+1\right]\frac{zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}$$
$$=\alpha(p-\beta)zP'(z)+(2\beta\alpha\delta-\alpha+1)(p-\beta)P(z)+\alpha\delta(p-\beta)^2P^2(z)+\left(\delta\alpha\beta^2-\alpha\beta+\beta\right)$$

$$=\phi\left(P(z),zP'(z);z\right), \tag{8}$$

where

(9)

$$\phi(r,s;t)=\alpha(p-\beta)\delta+(2\beta\alpha\delta-\alpha+1)(p-\beta)r+\alpha\delta(p-\beta)^2r^2+\left(\delta\alpha\beta^2-\alpha\beta+\beta\right)$$

for all real x and y satisfying $y=-n(1+x^2)/2$, we have

$$\text{Re}\phi(ix,y;z)=\alpha(p-\beta)y-\alpha\delta(p-\beta)^2x^2+\beta(\alpha\beta\delta-\alpha+1)$$
$$\leq-\frac{\alpha}{2}(p-\beta)n-\left\{\alpha\delta(p-\beta)^2+\frac{n\alpha}{2}(p-\beta)\right\}x^2+\beta(\alpha\beta\delta-\alpha+1)$$
$$=-\frac{\alpha}{2}(p-\beta)n-\frac{\alpha(p-\beta)}{2}(n+2p\delta-2\beta)x^2+\beta(\alpha\beta\delta-\alpha+1)$$

$$\leq\alpha\beta\left(\beta\delta+\frac{n}{2}-1\right)+\left(\beta-\frac{np\alpha}{2}\right). \tag{10}$$

Let

$$\Omega=\left\{w;\text{Re}w>0,\alpha\beta\left(\beta\delta+\frac{n}{2}-1\right)+\left(\beta-\frac{np\alpha}{2}\right)\right\},$$

then

$$\phi\left(P(z),zP'(z);z\right)\in\Omega\quad and\ \phi(ix,y;z)\notin\Omega.$$

For all real x and $y=-n(1+x^2)/2, z\in\triangle$, By application of lemma (1), the result (5) follows at once.

On taking $\delta=1,\ \mu=0$ in Theorem 1, we get

Corollary 1 *If $f(z) \in A_n(p)$ satisfies*

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha z\frac{f''(z)}{f'(z)}+1\right)\right\} > \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta+\frac{np\alpha}{2}\right), \tag{11}$$

$(z \in \triangle, 0 \le \alpha, 0 \le \beta < p)$ then $f(z) \in \mathcal{S}_n(p,\beta)$.

On taking $\beta = 0$, $n = 1$,$p = 1$ and $\beta = \alpha/2$, $n = 1, p = 1$ respectively we have a known result obtained by Ravichandran et al. [1]. On taking $\mu = 1$ and $\delta = 1$, we find an interesting result contained in the following corollary.

Corollary 2 *If $f(z) \in A_n(p)$, satisfies*

$$\text{Re}\left\{\frac{\alpha zf'(z)}{g(z)}+\frac{\alpha zf''(z)}{f'(z)}-\alpha\frac{zg'(z)}{g(z)}+1\right\}\frac{zf'(z)}{g(z)}$$

$$> \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{np\alpha}{2}\right) \qquad (z \in \Delta, 0 \le \alpha, \beta < p), \tag{12}$$

then $f(z) \in \mathcal{K}_n(p,\beta)$.

Theorem 2 *Let $0 \le \beta < p$,*

$$\lambda = \left(p-\beta+\frac{n}{2}\right)^2(p-\beta)^2, \quad \eta = \left\{\frac{n}{2}(p-\beta)-\left(\delta\beta^2-\beta\right)\right\}^2$$

$$v = (p-\beta)^2+\left(\delta\beta^2-\beta\right) \textit{ and } \quad \sigma = (p-\beta)^2(2\beta-1)^2$$

Also, suppose that t_0 to be the smallest positive root of the equation

$$\begin{aligned}&2\lambda(p-\beta)^2t^3+\left\{(p-\beta)^2(2\lambda+\eta-v+\sigma)+3\lambda\beta^2\right\}t^2\\&+2\beta^2(2\lambda+\eta-v+\sigma)t+(\lambda+2\eta-v+\sigma)\beta^2-(p-\beta)^2\eta=0\end{aligned} \tag{13}$$

and

$$\frac{(p-\beta)^2(1+t_0)}{(p-\beta)^2t_0+\beta^2}\left[\lambda t_0^2+(\lambda+\eta-v+\sigma)t_0+\eta\right]=\rho^2.$$

Now, if

$$\left|\left[\frac{\delta zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}-p\right]\left[\frac{\delta zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}\right]\right|\leq\rho, z\in\Delta$$

then $f(z)\in\mathcal{B}_n(p,\mu,\beta)$.

Proof. Define P(z) by

$$(p-\beta)P(z)+\beta=\left[\frac{zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}\right]$$

then $P(z)=1+c_nz^n+...$, is analytic in $\triangle$. A computation shows that

$$\frac{\delta zf'(z)}{[f(z)]^{1-\mu}[g(z)]^{\mu}}+\frac{zf''(z)}{f'(z)}-(1-\mu)\frac{zf'(z)}{f(z)}-\mu\frac{zg'(z)}{g(z)}$$
$$=\frac{(p-\beta)zP'(z)+\delta\{(p-\beta)P(z)+\beta\}^2-\{(p-\beta)P(z)+\beta\}}{(p-\beta)P(z)+\beta}$$
$$=\varphi\left(P(z),zP'(z);z\right),$$

where

$$\varphi(r,s;t)=\frac{(p-\beta)(r-1)}{(p-\beta)r+\beta}\left[(p-\beta)s+\delta\{(p-\beta)r+\beta\}^2-\{(p-\beta)r+\beta\}\right]$$

For all real x and $y=-n(1+x^2)/2$, we have

$$|\varphi(ix,y;z)|^2=\frac{(p-\beta)^2(1+x^2)}{(p-\beta)^2x^2+\beta^2}\left[\left\{(p-\beta)y-(p-\beta)^2x^2\delta+\delta\beta^2-\beta\right\}^2\right.$$
$$\left.+(p-\beta)^2x^2(2\delta\beta-1)^2\right]=g\left(x^2,y\right)$$

Now

$$\frac{\partial g}{\partial y}=\frac{2(p-\beta)^2(1+x^2)}{(p-\beta)^2x^2+\beta^2}\left[(p-\beta)y-(p-\beta)^2x^2\delta+\delta\beta^2-\beta\right]<0$$

then we have

$$h(t)=g(t,-n(1+t)/2)\leq g(t,y)$$

where

$$h(t)=\frac{(p-\beta)^2(1+t)}{(p-\beta)^2t+\beta^2}\Big[(p-\beta)^2\left(p-\beta+\tfrac{n}{2}\right)^2t^2+\Big\{(p-\beta)^2\left(p-\beta+\tfrac{n}{2}\right)^2+\left(\tfrac{n}{2}(p-\beta)-(\delta\beta^2-\beta)\right)^2$$

$$-\left((p-\beta)^2+(\delta\beta^2-\beta)\right)^2+(p-\beta)^2(2\beta-1)^2\Big\}t+\left(\tfrac{n}{2}(p-\beta)-(\delta\beta^2-\beta)\right)^2\Big]$$

$$=\frac{(p-\beta)^2(1+t)}{(p-\beta)^2t+\beta^2}\left[\lambda t^2+(\lambda+\eta-\upsilon+\sigma)t+\eta\right],$$

where

$$\lambda=\left(p-\beta+\tfrac{n}{2}\right)^2(p-\beta)^2,\quad \eta=\left\{\tfrac{n}{2}(p-\beta)-(\delta\beta^2-\beta)\right\}^2$$

and

$$\upsilon=\left\{(p-\beta)^2+(\delta\beta^2-\beta)\right\}^2\ and\quad \sigma=(p-\beta)^2(2\beta-1)^2$$

Now

$$h'(t)=\frac{(p-\beta)^2}{\left\{(p-\beta)^2t+\beta^2\right\}^2}\left[2(p-\beta)^2\lambda t^3+\left\{(p-\beta)^2(2\lambda+\eta-\upsilon+\sigma)+3\beta^2\lambda\right\}t^2\right.$$

$$\left.+2\beta^2(2\lambda+\eta-\upsilon+\sigma)t+(\lambda+2\eta-\upsilon+\sigma)\beta^2-(p-\beta)^2\eta\right]$$

Taking $h(t)=0$, we obtain (13) which is cubic in t. Let t_0 be the smallest positive root of the equation, then we have $h(t)=h(t_0)$, and hence

$$|\varphi(ix,y;z)|^2\geq h(t_0)=\rho^2$$

Define $\Omega = w:|w|<\rho$ then $\phi(P(z),zP(z);z)\in\Omega$ for all real x and $y=-n(1+x^2)/2$, $z\in\Delta$. Therefore by application of Lemma 1 the result follows On taking $\beta=0$ and $\mu=1$,$\delta=1$, $n=1$, $p=1$ in Theorem 1, we obtain the following interesting result.

Corollary 3 *If* $f(z) \in A_1(p,0) = A_1(p)$ *satisfies*

$$\left|\left(\frac{zf'(z)}{g(z)} - p\right)\left(\frac{zf'(z)}{g(z)} + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)\right| \leq \rho, (z \in \Delta)$$

where

$$\rho^2 = \frac{(1+t_0)}{t_0}\left[\frac{9}{4}t_0^2 + \frac{5}{2}t_0 + \frac{1}{4}\right]$$

t_0 *is the smallest positive root by the equation*

$$3t^3 + \frac{19}{4}t^2 - \frac{1}{4} = 0$$

then $f(z) \in \mathcal{K}_1(p,0)$.

Remark 1 *On taking* $\beta = 0$ *and* $\mu = 0$, $\delta = 1$, $n = 1$, $p = 1$ *in Theorem , we obtain a known result due to Ravichandran et. Al. [1].*

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