General Mathematics Vol. 17, No. 3 (2009), 161–172

A note on the Bernstein's cubature formula [1]

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Abstract

The Bernstein's cubature formula is revisited and the evaluation of it's remainder term is corrected.

2000 Mathematics Subject Classification: 65D32, 41A10, 41A63

Key words and phrases: Bernstein's operator, Bernstein's bivariate operator, Bernstein's cubature formula, remainder term

1 Preliminaries

Let us to denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Bernstein's bivariate operator $B_{m,n} : C([0,1] \times [0,1]) \to C([0,1] \times [0,1])$ is defined for any $f \in C([0,1] \times [0,1])$, any $(x, y) \in [0,1] \times [0,1]$ and any $m, n \in \mathbb{N}$ by:

$$(B_{m,n}f)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right), \tag{1}$$

[1] *Received 23 November, 2008*

Accepted for publication (in revised form) 26 December, 2008

where

$$p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k} \tag{2}$$

and

$$p_{n,j}(y)=\binom{n}{j}y^j(1-y)^{n-j} \tag{3}$$

are the fundamental Bernstein's polynomials.

Many approximation properties of the operator (1) are well known [1].

Let $f\in C([0,1]\times[0,1])$ be given. The following

$$f=B_{m,n}f+R_{m,n}f \tag{4}$$

is known as the "Bernstein bivariate approximation formula", $R_{m,n}f$ denoting the remainder term.

In [14], pp. 325, is mentioned the following:

"If $f\in C^{(2,2)}([0,1]\times[0,1])$ the remainder term of (4) can be expressed under the form

$$\begin{aligned}(R_{m,n}f)(x,y)=&-\frac{x(1-x)}{2m}f^{(2,0)}(x,\eta)-\frac{y(1-y)}{2n}f^{(0,2)}(\xi,y)\\&+\frac{xy(1-x)(1-y)}{4mn}f^{(2,2)}(\xi,\eta)."\end{aligned} \tag{5}$$

Next, using (4) with the expression of remainder term from (5), the Bernstein's cubature formula is constructed.

In our recent paper [4], was obtained the correct form for the remainder term of (4) when the approximated function f belong to $C([0,1]\times[0,1])$ and an upper bound estimation for $R_{m,n}f$ for the case when f is "sufficiently" differentiable on $[0,1]\times[0,1]$.

Let X be a linear space, $L_1, L_2 : X \to X$ be projectors, $I : X \to X$ be the identity operator and $R_1, R_2 : X \to X$ be the remainder operators associated to L_1 and respectively L_2. If L_1 and L_2 commute on X, the following decomposition of the identity operator

$$I = L_1 L_2 + R_1 \oplus R_2 \tag{6}$$

with

$$R_1 \oplus R_2 = R_1 + R_2 - R_1 R_2 \tag{7}$$

is well known [6], [7].

Suppose now that $X := C([0,1] \times [0,1])$, $L_1 := B_m^x$, $L_2 := B_n^y$, where B_m^x, B_n^y denote the parametrical extensions [1] of the Bernstein's univariate operator, i.e.

$$\left(B_m^x f\right)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k}{m}, y\right), \tag{8}$$

$$\left(B_n^y f\right)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(x, \frac{j}{n}\right). \tag{9}$$

It is well known [1] that (8) and (9) are not projectors. Is also well known [1] that for $f \in C^{2,2}([0,1] \times [0,1])$ the remainder operators associated to (8) and (9) are defined respectively by

$$\left(R_{m,n}^x f\right)(x,y) = -\frac{x(1-x)}{2m} f^{(2,0)}(x,\eta) \tag{10}$$

$$\left(R_{m,n}^y f\right)(x,y) = -\frac{y(1-y)}{2n} f^{(0,2)}(\xi,y) \tag{11}$$

for any $(x,y)\in[0,1]\times[0,1]$ and any $m,n\in\mathbb{N}$, where $(\xi,\eta)\in]0,1[\times]0,1[$. It is immediately that the operator (1) is the "tensorial product" [6], [7] of operators (10) and (11), i.e

$$B_{m,n}=B_m^xB_n^y. \tag{12}$$

Computing the boolean sum of operators (10) and (11) one arrives to the expression (5) which is false, because B_m^x, B_n^y are not projectors and the decomposition formula (6) doesn't holds.

By the above motives, we corrected (5) as follows.

Theorem 1 [4] *For any* $f\in C([0,1]\times[0,1])$ *and any* $(x,y)\in[0,1]\times[0,1]$ *the remainder term of* (4) *can be expressed under the form:*

(13)

$$\begin{aligned}(R_{m,n}f)(x,y)=&-\frac{x(1-x)}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{n}p_{m-1,k}(x)p_{n,j}(y)\left[\begin{matrix}x,\frac{k}{m},\frac{k+1}{m}\\ \frac{j}{k}\end{matrix};f\right]\\ &-\frac{y(1-y)}{n}\sum_{k=0}^{m}\sum_{j=0}^{n-1}p_{m,k}(x)p_{n-1,j}(y)\left[\begin{matrix}\frac{k}{m}\\ y,\frac{j}{n},\frac{j+1}{n}\end{matrix};f\right]\\ &+\frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)p_{n-1,j}(y)\left[\begin{matrix}x,\frac{k}{m},\frac{k+1}{m}\\ y,\frac{j}{k},\frac{j+1}{n}\end{matrix};f\right].\end{aligned}$$

Note that in (13) the brackets denote bivariate divided differences [2], [4].

In the Section 2, we use the following mean-value theorem for divided differences (see [8]).

Theorem 2 *Let* $m\in\mathbb{N}$, $a\le x_0<x_1<\cdots<x_m\le b$ *distinct knots and* $f:[a,b]\to\mathbb{R}$ *be a given function. If* f *is continuous on* $[a,b]$ *and has a* m^{th}

derivatives on (a,b), then there exists $\xi \in (a,b)$ such that

$$[x_0, x_1, \ldots, x_m; f] = \frac{1}{m!} f^{(m)}(\xi). \tag{14}$$

2 Main results

Theorem 3 *Let $p, q \in \mathbb{N}_0$, $p+q \geq 1$, $x_0, x_1, \ldots, x_p \in [a,b]$ and $y_0, y_1, \ldots, y_q \in [c,d]$ be a distinct knots and $f : [a,b] \times [c,d] \to \mathbb{R}$ be a function. If $f(\cdot, y) \in C([a,b])$ for any $y \in [c,d]$, $\frac{\partial^p f}{\partial x^p}(\cdot, y)$ exists on $]a,b[$ for any $y \in [c,d]$, $\frac{\partial^p f}{\partial x^p}(x,*) \in C([c,d])$ for any $x \in]a,b[$ and $\frac{\partial^{p+q}}{\partial x^p \partial y^q}(x,*)$ exists on $]c,d[$ for any $x \in]a,b[$, then there exists $(\xi, \eta) \in]a,b[\times]c,d[$ such that*

$$\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ & ; f \\ y_0, y_1, \ldots, y_q & \end{bmatrix} = \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q}(\xi, \eta), \tag{15}$$

where "$\cdot$" and "$*$" stand for the first and second variable.

Proof. Applying the method of parametric extension (see [3]) and the mean-value theorem for one dimensional divided differences, there exist $\xi \in]a,b[$ and respectively $\eta \in]c,d[$, such that

$$\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ & ; f \\ y_0, y_1, \ldots, y_q & \end{bmatrix} = [y_0, y_1, \ldots, y_q; [x_0, x_1, \ldots, x_p; f]_x]_y$$

$$= \left[y_0, y_1, \ldots, y_q; \frac{1}{p!} \frac{\partial^p f}{\partial x^p}(\xi, *)) \right]_y = \frac{1}{p!} \left[y_0, y_1, \ldots, y_q; \frac{\partial^p f}{\partial x^p}(\xi, *) \right]_y$$

$$= \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q}(\xi, \eta),$$

so the equality (15) holds.

Remark 1 *In the conditions of Theorem 3, if* $p = 0$ *then* $q \in \mathbb{N}$, *and we consider that* f *has the properties that* $f(x_0, *) \in C([c,d])$ *and* $\dfrac{\partial^q f}{\partial y^q}(x_0, *)$ *exists on* $]c,d[$. *If* $q = 0$, *then we consider similarly above conditions about function* f.

Theorem 4 *Let* $p, q \in \mathbb{N}_0$, $p + q \geq 1$, $x_0, x_1, \ldots, x_p \in [a,b]$ *and* $y_0, y_1, \ldots, y_q \in [c,d]$ *be a distinct knots. If* $f : [a,b] \times [c,d] \to \mathbb{R}$ *is a function with the property that* $f \in C^{(p,q)}([a,b] \times [c,d])$, *then exists* $(\xi, \eta) \in]a,b[\times]c,d[$ *such that*

$$\left[\begin{array}{l} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{array} ; f \right] = \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q} (\xi, \eta). \tag{16}$$

Proof. It results from Theorem 3.

Theorem 5 *Let* $f : [0,1] \times [0,1] \to \mathbb{R}$ *be a function.*
If $f(\cdot, y) \in C^1([0,1]$ *for any* $y \in [0,1]$, *exists* $\dfrac{\partial^2 f}{\partial x^2}(\cdot, y)$ *on* $]0,1[$ *for any* $y \in [0,1]$, $\dfrac{\partial^2 f}{\partial x^2}(x, *) \in C^1([0,1])$ *for any* $x \in]0,1[$, *exists* $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}(x, *)$ *on* $]0,1[$ *for any* $x \in]0,1[$, *then for any* $(x,y) \in [0,1] \times [0,1]$, *any* $m, n \in \mathbb{N}$, *there exist* $(\xi_i(k,j), \eta_i(k,j)) \in [0,1] \times [0,1]$, $i \in \{1,2,3\}$, *such that*

$$\begin{aligned} (R_{m,n} f)(x,y) = & -\frac{x(1-x)}{2m} \sum_{k=0}^{m-1} \sum_{j=0}^{n} \frac{\partial^2 f}{\partial x^2} (\xi_1(k,j), \eta_1(k,j)) \\ & - \frac{y(1-y)}{2n} \sum_{k=0}^{m} \sum_{j=0}^{n-1} \frac{\partial^2 f}{\partial y^2} (\xi_2(k,j), \eta_2(k,j)) \\ & + \frac{xy(1-x)(1-y)}{4mn} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} \frac{\partial^4 f}{\partial x^2 \partial y^2} (\xi_3(k,j), \eta_3(k,j)). \end{aligned} \tag{17}$$

If $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ *and* $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]0,1[\times]0,1[$, *the following inequalities*

(18)

$$|(R_{m,n}f)(x,y)| \le \frac{x(1-x)}{2m}\, M_1(f) + \frac{y(1-y)}{2n}\, M_2(f) + \frac{xy(1-x)(1-y)}{4mn}\, M_3(f) \le \frac{1}{8m}\, M_1(f) + \frac{1}{8n}\, M_2(f) + \frac{1}{64mn}\, M_3(f)$$

and

$$|(R_{m,n}f)(x,y)| \le \left(\frac{1}{8m} + \frac{1}{8n} + \frac{1}{64mn}\right) M(f) \tag{19}$$

hold, for any $(x,y) \in [0,1] \times [0,1]$ *and any* $m, n \in \mathbb{N}$, *where*

$$M_1(f) = \sup_{(x,y)\in]0,1[\times]0,1[} \left| \frac{\partial^2 f}{\partial x^2}\,(x,y) \right|, \tag{20}$$

$$M_2(f) = \sup_{(x,y)\in]0,1[\times]0,1[} \left| \frac{\partial^2 f}{\partial y^2}\,(x,y) \right|, \tag{21}$$

$$M_3(f) = \sup_{(x,y)\in]0,1[\times]0,1[} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}\,(x,y) \right| \tag{22}$$

and

$$M(f) = \max\{M_1(f), M_2(f), M_3(f)\}. \tag{23}$$

Proof. In the relation (13) we apply Theorem 3 and the relation (17) results. Because $x(1-x) \le \dfrac{1}{4}$, $y(1-y) \le \dfrac{1}{4}$,

$$\sum_{k=0}^{m-1} \sum_{j=0}^{n} p_{m-1,k}(x) p_{n,j}(y) = \sum_{k=0}^{m} \sum_{j=0}^{n-1} p_{m,k}(x) p_{n-1,j}(y) = \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} p_{m-1,k}(x) p_{n-1,j}(y) = 1$$

and transforming into modulus in the relation above and taking into account that the partial derivatives of f are bounded on $]0,1[\times]0,1[$, the inequalities from (18) are obtained.

Integrating the Bernstein's bivariate approximation formula (4) one arrives to the following Bernstein's cubature formula

$$\int_0^1\int_0^1 f(x,y)dx\,dy = \sum_{i=0}^{m}\sum_{j=0}^{n} A_{i,j} f\left(\frac{i}{m},\frac{j}{n}\right) + R_{m,n}[f]. \tag{24}$$

Theorem 6 [14] *The coefficients of the cubature formula* (24) *are given by the equalities:*

$$A_{ij} = \frac{1}{(m+1)(n+1)}, \quad i=\overline{0,m},\ j=\overline{0,n}. \tag{25}$$

Regarding the remainder term of (23), we have the following:

Theorem 7 *In the conditions of Theorem 5, the following upper-bound estimation for the remainder term of Bernstein's cubature formula* (24) *is*

$$|R_{m,n}[f]| \le \frac{1}{12m} M_1(f) + \frac{1}{12n} M_2(f) + \frac{1}{144mn} M_3(f), \tag{26}$$

where $M_1(f)$, $M_2(f)$ and $M_3(f)$ were defined at (20), (21) *and* (22).

Proof. The inequality (26) follows by integrating the Bernstein's bivariate approximation formula (4) and taking the first inequality (18) into account.

Theorem 8 *Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be a function. If $f\in C^{(2,2)}([0,1]\times[0,1])$, the relations* (17) *and* (26) *hold, where*

$$M_1(f) = \sup_{(x,y)\in[0,1]\times[0,1]} \left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|,$$

$$M_2(f) = \sup_{(x,y)\in[0,1]\times[0,1]} \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right|, \quad and$$

$$M_3(f) = \sup_{(x,y)\in[0,1]\times[0,1]} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|.$$

Proof. It results from Theorem 7

Remark 2 *In Theorem 7 we give a new proof for the known inequality (26)(see [14], pp.325). The inequality from (26) is demonstrate in [14] in the conditions of Theorem 8.*

Theorem 9 *In the conditions of Theorem 7 or Theorem 8, it follows that*

$$\lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{(m+1)(n+1)} f\left(\frac{i}{m}, \frac{j}{n}\right) = \int_0^1 \int_0^1 f(x,y)dx\,dy \tag{27}$$

and the convergence from (27) *is uniform.*

Proof. It results from inequality (26).

Remark 3 *Because the Bernstein's bivariate operator $B_{m,n}$ conserve only the lineares functions in x and respectively y, it follows that the degree of exactness for the cubature formula (24) is* $(1,1)$. *In the case when the approximated function f satisfies the hypotheses of Theorem 6, the above affirmation follows directly from the mentioned theorem.*

Acknowledgement. This paper is devoted to the memory of Luciana Lupaş and Alexandru Lupaş, remarkable representatives of the Romanian school of Approximation Theory and Numerical Analysis.

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