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Research Article

Microscopic Description of 2D Topological Phases, Duality, and 3D State Sums

Zoltán Kádár,[1] Annalisa Marzuoli,[1,2] and Mario Rasetti[1,3]

[1] *Institute for Scientific Interchange Foundation, Villa Gualino, Viale Settimio Severo 75, 10131 Torino, Italy*
[2] *Dipartimento di Fisica Nucleare e Teorica, Istituto Nazionale di Fisica Nucleare, Universita degli Studi di Pavia, Sezione di Pavia, via A. Bassi 6, 27100 Pavia, Italy*
[3] *Dipartimento di Fisica, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy*

Correspondence should be addressed to Annalisa Marzuoli, annalisa.marzuoli@pv.infn.it

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Doubled topological phases introduced by Kitaev, Levin, and Wen supported on two-dimensional lattices are Hamiltonian versions of three-dimensional topological quantum field theories described by the Turaev-Viro state sum models. We introduce the latter with an emphasis on obtaining them from theories in the continuum. Equivalence of the previous models in the ground state is shown in case of the honeycomb lattice and the gauge group being a finite group by means of the well-known duality transformation between the group algebra and the spin network basis of lattice gauge theory. An analysis of the ribbon operators describing excitations in both types of models and the three-dimensional geometrical interpretation are given.

1. Introduction

Topological quantum field theories (TQFTs) in three dimensions describe a variety of physical and toy models in many areas of modern physics. The absence of local degrees of freedom is a great simplification it often leads to complete solvability [1, 2]. Perhaps the most recent territory, where they appeared to describe real physical systems, is that of topological phases of matter, being, for example, responsible for the fractional quantum Hall effect [3]. Since the idea of fault-tolerant quantum computation appeared in the literature [4], TQFTs are also important in quantum information theory. These new applications also enhanced the mathematical research, and led to classification of the simplest models [5].

Due to their topological nature, TQFT's admit discretization yet remaining an exact description of the theory given by an action functional on a continuous manifold. One large class thereof is the so-called BF theories, whose Lagrangian density is given by the wedge product of a $(d-2)$ form B and the curvature 2-form F of a gauge field [6]. We will deal here with a special class of three-dimensional theories, which describe doubled topological phases and restrict our attention to discrete gauge groups G. The context they appeared in, in recent physics literature [4, 7], is Hilbert spaces of states in two dimensions and dynamics therein, which are boundary Hilbert spaces H of the relevant TQFT's. Operators acting on H correspond to three-dimensional amplitudes on the thickened surface. In this paper we will explain this correspondence, which was proved for the ground-state projection recently [8], and provide the geometric interpretation of the ribbon operators, which create quasiparticle excitations from the ground state. This is a step towards extending the correspondence to identify the ribbon operators as invariants of manifolds with coloured links embedded in them in the TQFT.

The emergence of topological phases from a description of microscopic degrees of freedom is modeled by the lattice models of Kitaev [4] and Levin and Wen [7]. Since they generically have degenerate ground states and quasi-particle excitations insensitive to local disturbances, they are also investigated in the theory of quantum computation [9], their continuum limit being closely related to the spin network simulator [10, 11]. The ground states were extensively studied in the literature; their MERA (multiscale entanglement renormalization ansatz) [12, 13] and tensor network representations [14] have been constructed to study for example, their entanglement properties [15, 16]. Finding the explicit root of these structures in lattice gauge theory and TQFT can help to understand their physical properties.

Lattice gauge theories admit seemingly very different descriptions. A state can be represented by assigning elements of the gauge group to edges of the lattice. The dual description in terms of spin network states where edges are labelled by irreducible representations (irreps) of the gauge group and vertices by invariant intertwiners are also well known since the publication of [17]. To name an application, this description turned out to provide a convenient basis for most approaches to modern quantum gravity theories [18, 19]. In this paper we will show in detail how these dual descriptions give rise to Kitaev's quantum double models in one hand and the spin net models of Levin and Wen on the other. Then the ribbon operators in both models and their identification will be discussed.

The organization of the paper is as follows. In the next section, we introduce the Turaev-Viro models via the example of BF theories. In Section 3, we briefly introduce the string net models of Levin and Wen on the honeycomb lattice Γ in the surface S and recall the proof [8] that the ground-state projection is given by the Turaev-Viro amplitude on $S \times [0, 1]$. The boundary triangulations of S are given by the dual graphs of Γ decorated by the labels inherited from the "initial" and "final" spin nets. In Section 4 the duality between the states of the Kitaev model and the string nets will be shown by changing the basis from the group algebra to the Fourier one. By using this duality and an additional projection, we will obtain the electric constraint operators of the string net models. The matrix elements of the magnetic constraints are also recovered provided that the local rules of Levin and Wen hold. We explain that they do in all BF theories, which is a strong motivation in their favor for the case when the gauge group is finite. In Section 5, we discuss ribbon operators and give their three-dimensional geometric interpretation in terms of framed links in the Turaev-Viro picture. Finally, a summary is given with a list of questions for future research.

2. Turaev-Viro Models

In three dimensions both the $F(A)$ (the field strength) and B fields of BF theory can be considered to be forms valued in the Lie algebra of the gauge group G. The action can then be written as $\int_M \mathrm{tr}(B \wedge F)$ with tr being an invariant nondegenerate bilinear form on the Lie algebra and M is a smooth, oriented, closed three-manifold. We may start from the case when G is a semisimple Lie group relevant in particle physics theories and gravity, A being the connection in the principal G-bundle over M. In three dimensions the "space-time" separated form of the Lagrangian has the structure $B_j dA_k + A_0 D_j A_k + B_0 F_{jk}(A)$ where j, k are spatial indices. (There is not necessarily physical time in the theory; one can do this decomposition for Euclidean signature as well.) The first term is the standard kinetic term, the second implies the (Gauss) constraint of gauge invariance, the third stands for the vanishing of the (two-dimensional) field strength (A_0 and B_0 are Lagrange multipliers), while D and d stand for the covariant and the exterior derivatives, respectively.

Since locally the solution of the constraints is given by a pure gauge $A_i = \tilde{g}^{-1}\partial_i\tilde{g}$ ($\tilde{g} : S \rightarrow G$ smooth function, with S being the spatial hypersurface), one may discretize the theory by introducing a lattice on the spatial surface and quantize the remaining degrees of freedom: the holonomies (elements of G) describing the coordinate change between faces of the lattice. They correspond to the edges of the dual lattice, which is constructed by placing a vertex inside each face and connecting new vertices, which were put inside neighbouring faces. This dual lattice is the starting point of the models in [4], the electric constraints are the remainders of the Gauss constraint and the magnetic ones are the remainders of the flatness constraint. For a detailed exposition see, for example, [20].

The partition function of the above BF theory is formally obtained by taking the functional integral over the fields A an B of the phase associated to the classical action

$$\mathbf{Z}_{\mathrm{BF}} = \int \mathfrak{D}A\mathfrak{D}B e^{i\int_M \mathrm{tr}\, B\wedge F(A)}. \tag{2.1}$$

It is not so easy to give this definition a precise sense, but for the moment it is not necessary to go into further details. What does matter is that there exists a consistent way to discretize the partition function by considering an oriented triangulation M_Δ of the manifold M by assigning two Lie algebra elements B_i, Ω_i to each edge i in M_Δ. The generator B_i can be thought as the integral of B along the edge i, whereas Ω_i as the logarithm of the group element corresponding to the holonomy around the edge i. (To be more precise, one needs to introduce the dual complex by putting vertices inside every tetrahedron, connecting those vertices which were put in neighbouring tetrahedra and a vertex should be singled out on the boundary of each dual face. Then the procedure to get Ω_i is the following: take the dual face corresponding to i. Multiply the holonomies along the boundary edges of this dual face starting from the vertex singled out in a circular direction determined by the orientation of i (say, by the right-hand rule). The logarithm of this group element is Ω_i.) Then the Feynman integral in (2.1) can be replaced by $\prod_i \int dg_i \int dB_i e^{i\int \mathrm{tr}\, B_i\Omega_i}$. The B integrals will yield Dirac deltas $\delta_{g_i,1}$ and one can now proceed with decompositions in terms of irreps of the gauge group. This way one ends up with a discrete state sum instead of the original Feynman integral, where each state is the triangulation coloured with irreps and its weight is given by the precise final form of the amplitude (examples are given below). The structure of the partition function (amplitude) for a prototypical theory, the Ponzano-Regge model [21] corresponding to $G = SU(2)$, reads

$$Z(M_\Delta) = \sum_{j_i}\prod_i d_{j_i}\prod_t (6j)_t, \tag{2.2}$$

where $(6j)$ is the Wigner $6j$ symbol of $SU(2)$ depending on the 6 irreps decorating the edges of the tetrahedron t, d_{j_i} is the dimension of the irreps j_i assigned to the edge i, and the sum ranges over all states, that is, all possible colourings of the edges with irreps. It turns out that this type of state sum is well-defined and independent of the chosen triangulation for a large class of models. (This is one way to define a TQFT rigorously.) For a systematic derivation of this state sum from action functionals, see [22] or [23, Section 2.3].

The Ponzano-Regge partition function (2.2) is formally independent of Δ, but is divergent. However, the Turaev-Viro (TV) model [24], a regularized version thereof, has a well defined partition function, given by

$$\mathbf{Z}_{\mathrm{TV}}\left[M^3;q\right] = \sum_j d^{-V}\prod_i d_{j_i}\prod_t \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_t, \tag{2.3}$$

where the underlying algebraic structure is the quasitriangular Hopf algebra $SU_q(2)$ with $q = \exp(2\pi i/k), k \in \mathbb{Z}$ fixed, $j \in [0, 1, \ldots, k-1]$ denotes the irreps of $SU_q(2)$ (with nonzero trace), $d_j \in \mathbb{C}$ is the so-called quantum dimension of j, the constant d is defined by $d = \sum_k d_k^2$, the quantity in the brackets is the quantum $6j$ symbol, and V is the number of vertices of the triangulation. One finds the precise definitions of all the quantities along with the algebraic properties assuring consistency and triangulation independence [21] of the amplitudes in [24]. We will briefly mention the origin of the latter property in the next section (Note, that in the case of the Turaev-Viro model based on $SU_q(2)$ these properties hold, as in the case of the Ponzano-Regge model, for arbitrary $q \in \mathbb{C}$, but it is only for q being a root of unity, when the partition function is finite.). The final fact for this introductory section is about the form of the amplitude (2.3) for manifolds with nonempty boundary. The associated boundary triangulation, whose edges are decorated by labels $\{j'\}$, derives from a given triangulation in the 3D bulk and is kept fixed. The amplitude reads

$$\mathbf{Z}_{\mathrm{TV}}\left[M^3,\{j'\};q\right] = \sum_{j\ \mathrm{int}} d^{-V}\prod_i d_{j_i}\prod_{i'} d_{j_{i'}}^{-1/2}\prod_{v'} d_{j_{v'}}\prod_t \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_t, \tag{2.4}$$

where V is the number of internal vertices, the index i ranges over internal edges, i' over boundary edges, v' over boundary vertices (each boundary vertex is the endpoint of an internal edge; $j_{v'}$ is its colour), and t over all tetrahedra, and the summation is done for internal edge labels only, while those on the boundary are kept fixed.

Note that there is a quasi-triangular Hopf algebra associated to finite groups as well, the so-called Drinfeld (quantum) double $\mathfrak{D}(G)$ [25]. There, the dimensions d_j as well as the $6j$ symbols can be obtained from the representation theory of the group G.

3. String Nets

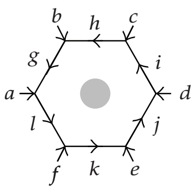
Levin and Wen [7] started off from the algebraic structure underlying the above models (consistent set of $6j$ symbols and quantum dimensions), which serves as the algebraic data in defining TQFT's. Taking these data for granted, they constructed a two-dimensional lattice model, which we will now introduce briefly. Consider a surface S with a fixed oriented honeycomb lattice Γ embedded in it. The Hilbert space is spanned by all possible decorations of the edges with labels j; we will refer to them as irreps (of $SU_q(2)$ or the finite group G) as we will not need to treat the most general TQFT's. The Hamiltonian is a sum of two families of mutually commuting constraint operators:

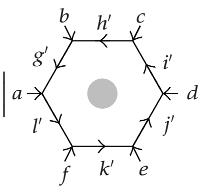
$$H = -\sum_v A_v - \sum_p B_p \tag{3.1}$$

where the first sum is over all vertices and the second is over all plaquettes of the lattice. $A_v \equiv N_{ijk}$ for i, j, k being the irreps decorating the edges adjacent to the vertex v (the numbers $N_{ijk} \in \mathbb{N}$ are referred to as fusion coefficients between the irreps: $i \otimes j = \sum_k N_{ijk} k$). The magnetic constraints are written as a sum

$$B_p = \frac{1}{d} \sum_s d_s B_p^s \tag{3.2}$$

over irreps and the action of the individual terms is

$B_p^s \Big|$ 

$\Big\rangle = \sum_{g'h'i'j'k'l'} F^{bg^*h}_{s^*h'g'^*} F^{ch^*i}_{s^*i'h'^*} F^{di^*j}_{s^*j'i'^*} F^{ej^*k}_{s^*k'j'^*} F^{fk^*l}_{s^*l'k'^*} F^{al^*g}_{s^*g'l'^*} \Big|$ 

$\Big\rangle \qquad (3.3)$

while its action on the rest of the state supported on the honeycomb lattice Γ is trivial. The numbers F^{ijk}_{lmn} are the $6j$ symbols, part of the algebraic data of a TQFT; i^* denotes the irreps dual to i. Changing the orientation of an edge is equivalent to changing its label i to its dual i^*. Levin and Wen use a different normalisation that of (2.3), (2.4):

$$d_n \begin{Bmatrix} i & j & m \\ k & l & n \end{Bmatrix} = F^{ijm}_{kln}. \tag{3.4}$$

Before proceeding, let us write down an important algebraic property of the F symbols:

$$\sum_{n=0}^{N} F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr^*} F^{riq^*}_{mls^*}. \tag{3.5}$$

This identity is called the Biedernharn-Elliot identity or pentagon equation, which holds in every TQFT. In the concrete examples mentioned above, they can be proved by the definition of the $6j$ symbols as connecting the two different fusion channels of recoupling irreps (graphically encoded by (4.21) in Section 4.2) (by means of using two different ways of coupling five irreps.)

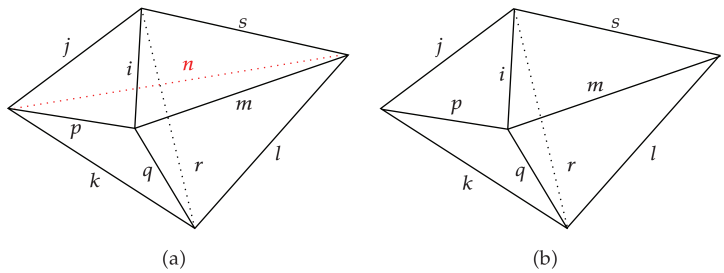


Figure 1

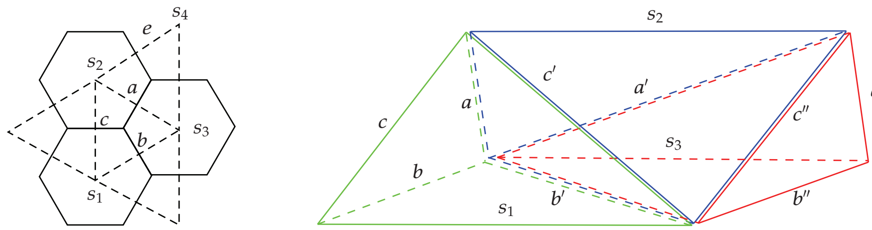
3.1. Reconstructing 3D Geometry

In our work [8] we recovered a three-dimensional Turaev-Viro invariant [26, 27] from the algebra of Levin and Wen. We associated geometric tetrahedra to the algebraic $6j$ symbols, where the edges are decorated with irreps from the $6j$ symbols. In that a convention needs to be adopted; for example, the upper row should correspond to a (triangular) face of the tetrahedron and labels in the same column should correspond to opposite edges. In the examples we are looking at there is always a normalization of the $6j$'s such that they have the same symmetry as the tetrahedron. Orientation of edges can also be taken care of in a consistent manner; we will however omit them for most of what follows. Now we can translate the Biedernharn-Elliot identity to geometry. Then Figure 1 arises: where the two configurations (three tetrahedra joined at the edge n and two tetrahedra glued along the triangle (irq)) correspond to the left-hand and right-hand sides of (3.5), respectively. This is a cornerstone in proving triangulation independence of the amplitudes (2.3) and (2.4) and shows that whenever tetrahedra are glued labels corresponding to internal edges have to be summed over.

Now, one can try to find the geometric counterpart of the operator (3.3). Constructing the dual (triangle) graph $\widetilde{\Gamma}$ of the honeycomb lattice Γ such that a dual edge inherits the label of the original edge it corresponds to (recall that edges of the original and the dual graph are in 1-1 correspondence in two dimensions), we proved the equality [8]

$$\left\langle \Gamma_1^{\{j\}} \middle| \prod_p B_p \middle| \Gamma_0^{\{j''\}} \right\rangle = Z_{\mathrm{TV}}\left[S \times [0,1], \widetilde{\Gamma}_0^{\{j\}}, \widetilde{\Gamma}_1^{\{j''\}} \right]. \tag{3.6}$$

The left-hand side. means the matrix element of the operator $\prod_p B_p$ between two spin nets, that is, the honeycomb lattice $\Gamma_0 \equiv \Gamma$ decorated by labels $\{j\}$ and another copy of $\Gamma_1 \equiv \Gamma$ decorated by $\{j''\}$. The right-hand side. is the Turaev-Viro amplitude (2.4) of the three-dimensional manifold $S \times [0,1]$ with fixed triangulations on the two boundaries given by the dual graph $\widetilde{\Gamma}_i$ with the labels inherited from Γ_i. In Figure 2(a) below, the dashed lines show a part of $\widetilde{\Gamma}_0^{\{j\}}$. Let us concentrate on the middle vertex in Figure 2 corresponding to the dual triangle abc. There is an F symbol corresponding to that vertex from all three operators B_p of the three plaquettes sharing that vertex. To each F we associate a tetrahedron and they induce the internal triangulation of $S \times [0,1]$ depending on the order how the B_p operators are multiplied one after the other. These different orders of multiplication correspond to different decompositions of the prism (built from translating the triangle abc in $\widetilde{\Gamma}_0$ to the corresponding one $a''b''c''$ in $\widetilde{\Gamma}_1$) into three tetrahedra. The fact that they commute is nicely reflected by the



(a) (b)

Figure 2

independence of the TV amplitude on the internal triangulation (Note that for proving this equality it is necessary that the coefficients of B_p^s be given by (3.2).) Figure 2(b) shows a part of the amplitude corresponding to $F^{acb}_{s_1 b' c'} F^{b' a c'}_{s_2 c'' a'} F^{c'' b' a'}_{s_3 a'' b''}$. The labels with one bar a', b', c' are to be summed over when the full amplitude is written in accordance with the fact that the underlying edges are the internal edges of the triangulation of $S \times [0,1]$. Summation of the s_i labels comes from the sum in (3.2).

Note that $\prod_p B_p = \prod_p B_p \prod_v A_v$ if we naturally define the $6j$ symbols to be zero whenever a triple (ijk) corresponding to a dual triangle of a tetrahedron has $N_{ijk} = 0$. This way, what we found is that the TV amplitude gives the ground-state projection. For the precise matching of all the weights to those of (2.4) and the consistency of the full amplitude, see [8, the last Section].

4. Duality and the Quantum Double Lattice Models

Now, we will restrict our attention to the case when the TQFT is given by the structure of the double $\mathfrak{D}(G)$ of a finite group G and show the equivalence between the lattice models of Kitaev [4] based on that structure and the corresponding string net models. Our method relies on the duality in the underlying lattice gauge theory [17]. Essentially the same idea was employed also in the very recent paper in [28].

The Hilbert space of the Kitaev model (and that of a lattice gauge theory) is spanned by the group algebra basis $\{|g_1, g_2, \ldots, g_E\rangle, g_i \in G\}$, supported on an oriented lattice Γ with E edges, V vertices, and F faces. The scalar product is given by

$$\langle g_1, g_2, \ldots, g_E \mid h_1, h_2, \ldots, h_E \rangle = \delta_{g_1,h_1}\delta_{g_2,h_2}\cdots\delta_{g_E,h_E}. \tag{4.1}$$

The Hamiltonian consists of two families of sums of constraint operators, which are projections and mutually commuting. We will follow the strategy of imposing the electric constraints first and find the corresponding operators in the string net model of Levin and Wen. Then we will study the action of the magnetic constraints in the range of the set of electric constraints, and determine their matrix elements in the dual basis, recovering the magnetic operators in the string net model this way.

The basic idea is the well-known expansion of any function $f \in L_2(G)$ with G being any compact Lie group

$$f(g) = \sum_j \sum_{m,n}^{d_j} c^j_{mn} D^j_{mn}(g) \tag{4.2}$$

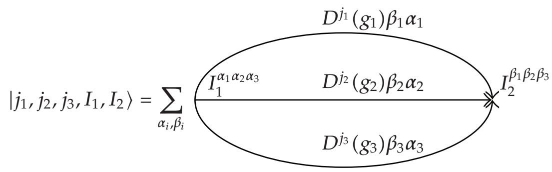


Figure 3

in terms of irreps j (D^j are the representation matrices and c^j_{mn} are coefficients). The statement is known as the Peter-Weyl theorem. We now define a new basis

$$\{|j_1, j_2, \ldots, j_E, \alpha_1, \alpha_2, \ldots \alpha_E, \beta_1, \beta_2, \ldots, \beta_E\rangle\} \tag{4.3}$$

by means of the scalar product

$$\begin{aligned} &\langle g_1, g_2, \ldots, g_E \mid j_1, j_2, \ldots, j_E, \alpha_1, \alpha_2, \ldots \alpha_E, \beta_1, \beta_2, \ldots, \beta_E\rangle \\ &= D^{j_1}(g_1)_{\alpha_1\beta_1} D^{j_2}(g_2)_{\alpha_2\beta_2} \cdots D^{j_E}(g_E)_{\alpha_E\beta_E}. \end{aligned} \tag{4.4}$$

The α_i (β_i) denote the target (source) index of the oriented edge i and they range over the dimensions of the irreps j. We will need a linear combination of this basis defined in the following way. Consider all elements with fixed irreps $j_1, j_2, \ldots, j_E$. For every vertex v of Γ take a three-index tensor I_v, where the indices range over the dimension of the irreps associated to the three edges (the honeycomb lattice is trivalent) incident to v. Then contract all indices with the corresponding ones in (4.3). A simple example corresponding to the theta-graph is given in Figure 3. For these states (Figure 3)associated to the graph Γ we use the notation

$$|j_1, j_2, \ldots, j_E, I_1, I_2, \ldots, I_V\rangle. \tag{4.5}$$

4.1. The Electric Constraints

Let us recall the electric constraints of the Kitaev model. They are written in terms of the following operators:

$$L^g(i, v) : |\cdots h_i \cdots\rangle \longmapsto \begin{cases} |\cdots g h_i \cdots\rangle & \text{if } i \text{ points towards } v, \\ |\cdots h_i g^{-1} \cdots\rangle & \text{otherwise.} \end{cases} \tag{4.6}$$

The local gauge transformation acting at vertex v reads

$$A_g(v) = \prod_{i \in v} L^g(i, v), \tag{4.7}$$

(the product is over edges incident to the vertex v) and the electric constraint is the projection defined as the average of the latter over the group

$$A(v) = \frac{1}{|G|}\sum_{g\in G} A_g(v). \tag{4.8}$$

Note that the range of the set of electric constraints are gauge invariant states, that is, they are invariant under $\prod_v A_{g_v}(v)$ with arbitrary tuple $(g_1, g_2, \ldots, g_V) \in G^V$, as shown by the following calculation:

$$A_g(v)\frac{1}{|G|}\sum_{h\in G} A_h(v) = \frac{1}{|G|}\sum_{h\in G} A_{gh}(v) = \frac{1}{|G|}\sum_{h'\in G} A_{h'}(v). \tag{4.9}$$

Hence, at each vertex, the projection (4.8) implements gauge invariance. Let us see how this is done in the general set of states (4.5) also called spin networks. The action of a gauge transformation $A_v(g)$ on a spin network can be determined by rewriting the scalar product as

$$\left\langle g_1, g_2, \ldots, g_E \middle| \sum_{S'} \widetilde{A}^g(v)_{S,S'} \middle| S' \right\rangle = \langle S | A^g(v)^\dagger \big| g_1, g_2, \ldots, g_E \rangle. \tag{4.10}$$

Let us write down this action explicitly for a vertex v whose incident edges are oriented outwards and use a simpler notation $|g_1, g_2, g_3\rangle$ for a generic state supported on Γ with $1, 2, 3$ being the labels of the edges incident to v. Let us also use a similar abbreviation $|j_1, j_2, j_3, I_v\rangle$ for $|S\rangle$ as the remaining parts are not important for the case at hand. Since $A^g(v)^\dagger = \prod_{i\in v} L^g(i, v)^\dagger = \prod_{i\in v} L^g(i^*, v)$ with i^* denoting the opposite orientation for the edge i, we can write

$$\begin{aligned}
\sum_{\alpha_1,\alpha_2,\alpha_3} & I_v^{\alpha_1\alpha_2\alpha_3} D^{j_1}(g_1)_{\alpha_1\beta_1} D^{j_2}(g_2)_{\alpha_2\beta_2} D^{j_3}(g_3)_{\alpha_3\beta_3} \cdots \longmapsto \left\langle j_1, j_2, j_3, I_v \middle| A^g(v)^\dagger \middle| g_1, g_2, g_3 \right\rangle \\
&= \langle j_1, j_2, j_3, I_v \mid g g_1, g g_2, g g_3 \rangle \\
&= \sum_{\alpha_1,\alpha_2,\alpha_3} I_v^{\alpha_1\alpha_2\alpha_3} D^{j_1}(g g_1)_{\alpha_1\beta_1} D^{j_2}(g g_2)_{\alpha_2\beta_2} D^{j_3}(g g_3)_{\alpha_3\beta_3} \cdots \\
&= \sum_{\gamma_1,\gamma_2,\gamma_3} \left(\sum_{\alpha_1,\alpha_2,\alpha_3} I_v^{\alpha_1\alpha_2\alpha_3} D^{j_1}(g)_{\alpha_1\gamma_1} D^{j_2}(g)_{\alpha_2\gamma_2} D^{j_3}(g)_{\alpha_3\gamma_3} \right), \\
D^{j_1}&(g_1)_{\gamma_1\beta_1} D^{j_2}(g_2)_{\gamma_2\beta_2} D^{j_3}(g_3)_{\gamma_3\beta_3} \cdots \\
&= \sum_{\gamma_1,\gamma_2,\gamma_3} I_v^{g\ \gamma_1\gamma_2\gamma_3} D^{j_1}(g_1)_{\gamma_1\beta_1} D^{j_2}(g_2)_{\gamma_2\beta_2} D^{j_3}(g_3)_{\gamma_3\beta_3} \cdots .
\end{aligned} \tag{4.11}$$

In the above the dots $\cdots$ stand for the remaining part of the spin network, which is not affected. In the fourth equality the group homomorphism property of the matrices ($D(gh) = D(g)D(h)$) wass used. The last equality is the definition of I_v^g as the quantity in the big parenthesis.

The transformation rule of $I_v \to I_v^g$ means that $I_v \in j_1 \otimes j_2 \otimes j_3$. The Clebsch-Gordan series $i \otimes j = \sum_k N_{ijk} k$ shows that (after a suitable unitary transformation) I_v is block diagonal with $j_1 \otimes j_2 \otimes j_3 = \sum_{kl} N_{j_1 j_2 k} N_{k j_3 l} l$ and each block transforms according to an irreps l of G. The case $I_v^g = I_v, g \in G$ for all $g \in G$ corresponds to the trivial representation, which appears in the block decomposition if and only if thus but $N_{j_1 j_2 j_3} \neq 0$. We see now that gauge invariance at vertex v is achieved by acting with the projection that projects into the invariant subspace of the decomposition. This should correspond to the projection of Levin and Wen. Assuming that $N_{ijk} < 2$ for every triple of irreps ensures that there is one unique gauge invariant tensor I (for coupling three irreps), an intertwiner, so that the invariant subspace of $\prod_v A_v^{g_v}$ is spanned by

$$\left\{ \left| j_1, j_2, \ldots, j_E \right\rangle : N_{j_i, j_k, j_l} = 1 \ \forall (i, k, l) \text{ incident to a vertex} \right\} \tag{4.12}$$

and I is understood to be at every vertex contracting all α, β indices of (4.3). For these states associated to the honeycomb graph Γ and only for these, we will use the notation $|S\rangle$.

A shorter way to arrive at invariant spin network states is to consider a generic gauge invariant state supported on Γ in the group algebra basis. These are the so-called cylindrical functions $\Psi \in L^2(G^E)$ with the invariance property

$$\langle \Psi \mid g_1, g_2, \ldots, g_E \rangle \equiv \Psi(g_1, g_2, \ldots g_E) = \Psi\Big(h(t_1) g_1 h(s_1)^{-1}, h(t_2) g_2 h(s_2)^{-1}, \ldots, h(t_E) g_E h(s_E)^{-1} \Big) \tag{4.13}$$

for every $(h_1, h_2, \ldots, h_V) \in G^V$ where $t(i)$ $(s(i))$ denotes the target (source) vertex of the edge i. It can be shown that the spin network states constitute an orthonormal basis in this Hilbert space [29].

4.2. The Magnetic Constraints

To recall the construction of the magnetic operators of the Kitaev model, we define auxiliary operators associated to pairs (i, p) where p is a face (plaquette) of Γ and $i \in \partial p$ is an edge on the boundary of p:

$$T^g(j, p) : \left| g_1, g_2, \ldots, g_E \right\rangle \longmapsto \delta_{g^{\pm 1} g_j} \left| g_1, g_2, \ldots, g_E \right\rangle, \tag{4.14}$$

where we have g (g^{-1}) in the argument of the Dirac delta, when p is to the right (left) of the edge i oriented forward. The magnetic constraint is the special case $g = 1$ of the operator

$$B_g(p) = \sum_{\substack{h_i \in \partial p \\ h_1 \cdots h_6 = g}} \prod_{m=1}^{6} T^{h_m}(j_k, p). \tag{4.15}$$

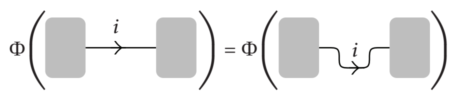
To adapt to the string net model we took Γ to be the honeycomb lattice. After straightforward calculation one finds the action of $B_1(p)$ to be given by

$$\left| g_1, g_2, \ldots, g_E \right\rangle \longmapsto \delta_{g_{p(1)} g_{p(2)} \cdots g_{p(6)}, 1} \left| g_1, g_2, \ldots, g_E \right\rangle \tag{4.16}$$

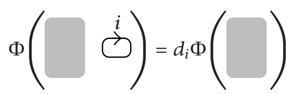
whenever all edges $p(1), p(2), \ldots, p(6)$ bounding the hexagonal face consecutively point to the counterclockwise direction. Should a boundary edge $p(l)$ point to the opposite direction, $g_{p(l)}$ needs to be replaced by $g^{-1}_{p(l)}$ in the above expression. To proceed we write down the Plancherel decomposition of the Dirac delta function, which reads

$$\delta_{g_1 g_2 \cdots g_6, 1} = \sum_j d_j \operatorname{tr}\Big(D^j(g_1 g_2 \cdots g_6)\Big) = \sum_j d_j \operatorname{tr}\Big(D^j(g_1) D^j(g_2) \cdots D^j(g_6)\Big). \tag{4.17}$$

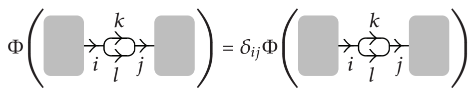
Each term in the above sum is the scalar product of a spin network based on a two-valent graph, the hexagon with $|g_1, g_2, \ldots, g_6\rangle$. This is a spin network of only bivalent vertices (the only intertwiner for bivalent vertex is the trivial Dirac delta connecting identical representations) "evaluated" on the same group elements that appear in the bounding plaquette p of the spin network $|S\rangle$. So we may write the action of $B_p : |S\rangle \mapsto \sum_j d_j |S, p, j\rangle$, where the state $|S, p, j\rangle$ is a generalized spin network with double lines inside the plaquette p. The notion, used also in [7], nonetheless, still requires proper definition. Were the use of the local rules



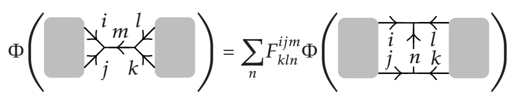
(4.18)



(4.19)



(4.20)



(4.21)

of [7] allowed, we could just refer to the calculation given by formula (C1) in that article, which gives the expansion of $|S, p, j\rangle$ in terms of bona fide spin networks $\langle S | g_1, g_2, \ldots, g_E\rangle$. We could then just take it as the definition and we would be done. However, to argue in favour of these local rules in the Kitaev model, we need to get back to the theory in the continuum. It has been mentioned that in the case when the group G is a Lie group the electric constraints are the lattice versions of the Gauss constraint that imposes local gauge invariance. This was explicitly justified in the previous section. Turning to the flatness constraint, any flat connection has trivial holonomy $g_\gamma[A] \equiv \mathcal{P} \exp(\int_\gamma A)$ along a closed curve γ that is contractible (otherwise we could contract the curve to the point, whose curvature would be proportional to the generator of the holonomy). The converse is also true, to every decoration of Γ with group elements satisfying the constraint (4.16) for all plaquettes; there exist smooth flat connection(s) in the manifold Γ is embedded into. Suppose that we have constructed one for the embedding surface of the honeycomb lattice. Then a spin network state

$$\Phi(S_{\Gamma'}) \equiv \langle S_{\Gamma'} \mid g_1[A], g_2[A], \ldots, g_{E'}[A] \rangle \tag{4.22}$$

with any graph Γ' makes sense and it is invariant of the homotopy class of the graph Γ'. This justifies (4.18). The connection is flat, so the holonomy along a contractible curve is 1, $\mathrm{Tr}D^j(1) = d_j$, which gives (4.19) There is no nontrivial intertwiner between two different irreps, whereas the left-hand side. of (4.20) is a composition of invariant maps with $i \to j$ included, so that rule also holds. Finally, we can smoothly contract the edge with label m in (4.21) without changing the value of (4.22), we have

$$\begin{aligned}
\sum_{\alpha_m,\alpha'_m} I^{\alpha_i\alpha_j\alpha_m} I^{\alpha_l\alpha_k\alpha'_m} D^m(1)_{\alpha_m\alpha'_m} &= \sum_{\alpha_m} I^{\alpha_i\alpha_j\alpha_m} I^{\alpha_l\alpha_k\alpha_m} \\
&= \sum_{n,\alpha_n} F^{ijm}_{kln} I^{\alpha_i\alpha_l\alpha_n} I^{\alpha_j\alpha_k\alpha_n} = \sum_{n,\alpha_n\alpha'_n} F^{ijm}_{kln} I^{\alpha_i\alpha_l\alpha_n} I^{\alpha_j\alpha_k\alpha'_n} D^n(1)_{\alpha_n\alpha'_n},
\end{aligned} \tag{4.23}$$

where the middle equality is a property of intertwiners and the rightmost formula coincides with the inner part of the rhs. of (4.21), when its middle edge with label n is contracted. Note that we have omitted also the representation matrices for the irreps i, j, k, l as they are not affected by the above, as well as the other parts of the spin networks.

Let us summarize what we have achieved. If we have a Lie group G and impose gauge invariance on the honeycomb lattice Γ, the matrix elements of the magnetic operators in the Kitaev model in the spin network basis $\{|S\rangle\}$ can be done in two steps. First, one constructs a smooth connection in the manifold in which Γ is embedded. Then one uses the local rules for transforming the spin network (4.22) as in [7, formula (C1)]. During this process, the group elements also change as we deform the edges, whose holonomies are these group elements, but in the end, we can deform all edges to their original location. This way we find a linear combination of the spin network states $\{|S\rangle\}$ corresponding to the magnetic constraint given by the expression (3.3).

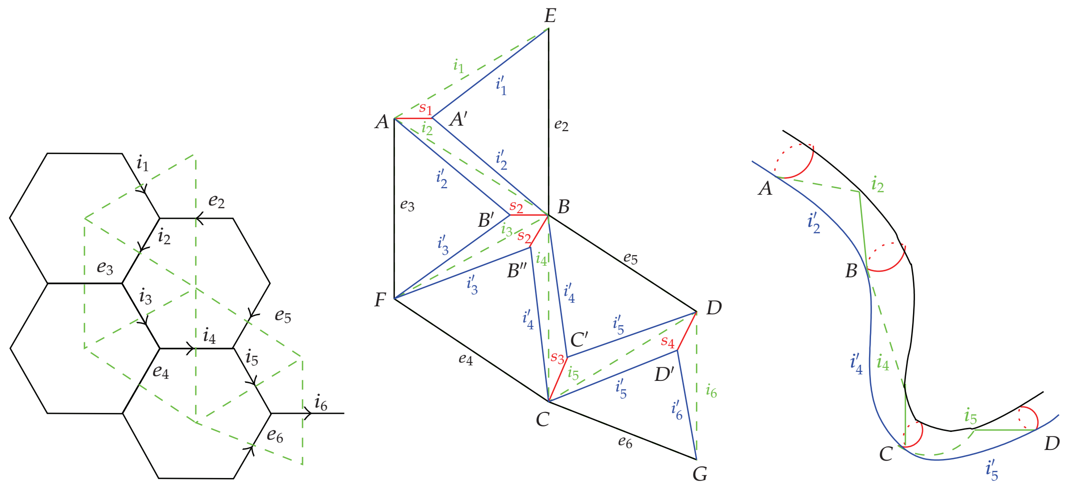
Nevertheless, for finite groups the local rules are, even if well motivated, postulates. The magnetic operator has been derived in a more direct way by introducing some auxiliary degrees of freedom in the very recent paper in [28].

5. Ribbon Operators

In Section 3.1 we have been studying the ground state, the constraints that it stabilizes and the projection from the Hilbert space into the ground state as a three-dimensional TV amplitude. One of the main physical interests, however, is the string-like excitations, the ribbon operators, which correspond to quasiparticles. We are going to sketch the corresponding preliminary results to illustrate that the logic which worked for the ground-state projection, provides us with the three-dimensional interpretation of these quantities as well.

A general ribbon in the spin net model is a string running along a certain path in the honeycomb lattice. The corresponding operator has the following structure:

$$W^{i'_1 i'_2 \cdots i'_N}_{i_1 i_2 \cdots i_N}(e_1 e_2 \cdots e_N) = \sum_{\{s_k\}} \left(\prod_{k=1}^{N} F^{s_k}_k \right) \mathrm{Tr}\left(\prod_{k=1}^{N} \Omega^{s_k}_k \right), \tag{5.1}$$



(a) (b) (c)

Figure 4

where k runs through the vertices of the string, and e_k is the label of the third edge adjacent to the kth vertex, which is not part of the string. The label s_i is the "type" of the string. The index structure of the $6j$ symbols is given by

$$F_k^s = \begin{cases} F^{e_k i_k^* i_{k-1}}_{s^* i'_{k-1} i'^*_k}, & \text{if } P \text{ turns left at } \mathbf{I_k}; \\[2ex] F^{e_k i^*_{k-1} i_k}_{s i'_k i'^*_{k-1}}, & \text{if } P \text{ turns right at } \mathbf{I_k}; \mathrm{c} \end{cases} \tag{5.2}$$

where $\mathbf{I_k}$ is the kth vertex of the string. The Ω matrices in a string operator have the index structure

$$\Omega_k^{s_k} = \begin{cases} \dfrac{v_{i_k} v_{s_k}}{v_{i'_k}} \quad \Omega^{i'_k}_{s_k s_{k+1} i_k}, & \text{if } P \text{ turns right, left at } \mathbf{I_k}, \mathbf{I_{k+1}}, \\[2ex] \dfrac{v_{i_k} v_{s_k}}{v_{i'_k}} \quad \overline{\Omega}^{i'_k}_{s_k s_{k+1} i_k}, & \text{if } P \text{ turns left, right at } \mathbf{I_k}, \mathbf{I_{k+1}}, \\[2ex] \delta_{s_k s_{k+1}} \cdot \mathrm{Id}, & \text{otherwise}, \end{cases} \tag{5.3}$$

and generically Ω^i_{jkl} are matrices (so they have two more indices, which are suppressed above). We would like to proceed as in Section 3 and find a TV amplitude that a string operator describes. Before asking what the Ω matrices correspond to, let us see what geometry we find by passing the description to the dual graph and gluing a tetrahedron whenever there is an F symbol.

In Figure 4(a) we have depicted a part of a string, indicating the dual graph along. In the following we will mean this line when referring to the string, and we will mean the collection of dual triangles (shown by green dashed lines in Figure 4(a)) when referring to the ribbon. In Figure 4(b) we took the ribbon and drew a tetrahedron over each triangle it

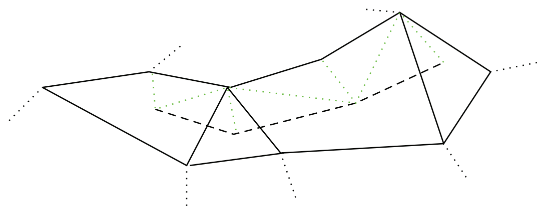


Figure 5

consists of, as dictated by (5.1). The decoration of the edges follow the index structure of the operator. The edges with the same label belong to the same edge of the spin net, so they are to be glued. This results in the Figure 4(c).

We may interpret the above in the following way. There is the path $ABCD\cdots$ in the dual graph $\widetilde{\Gamma}_0^{\{j\}}$, which is a continuous line of dual edges of the ribbon that correspond to edges of Γ, which connect vertices with different turning directions of the string (Figures 4(a) and 4(c)). There is an analogous path $A'B'C'D'\cdots$ in $\widetilde{\Gamma}_1^{\{j'\}}$. The gluing dictated by the algebraic structure of the operator is such that the line in $\widetilde{\Gamma}_1^{\{j'\}}$ winds around the one in $\widetilde{\Gamma}_0^{\{j\}}$ exactly once during each segment of the line. For each such segment an Ω matrix is present in the form of the operator and the notation $\widetilde{\Gamma}_i$ $(i = 0, 1)$ refers to the initial and the final string nets.

The observables in the TV model are typically ribbon graphs, fat graphs or links embedded in a manifold, over the labels of which, there is no summation in the amplitude [26, 27]. They are invariant under isotopy transformations. This property is ensured by the precise form of the braiding matrices, which then satisfy the Yang-Baxter equation. The latter equation seems to be related to equation (22) of [7] in the spin nets. However, the precise relation and the identification of the braiding matrix in the TV models with an expression of Ω as, for example, the work in [30] suggests should be found for a complete equivalence.

5.1. Kitaev's Ribbons

The ribbon operators are present in the Kitaev model as well [4].

A prototypical example shown in the Figure 5 is given by a strip between a path along the edges of the original lattice (thick lines) and a neighboring path in the dual lattice (dashed lines). It can be composed of elementary operators associated to triangles, which connect sites, that is, pairs of a plaquette and a vertex on its boundary. In Figure 5 sites are indicated by green dotted lines. One elementary building block (i, p) is a triangle, which is composed of an edge i and a dual vertex (which corresponds to a plaquette p). The other elementary building block (j, v) is also a triangle composed of a dual edge (which corresponds to the original edge j) and a vertex v. The associated operators depend on elements of the double $\mathfrak{D}(G)$, which can be represented by pairs of group elements (g, h). The two types of elementary ribbon operators read

$$W^{(h,g)}(i,v) = \delta_{g,1}L^h(i,v) \qquad {W'}^{(h,g)}(j,p) = T^{g^{-1}}(j,l). \tag{5.4}$$

Recall that the lattice is assumed to be oriented, so these formulae make sense. The composition of these elementary operators into a long ribbon is done by the comultiplication,

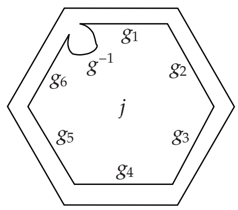


Figure 6

which is given by

$$W^{(h,g)} = \sum_{h_i,g_i} W^{(h_1,g_1)} W^{(h_2,g_2)} \omega^{(h,g)}_{(h_1,g_1)(h_2,g_2)} \quad \text{with } \omega^{(h,g)}_{(h_1,g_1)(h_2,g_2)} = \delta_{g,g_1g_2}\delta_{h_1,h}\delta_{h_2,g_1^{-1}hg_1}. \tag{5.5}$$

It is desirable to express these ribbons in the spin network basis to recover their corresponding matrix elements in the spin net model. However, there are several obstacles, which should be overcome to accomplish this task. In [7], there are additional local rules to reduce a generalized spin net containing ribbons, to the basis $\{|S\rangle\}$; see the beginning of Section 4. In order for this to work, one should find a generalized spin network representation of the above operators. Another difficulty comes about when the dual string crosses the original one. In this case, the elementary triangles overlap and the corresponding comultiplication operations do not commute. One needs to find a consistent rule to define their comultiplication in a non-ambiguous way. Note that the simplest ribbon operator in the spin net model, which is the one that winds around one hexagon, is easily found to correspond to (4.15). We find the following equality:

$$B_g(p) = \sum_j \operatorname{tr}\Big(D^j(g)\Big) B_p^j. \tag{5.6}$$

The procedure to get it is doing the comultiplication for the six elementary operators, all corresponding to the second type in (5.4) to arrive at $\delta_{g_1g_2g_3g_4g_5g_6g^{-1},1}$. Here the g_i are the group elements corresponding to the edges in the group algebra basis $\{|g_1, g_2, \ldots, g_E\rangle\}$. Then one draws a generalized spin network representation corresponding to the Plancherel decomposition of the Dirac delta as shown in Figure 6 (similarly to those for the magnetic constraints) and resolves it to the spin network basis by using the local moves. It is, however, not straightforward to generalize it (Furthermore, the argument given in Section 4.2 in favour of the local rules is also lost, since the underlying connection here is not flat.)

6. Summary and Outlook

In this paper we have been studying the lattice models of Levin, Wen, and Kitaev from two perspectives. On one hand we identified the ground states and the constraint operators of these models in case the underlying lattice is the honeycomb and the gauge group is a finite group. This has been achieved by changing the basis from that of the group algebra, that is, when edges are decorated by group elements, to the Fourier basis. This basis is spanned by the matrix elements of the irreps. A special linear combination by means of invariant intertwiners at the vertices has been shown to provide the range of all electric constraints and the projection at individual vertices has been identified with the projection to the invariant subspace. Then, the magnetic operators in the group algebra basis have been shown to correspond to those in the spin net model once the local rules postulated in the latter are satisfied. We gave an argument in favour of them from lattice gauge theory with continuous gauge group.

A second focus of the paper was on mapping the spin net to the Turaev-Viro state sum. We have used the idea of building up simplicial manifolds by tetrahedra with edges decorated with irreps corresponding to $6j$ symbols in the algebraic expressions of operators in the spin net model. This provided the three-dimensional geometric interpretation for the ribbon operator. Having a precise TV amplitude identified with the ribbon operator in the spin net needs further investigation.

One would also like to match these ribbon operators also in the model of Kitaev and the spin net of Levin and Wen. However, finding generalized spin network representations of the previous so that one could reduce them to the spin network basis is not straightforward.

In a series of papers [31–33], families of q-deformed "spin network automata" were implemented for processing efficiently classes of computationally-hard problems in geometric topology in particular, approximate calculations of topological invariants of links (collections of knots) and of closed 3-manifolds. A prominent role was played there by "universal" unitary braiding operators associated with suitable representations of the braid group in the tensor algebra of $(SU(2)_q)$. Traces of matrices of these representations provide polynomial invariants of $SU(2)_q$-colored links (actually framed links), while weighted sums of the latter give topological invariants of 3-manifolds presented as complements of framed knots in the 3-sphere. These invariants are in turn recognized as partition functions and vacuum expectation values of physical observables (Wilson loop operators) in 3-dimensional Chern-Simons-Witten (CSW) Topological Quantum Field Theory [1]. As is well known (see, e.g., [6], the review in [34], and the original references therein), any 3D TQFT of BF type can be presented as a "double" CSW model, on one hand; and the square modulus of the Witten invariant for a closed oriented 3-manifold equals the TV invariant for the same manifold, on the other.

The remarks above make it manifest that the efficient (approximate) quantum algorithms proposed in [31–33] could be extended in a quite straightforward way to the string-net ground states and ribbon-like excitations framed in the "naturally discretized" double $SU(2)$ CSW environment given by the TV approach, as we have done in the present paper. Work is in progress in this direction.

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