Absorbers: Definitions, Properties and Applications

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Roughly speaking, the absorber is a set, which includes, after finite number of initial states, each trajectory of a transformation of space into itself. This paper deals with the exact definition of absorbers for linear operators, the study of the properties, the applications to "classical" dynamics and to solvability of operator equations. It is expected that the description of the structure of absorbers will add new insights to the recent discussion of nature and content of notion of attractiveness for nonlinear dynamics.

Keywords: Absorbers, Induced dynamics, Functional equations

1 INTRODUCTION

We consider linear operators in the spaces of continuous or smooth functions on topological spaces or, correspondingly, on smooth manifolds.

More exactly, let X and Y be topological spaces, and $\mathcal{E}_1, \mathcal{E}_2$ be linear topological spaces. Denote by ξ_0 and η_0 the sheaves of germs of continuous mappings $X \to \mathcal{E}_1$ and $Y \to \mathcal{E}_2$ correspondingly. It means (see [6]) that the space $\Gamma(\xi_0|U)$ of local sections on $U \subset X$ is the set of all continuous mappings $U \to \mathcal{E}_1$. Similarly, $\psi \in \Gamma(\eta_0|V)$, $V \subset Y$ means that ψ is a continuous mapping $V \to \mathcal{E}_2$.

Further, we consider the *subsheaves* $\xi \subset \xi_0$ and $\eta \subset \eta_0$. It means that $\Gamma(\xi|U)$, $U \subset X$, and $\Gamma(\eta|V) \subset \Gamma(\eta_0|V)$, $V \subset Y$, are corresponding subspaces.

We do not assume the spaces $\Gamma(\xi)$ and $\Gamma(\eta)$ of global sections to be endowed with any topology.

Consider some linear mapping $T: \Gamma(\xi) \to \Gamma(\eta)$. We show that T induces some "dynamics" on the set of all subsets $S \subset X$. In terms of this dynamics, we single out the definition of *absorbers*. The main property of these subsets is that they contain all obstacles to the solvability of the equation

$$T\varphi = \gamma, \tag{1.1}$$

with given $\gamma \in \Gamma(\eta)$ and unknown $\varphi \in \Gamma(\xi)$. Connections between absorbers and "classical" dynamics of mappings are also discussed.

2 DEFINITIONS

2.1 Induced Dynamics

For a section $\varphi \in \Gamma(\xi)$ we set

$$N(\varphi) = \{y \in Y \mid (T\varphi)(y) = 0\}.$$

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Obviously, $N(\varphi) \subset Y$ is a closed subset. For a subset $S \subset Y$ set

$$T_*(S) = \bigcap N(\varphi), \quad \varphi|S = 0.$$

In other words, $y \in T_*(S)$ *if and only if* $(T\varphi)(y) = 0$ *for all sections* $\varphi \in \Gamma(\xi)$ *vanishing on* S.

It may happen that $T_*(S) = \emptyset$ for a subset $S \subset Y$. Obviously,

$$T_*(X) = Y, \qquad (\text{id})_*(S) = S.$$

Here id$: \Gamma(\xi) \to \Gamma(\xi)$ is identity mapping.

Example 2.1   Let

$$\Gamma(\xi) = C(X), \qquad \Gamma(\eta) = C(Y).$$

It means that $\xi = \xi_0$, $\eta = \eta_0$ with $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}$ (see Introduction). Given continuous $f: Y \to X$ let us set

$$(T\varphi)(y) = \varphi(f(y)), \quad \varphi \in C(X).$$

Then

$$T_*(S) \supset f^{-1}(S).$$

Example 2.2   More generally, let $f_t: Y \to X$, $t \in I \subset \mathbb{R}$ be a family of continuous mappings. Assume that measure μ on I and continuous mapping $g: Y \times \mathbb{R} \to \mathbb{R}$ are such that

$$(T\varphi)(y) = \int g(t, y) \cdot \varphi(f_t y)\, d\mu \in C(Y)$$

for all $C(X)$. In other words, T acts from $C(X)$ to $C(Y)$. Then

$$T_*(S) \supset \bigcap_{t \in I} f_t^{-1}(S).$$

2.2   Some Properties of the Induced Operation

PROPOSITION 2.3   (a) *The subset* $T_*(S) \subset Y$ *is closed for any* $S \subset X$;
(b) $S_1 \subset S_2 \Rightarrow T_*(S_1) \subset T_*(S_2)$;
(c) $T_*(S_1 \cup S_2) \supset T_*(S_1) \cup T_*(S_2)$;
(d) $T_*(S_1 \cap S_2) \subset T_*(S_1) \cap T_*(S_2)$.

These properties are obvious.

Consider three sheaves ξ, η, ζ on X, Y and Z correspondingly. Let $T: \Gamma(\xi) \to \Gamma(\eta)$ and $H: \Gamma(\eta) \to \Gamma(\zeta)$ be some mappings.

PROPOSITION 2.4   *The inclusion*

$$(TH)_*(S) \supset T_* H_*(S), \quad S \subset X,$$

holds.

Indeed,

$$\varphi_1|S = \varphi_2|S \Rightarrow H\varphi_1|H_*(S) = H\varphi_2|H_*(S).$$

Hence

$$TH\varphi_1|T_* H_*(S) = TH\varphi_2|T_* H_*(S).$$

It follows from this that

$$T_* H_*(S) \subset (TH)_*(S).$$

In particular, in $\xi = \eta$, then one can consider powers of the mappings T and T_*. It follows from Proposition 2.4 that

$$(T^n)_*(S) \supset (T_*)^n(S), \quad n = 0, 1, 2, \ldots .$$

2.3   Absorbers

DEFINITION 2.5   A pair (A, L) where $A \subset Y$ is a closed subset and $L: \Gamma(\eta) \to \Gamma(\xi)$ is a linear mapping is called an absorber for the mapping T if the following properties hold:
(a) Invariance:

$$(TL)_*(A) \supset A.$$

(b) Absorption: for any $y \in Y$ there exist a neighborhood $W \ni y$ and a number $n_0 = n_0(y)$ such that

$$W \subset ((\text{id} - TL)^n)_*A, \quad n \geq n_0.$$

Example 2.6   Let $\Gamma(\xi) = \Gamma(\eta) = C(X)$ and

$$f: X \to X, \qquad g: X \to \mathbb{R}$$

are continuous mappings. Then the mapping

$$(T\varphi)(x) = g(x) \cdot \varphi(f(x))$$

acts in $C(X)$. Let us fix some open covering

$$X = \bigcup_\alpha U_\alpha$$

and a numerical function $q(\alpha) \in \mathbb{R}$. Set

$$A = A(\{q(\alpha)\}, \{U_\alpha\}) = \overline{\bigcup_\alpha \bigcup_{n \geq q(\alpha)} f^n(U_\alpha)}.$$

Then the pair (A, id) be an absorber for T.

Example 2.7 Let

$$(T\varphi)(y) = \sum_{i=1}^{l} a_i(y)\varphi(f_i y), \quad a_i \in C(Y),$$

where $f_i \colon Y \to X$ are continuous. Assume that $a_1(y) \neq 0$ ($y \in Y$) and that f_1 is a homeomorphism. Set

$$A = \overline{\bigcup_\alpha \bigcup_{n \geq q(\alpha)} (f_1^{-1}f_{i_1} \cdots (f_1^{-1}f_{i_n})(U_\alpha)}, \quad i_\nu \in \overline{1, l},$$

$$(L\psi)(x) = \frac{1}{a_1(f_1^{-1}x)}\psi(f_1^{-1}x), \quad \psi \in C(Y).$$

Then (A, L) is an absorber for T.

Consider the set $\{(A, L)\}$ of all absorbers for T and denote

$$\mathcal{A}(T) = \bigcap A.$$

3 EXTENSION OF SOLUTIONS FOR THE OPERATOR EQUATIONS

The section $\varphi_0 \in \Gamma(\xi)$ is said to be a *solution of* Eq. (1.1) *on a subset* $V \in Y$ if the difference $\gamma - T\varphi_0$ vanishes on V.

THEOREM 3.1 *Let (A, L) be an absorber. Then for any solution φ_0 of Eq. (1.1) on A there exists a solution (on X) which coincides with φ_0 on the subset L_*A.*

Proof Consider the equation

$$TL\psi = \tilde{\gamma}, \quad \tilde{\gamma} = \gamma_0 - T\varphi_0. \tag{3.1}$$

It is sufficient to prove that this equation has a solution $\psi \in \Gamma(\eta)$, vanishing on A. Then the section $\varphi = \varphi_0 + L\psi$ is a solution we need.

To this end let us set

$$\psi_n = \tilde{\gamma}_+ + \widetilde{T}\psi_{n-1}, \quad n = 1, 2, \ldots, \quad \psi_0 = 0,$$

with $\widetilde{T}\psi = \psi - TL\psi$.

First of all note that

$$\psi_n | A = 0, \quad n = 0, 1, \ldots . \tag{3.2}$$

Indeed, $\psi_1 = \gamma - T\varphi_0$. Since φ_0 is a solution on A, we have $(\gamma - T\varphi_0)|A = 0$. Hence, $\psi_1|A = 0$. Now we obtain (3.2) by induction. Namely, assume that $\psi_{n-1}|A = 0$. Then $TL\psi_{n-1}$ vanishes on the subset $(TL)_*A \supset A$. Therefore, $\psi_n = \tilde{\gamma} + \psi_{n-1} - TL\psi_{n-1}$ coincides with ψ_{n-1} on A, i.e. $\psi_n|A = 0$.

Further, let $y_0 \in Y$. Choose a neighborhood $W \ni y_0$ and a number $n_0(y)$ such that

$$W \subset (\widetilde{T}^n)_* A, \quad n \geq n_0.$$

Consider a local section $h \in \Gamma(\eta|W)$, setting

$$h = \psi_n | W, \quad n \geq n_0.$$

It is well defined. Indeed, let $n \geq n_0$. Then

$$\psi_n = \sum_{k=0}^{n_0 - 1} \widetilde{T}^k \gamma + \widetilde{T}^{n_0} \psi_{n-n_0}.$$

Since

$$\psi_{n-n_0} | A = 0,$$

the section ψ_n coincides with $\sum_{k=0}^{n_0-1} \widetilde{T}^k \gamma$ on $(\widetilde{T}^{n_0})_* A \supset W$. In other words, ψ_n is *stabilized* on W under $n \geq n_0$.

Continuing this construction, we will build some open covering $Y = \bigcup_\alpha W_\alpha$ and local sections $h_\alpha \in \Gamma(\eta|W_\alpha)$. Obviously, these sections coincide

on the intersection $W_\alpha \cap W_\beta$. Since η is a sheaf, there exists a global section ψ such that $\psi|W_\alpha = h_\alpha$. The section ψ satisfies (3.1) and $\psi|A = 0$.

In particular, assume that $\gamma|A = 0$. Then the *zero section* $\varphi_0 = 0$ is a solution on A. Hence, we obtain

COROLLARY 3.2 *Let (A, L) be an absorber. If the section γ vanishing on A, then Eq. (1.1) has a solution φ vanishing on $L_* A$.*

Further, let (A_i, L_i), $i \in \overline{1, q}$ be some finite number of absorbers. Assume that φ_0 is a solution on the intersection $\bigcap_i A_i$ in some more strong sense. Namely,

$$\gamma - T\varphi_0 = \gamma_1 + \cdots + \gamma_q,$$

where $\gamma_i|A_i = 0$, $i \in \overline{1, q}$. Then we have

COROLLARY 3.3 *There exists a solution of (1.1) coinciding with φ_0 on the intersection $\bigcap_{i=1}^q (L_i)_* A_i$.*

Proof Consider the following equations:

$$T\varphi_i = \gamma_i, \quad i \in \overline{1, q}.$$

Since $\gamma_i|A_i = 0$, there exists a solution φ_i vanishing on $(L_i)_* A_i$. Then the section $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_q$ is a solution of (1.1) coinciding with φ_0 on the intersection $\bigcap_i (L_i)_* A_i$.

In the case that any section $\delta \in \Gamma(\eta)$ vanishing on $\bigcap_{i=1}^q A_i$ may be decomposed to the sum $\sum_{i=1}^q \gamma_i$, $\gamma_i|A_i = 0$, the solvability on $\bigcap_i A_i$ implies a global one. In particular, if $\bigcap_{i=1}^q A_i = \emptyset$, then we obtain an existence theorem: for any $\gamma \in \Gamma(\eta)$ Eq. (1.1) has a solution $\varphi \in \Gamma(\xi)$.

4 APPLICATIONS

4.1 Localization of Linear Functionals

Let $T: \Gamma(\xi) \to \Gamma(\eta)$ be a linear mapping. Then one can define a *conjugate* mapping

$$T^* : (\Gamma(\eta))^0 \to (\Gamma(\xi))^0$$

between dual spaces of all linear functionals:

$$(T^*\varphi)(\mu) = \mu(T\varphi), \quad \mu \in (\Gamma(\eta))^0.$$

Obviously,

$$\gamma \in \operatorname{Im} T \;\Rightarrow\; \mu(\gamma) = 0 \quad (\mu \in \operatorname{Ker} T^*).$$

In other words, $\operatorname{Im} T \subset (\operatorname{Ker} T^*)_\perp$.

We will say that a point $y_0 \in Y$ *belongs to the support of a functional* $\mu \in (\Gamma(\eta))^0$ if for any neighborhood $U \ni y_0$ there exists a section $\gamma \in \Gamma(\eta)$ vanishing outside of U such that

$$\mu(\gamma) \neq 0.$$

The set $\operatorname{supp}\mu \subset Y$ of all such points y_0 is a closed subset. It may happen that $\operatorname{supp}\mu = \emptyset$. However, in any case we have

THEOREM 4.1 *The inclusion*

$$\operatorname{supp}\mu \subset \mathcal{A}(T) \quad (\mu \in \operatorname{Ker} T^*)$$

holds.

Proof Let $y_0 \notin A$. Since A is closed, there exists a neighborhood $U \ni y_0$ such that $U \cap A = \emptyset$. Assume that $y_0 \in \operatorname{supp}\mu$, $\mu \in \operatorname{Ker} T^*$. Consider a section $\gamma \in \Gamma(\eta)$ vanishing outside of U and such that $\mu(\gamma) \neq 0$. Since $\gamma|A = 0$, $\gamma \in \operatorname{Im} T$ (see Corollary 3.2). The latter contradicts with inequality $\mu(\gamma) \neq 0$.

In the most interesting cases both of the spaces $\Gamma(\xi)$ and $\Gamma(\eta)$ are endowed by some topology. Then one can consider subspaces $(\Gamma(\xi))^* \subset (\Gamma(\xi))^0$ and $(\Gamma(\eta))^* \subset (\Gamma(\eta))^0$ of *continuous* linear functionals. If the linear operator $T: \Gamma(\xi) \to \Gamma(\eta)$ is a continuous one, then the conjugated T^* is so.

For instance, let X and Y be a locally compact topological spaces or smooth manifolds and let $\Gamma(\xi) = C^k(X)$, $\Gamma(\eta) = C^p(Y)$ with some $p, k \in [0, \infty]$. They are endowed by a natural topology with the convergence of all derivatives up to p (or k) on compact subsets. Then $(\Gamma(\xi))^*$ and $(\Gamma(\eta))^*$ are spaces of corresponding distributions with compact supports. In particular,

$$\mu \in (\Gamma(\xi))^*, \quad \operatorname{supp}\mu = \emptyset \;\Rightarrow\; \mu = 0.$$

It is well known (see [7]) that

$$\overline{\operatorname{Im} T} = (\operatorname{Ker} T^*)_\perp.$$

Assume that $\gamma|\mathcal{A}(T) = 0$. It follows from Theorem 4.1 that $\gamma \in (\operatorname{Ker} T^*)_\perp$. Hence, we obtain

COROLLARY 4.2 *If a function* $\gamma \in (\Gamma(\eta))^*$ *vanishes on* $\mathcal{A}(T)$ *then* $\gamma \in \overline{\operatorname{Im} T}$. *In particular, if* $\mathcal{A}(T) = \emptyset$, *then* $\operatorname{Im} T$ *is dense in the space* $\Gamma(\eta)$.

On the other hand, as considerations after Corollary 3.3 show, in "regular cases" section γ, vanishing in a neighborhood of the intersection of a *finite number of absorbers* belongs to $\operatorname{Im} T$.

Example 4.3 Let $X = \mathbb{R}^1$ and

$$(T\varphi)(x) = \sum_{i=1}^{l} a_i(x)\varphi(\alpha_i x + \beta_i), \quad \varphi \in C(\mathbb{R}^1),$$

with $\alpha_i > 0$ and continuous coefficients $a_i(x) \neq 0$ $(x \in \mathbb{R}^1)$. Let

$$\alpha_1 = \cdots = \alpha_q = \alpha > \alpha_i, \quad i \in \overline{1, q},$$

and

$$\beta_1 < \cdots < \beta_q,$$

with some $q \in \overline{1, l}$. Then T has two families of absorbers $(A_\pm(c), L_\pm)$. Namely,

$$A_-(c) = [c, +\infty), \quad c < c_- \equiv \min_{i > q} \frac{\beta_i - \beta_1}{\alpha - \alpha_i},$$

with

$$(L_-\psi)(x) = \frac{1}{a_1(\alpha^{-1}(x - \beta_1))} \psi(\alpha^{-1}(x - \beta_1))$$

and

$$A_+(c) = (-\infty, c], \quad c > c_+ \equiv \max_{i > q} \frac{\beta_i - \beta_q}{\alpha - \alpha_i},$$

with

$$(L_+\psi)(x) = \frac{1}{a_q(\alpha^{-1}(x - \beta_q))} \psi(\alpha^{-1}(x - \beta_q)).$$

Hence,

$$\mathcal{A}(T) \subset [c_-, c_+].$$

In the case $c_+ < c_-$ it means that $\mathcal{A}(T) = \emptyset$. Thus, any function γ vanishing on the interval $[c_-, c_+]$, belongs to the closer of $\operatorname{Im} T$. In other words,

$$\mu \in \operatorname{Ker} T^* \Rightarrow \operatorname{supp} \mu \subset [c_-, c_+].$$

Moreover, if Eq. (1.1) is solvable on an inverval $[a, b]$ with $a < c_-$, $b > c_+$, then $\gamma \in \operatorname{Im} T$. In particular, in the case $c_- > c_+$ the operator $T: C(\mathbb{R}) \to C(\mathbb{R})$ is surjective (see [2]).

4.2 Absorbers and Dynamics of Mappings

Let $f: X \to X$ be a continuous mapping. Consider the linear operator

$$H_f(\varphi)(x) = \varphi(x) - \varphi(f(x)), \quad \varphi \in C(X).$$

There is a connection between absorbers for H_f and dynamical behavior of the mapping f: any absorber contains all periodic points $\operatorname{Per}(f)$ and, moreover, all "attracting" points. Namely, let us remind that a point $x_0 \in X$ is called ω-*limited for* f if there is some $z \in X$ and a sequence $\{n_k\} \subset \mathbb{Z}_+$ such that

$$f^{n_k}z \to x_0, \quad k \to \infty.$$

If f is a homeomorphism, then α-*limited point is a* ω-*limited one for* f^{-1}. Denote by $\Omega_+(f)$ the set of all ω-limited points. In the case f is a homeomorphism denote by $\Omega_-(f)$ the set of all α-limited points.

THEOREM 4.4 *Let* X *be a normal topological space. Then the inclusion*

$$\Omega_+(f) \subset \mathcal{A}(H_f)$$

holds.

Proof Let $x_0 \in \Omega_+(f)$ and assume that $x_0 \notin A$ for some absorber (A, L). Then there exists a

neighborhood $U \ni x_0$ such that $\overline{U} \cap A = \emptyset$. Let $\gamma \in C(X)$ be a nonnegative function such that

$$\gamma|\overline{U} = 1, \qquad \gamma|A = 0.$$

Due to Corollary 3.2, $\gamma \in \operatorname{Im} T$. Let $\varphi \in C(X)$ be a solution, i.e.

$$\varphi(x) - \varphi(f(x)) = \gamma(x) \quad (x \in X). \tag{4.1}$$

Then

$$\varphi(x) = \varphi(f^n(x)) + \sum_{j=0}^{n-1} \gamma(f^j(x)); \quad x \in X,$$

$$n = 1, 2, \ldots.$$

Now let $f^{n_k} z \to x_0$. Then we obtain

$$\varphi(z) = \varphi(f^{n_k} z) + \sum_{j=0}^{n_k-1} \gamma(f^j(z)). \tag{4.2}$$

One can assume that φ is bounded in the neighborhood U: $|\varphi(x)| \leq M$, $x \in \overline{U}$. Then (4.2) gives

$$\sum_{i=1}^{k-1} \gamma(f^{n_i} z) \leq \sum_{j=0}^{n_k-1} \gamma(f^j(z)) \leq 2M.$$

Since $\sum_{i=1}^{k-1} \gamma(f^{n_i} z) \to \infty$, $k \to \infty$, we obtain a contradiction.

COROLLARY 4.5   *Let* $f: X \to X$ *be a homeomorphism of a normal topological space. Then*

$$\Omega_+(f) \cup \Omega_-(f) \subset \mathcal{A}(f).$$

Indeed, let (A, L) be some absorber for H_f. Set $(L_0\psi)(x) = -\psi(f^n(x))$. Then $H_{f^{-1}} L_0 = H_f$. Hence, the pair $(A, L_0 L)$ is an absorber for $H_{f^{-1}}$.

4.3   Applications to the Solvability Problem of Functional Equation

Consider the functional equation

$$(T\varphi)(x) \equiv P(x)\varphi(x) + Q(x)\varphi(fx) = \gamma(x), \tag{4.3}$$

where $f: X \to X$ is a homeomorphism and

$$P: X \to \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m),$$
$$Q: X \to \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m),$$
$$\gamma: X \to \mathbb{C}^m$$

are given continuous mappings.

Assume that $\operatorname{rank} P(x) = m$ $(x \in X)$. Then the pair (A_1, L_1) where

$$A_1 = \overline{\bigcap_\alpha \bigcap_{n \geq q_\alpha} f^n(U_\alpha)}$$

(see Example 2.6) and

$$(L_1\psi)(x) = P^*(x)(P(x)P^*(x))^{-1}\psi(x)$$

is an absorber for T.

Analogously, assume that $\operatorname{rank} B(x) = m$ $(x \in X)$. Then the pair (A_2, L_2), where

$$A_2 = \overline{\bigcap_\alpha \bigcap_{n \geq q_\alpha} f^{-n}(U_\alpha)}$$

and

$$(L_2\psi)(x) = Q^*(x)(Q(x)Q^*(x))^{-1}\psi(x)$$

is an absorber.

Let us call Eq. (4.2) *nondegenerated* if

$$\operatorname{rank} P(x) = \operatorname{rank} Q(x) = m \quad (x \in X).$$

In this case solvability on intersection $A_1 \cap A_2$ implies a global one, at least if X is a normal topological space. In particular, if there are A_1 and A_2 with empty intersection, then (4.2) is solvable for any γ, i.e. T is surjective. Moreover

THEOREM 4.6   *Let* $f: X \to X$ *be a homeomorphism of a normal topological space. Then the following conditions are equivalent*:
(a) *the Abel equation*

$$\tau(x) - \tau(f(x)) = 1$$

has a continuous solution $\tau \in C(X)$;

(b) *any nondegenerated equation (4.3) has a continuous solution* $\varphi : X \to \mathbb{C}^n$;

(c) *there exists an open countable covering* $X = \cup_{k=1}^{\infty} U_k$, $U_{k+1} \supset U_k$ *such that*

$$f^n(\overline{U}_k) \cap \overline{U}_k = \emptyset, \quad n \geq n_k.$$

If X is a locally compact space countable at infinity, the condition (c) is equivalent to

(c') *for any compact subset* $K \subset X$ *there is a number* $n_0 = n_0(K)$ *such that*

$$f^n(K) \cap K = \emptyset, \quad n \geq n_0.$$

The same result is true in classes of smooth vector functions on a smooth manifold X.

Theorem 4.6 was proved in [5]. For another application of our general approach to functional equations see [1,2,3].

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