GLOBAL STABILITY OF DELAYED HOPFIELD NEURAL NETWORKS UNDER DYNAMICAL THRESHOLDS

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We study dynamical behavior of a class of cellular neural networks system with distributed delays under dynamical thresholds. By using topological degree theory and Lyapunov functions, some new criteria ensuring the existence, uniqueness, global asymptotic stability, and global exponential stability of equilibrium point are derived. In particular, our criteria generalize and improve some known results in the literature.

1. Introduction

Since Hopfield neural networks were introduced by Hopfield [9], they have been widely developed and studied both in theory and applications, including both continuous-time and discrete-time settings. Meanwhile, they have been successfully applied to associative memories, signal processing, pattern recognition, and optimization problems, and so on. Many essential features of these networks, such as qualitative properties of stability, oscillation, and convergence issues have been investigated by many authors, see [1, 2, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17] and the references cited therein. As is well known, the use of constant fixed delays in Hopfield neural networks provide a good approximation in simple circuits. Unfortunately, due to the existence of parallel pathways with a variety of axon sizes and lengths, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. Under these environments, the signal propagation is not instantaneous and cannot be described with discrete delays. Thus, a suitable way is to introduce continuously distributed delays determined by a delay kernel. Moreover, Hopfield neural networks with dynamic thresholds have not received wide attention. Motivated by this, Gopalsamy and Leung [6] considered the following delayed neural networks under thresholds

$$x'(t) = -x(t) + a\tanh\left[x(t) - b\int_0^\infty k(s)x(t-s)ds - c\right], \quad t \geq 0, \tag{1.1}$$

where $a > 0$, $b \geq 0$, $a(1-b) < 1$, $a(1+b) < 1$, $x \in C(\mathbb{R},\mathbb{R})$, and $k \in C(\mathbb{R}^+,\mathbb{R}^+)$ is delayed

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ker-function with the following property:

$$\int_0^\infty k(s)ds = 1, \tag{1.2}$$

$$\int_0^\infty sk(s)ds < +\infty. \tag{1.3}$$

For the physical meaning of signs in (1.1), one can refer to Gopalsamy and Leung [6]. If the delayed ker-function satisfies (1.2) and (1.3), then, using Lyapunov function, they established a sufficient condition ensuring global asymptotic stability of the unique equilibrium $x^* = 0$ of the system (1.1) with $c = 0$.

Cui [4] further considered the system (1.1). Using differential inequality and variations of constants, he obtained new criteria for global asymptotic stability of equilibrium $x^* = 0$ of system (1.1) with $c = 0$.

In this paper, our aim is to consider the multineurons model with delayed-ker-functions under dynamic thresholds. That is to say, we will consider the following more general multineurons model with delayed ker-functions under dynamic thresholds

$$x_i'(t) = -g_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j\left[x_j(t) - b_{ij}\int_0^\infty k_{ij}(s)x_j(t-s)ds - c_j\right], \quad t \geq 0, \tag{1.4}$$

where $i = 1,2,\dots,n$, n denotes the number of units in the neural networks (1.4), $x_i(t)$ represents the states of the ith neuron at time t, a_{ij} and d_j are positive constants, b_{ij} and c_j are nonnegative constants, a_{ij} denotes the strength of the jth neuron on the ith neuron, b_{ij} denotes a measure of the inhibitory influence of the past history of the jth neuron on the ith neuron, c_j denotes the neural threshold of the jth neuron, and $g_j : \mathbb{R} \to \mathbb{R}$ is continuous function, which denotes the rate with which the jth neuron will rest its potential to the resting state in isolation when disconnected from networks and external inputs, and satisfies the following hypothesis:

(H1) $g_j : \mathbb{R} \to \mathbb{R}$ *is differentiable and strictly monotone increasing, that is,*

$$d_j = \inf_{x\in\mathbb{R}} \{g_j'(x)\} > 0, \quad g_j(0) = 0, \; j = 1,2,\dots,n. \tag{1.5}$$

f_j denotes the output of the ith neuron at time t and satisfies the following hypothesis:

(H2) *for each* $j \in \{1,2,\dots,n\}$, $f_j : \mathbb{R} \to \mathbb{R}$ *is globally Lipschitz with Lipschitz constant* $L_j > 0$, *that is,*

$$|f_j(u) - f_j(v)| \leq L_j|u - v| \quad \forall u,v \in \mathbb{R}. \tag{1.6}$$

$k_{ij} : [0,+\infty) \to [0,+\infty)$ is a continuous delayed ker-function satisfying (1.2) and (1.3).

Using the topological degree theory and Lyapunov functions, some new criteria ensuring the existence, uniqueness, global asymptotic stability, and global exponential stability of equilibrium point of (1.4) are derived. In these results, we do not require the activation function f_j to be bounded, differentiable, and monotonic nondecreasing. Moreover, the symmetry of the connection matrix is not also necessary.

The initial condition associated with (1.4) is of the form

$$x_{0_i}(t) = \phi_i(t), \quad t \in (-\infty, 0],\ i = 1, 2, \dots, n, \tag{1.7}$$

where $\phi_i \in C((-\infty,0],\mathbb{R})$, $\phi_i(t)$ is bounded on $(-\infty,0]$, and the norm of $C((-\infty,0],\mathbb{R})$ is denoted by

$$||\phi(t)|| = \sup_{t\in(-\infty,0]} \sum_{i=1}^{n} |\phi_i(t)|, \tag{1.8}$$

where $\phi(t) = (\phi_1(t), \dots, \phi_n(t))$.

2. Existence and uniqueness of the equilibrium

In this section, we will consider existence and uniqueness of the equilibrium of the system (1.4). Before starting our main results, we first give the definitions of topological degree and homotopy invariance principle.

Definition 2.1 [3]. Assuming that $f(x) : \Omega \to \mathbb{R}^n$ is a continuous and differentiable function, if $p \notin f(\partial\Omega)$ and $J_f(x) \neq 0$, for all $x \in f^{-1}(p)$, then

$$\deg(f, \Omega, p) = \sum_{x\in f^{-1}(p)} \operatorname{sgn} J_f(x), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open set, $J_f(x) = \det(f_{ij}(x))$, $f_{ij}(x) = \partial f_i/\partial x_j$. Suppose $f(x) : \Omega \to \mathbb{R}^n$ is a continuous function, $g(x) : \Omega \to \mathbb{R}^n$ is a continuous and differentiable function, if $p \notin f(\partial\Omega)$ and $\|f(x) - g(x)\| < \rho(p, f(\partial\Omega))$, then

$$\deg(f, \Omega, p) = \deg(g, \Omega, p). \tag{2.2}$$

Lemma 2.2 (homotopy invariance principle) [3]. *Assuming that $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ is a continuous function, let $h_t(x) = H(x,t)$ and let $p : [0,1] \to \mathbb{R}^n$ be a continuous function satisfying $p(t) \notin h_t(\partial\Omega)$ if $t \in [0,1]$. Then, $\deg(h_t, \Omega, p(t))$ is independent of t.*

In the following, we will consider the existence and uniqueness of the equilibrium of system (1.4).

Theorem 2.3. *Assume that (H1), (H2), and (1.2) hold and that there exist positive constants $\xi_i > 0$ such that*

$$\xi_i d_i - \sum_{j=1}^{n} \xi_j a_{ji} L_i |1 - b_{ji}| > 0, \quad i = 1, 2, \dots, n. \tag{2.3}$$

Then, system (1.4) has a unique equilibrium x^*.

Proof. From (1.2), it is easy to see that $x^* = (x_1^*,\dots,x_n^*)$ is an equilibrium of the system (1.4) if and only if the following condition holds:

$$g_i(x_i^*) = \sum_{j=1}^{n} a_{ij} f_j[(1-b_{ij})x_j^* - c_j], \quad i = 1,2,\dots,n. \tag{2.4}$$

Let $h(x) = (h_1(x),\dots,h_n(x))$, where

$$h_i(x) = g_i(x_i) - \sum_{j=1}^{n} a_{ij} f_j[(1-b_{ij})x_j - c_j], \quad i = 1,2,\dots,n. \tag{2.5}$$

Obviously, the solutions of $h(x) = 0$ are equilibrium points of the system (1.4). We define a homotopic mapping

$$F(x,\lambda) = \lambda h(x) + (1-\lambda)g(x), \tag{2.6}$$

where $\lambda \in [0,1]$, $F(x,\lambda) = (F_1(x,\lambda),\dots,F_n(x,\lambda))$, and

$$F_i(x,\lambda) = \lambda h_i(x) + (1-\lambda)g_i(x_i). \tag{2.7}$$

Then, from (H1), (H2), and (1.2), it follows that

$$\begin{aligned}
|F_i(x,\lambda)| &= |\lambda h_i(x) + (1-\lambda)g_i(x)| \\
&= \left|\lambda g_i(x) - \lambda \sum_{j=1}^{n} a_{ij} f_j[(1-b_{ij})x_j - c_j]\right| \\
&\geq |g_i(x_i)| - \lambda \sum_{j=1}^{n} a_{ij} |f_j[(1-b_{ij})x_j - c_j]| \\
&= |g_i(x_i)| - \lambda \sum_{j=1}^{n} a_{ij} |f_j[(1-b_{ij})x_j - c_j] - f_j(-c_j) + f_j(-c_j)| \\
&\geq |g_i(x_i)| - \lambda \sum_{j=1}^{n} a_{ij} L_j |1-b_{ij}| |x_j| - \lambda \sum_{j=1}^{n} a_{ij} |f_j(-c_j)| \\
&\geq d_i |x_i| - \lambda \sum_{j=1}^{n} a_{ij} L_j |1-b_{ij}| |x_j| - \lambda \sum_{j=1}^{n} a_{ij} |f_j(-c_j)| \\
&\geq \lambda\left(d_i |x_i| - \lambda \sum_{j=1}^{n} a_{ij} L_j |1-b_{ij}| |x_j|\right) - \lambda \sum_{j=1}^{n} a_{ij} |f_j(-c_j)|.
\end{aligned} \tag{2.8}$$

Further, by (2.3), we have

$$\begin{aligned}\sum_{i=1}^{n}\xi_i|F_i(x,\lambda)| &\geq \lambda\sum_{i=1}^{n}\xi_i\left(d_i|x_i| - \lambda\sum_{j=1}^{n}a_{ij}L_j|1-b_{ij}||x_j|\right)\\ &\quad -\lambda\sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{n}a_{ij}|f_j(-c_j)|\right)\\ &= \lambda\sum_{i=1}^{n}\left[\xi_i d_i - \sum_{j=1}^{n}\xi_j a_{ji}L_i|1-b_{ji}|\right]|x_i|\\ &\quad -\lambda\sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{n}a_{ij}|f_j(-c_j)|\right).\end{aligned} \tag{2.9}$$

Define

$$\begin{aligned}\xi_0 &= \min_{1\leq i\leq n}\left\{\xi_i d_i - \sum_{j=1}^{n}\xi_j a_{ji}L_i|1-b_{ji}|\right\},\\ a_0 &= \max_{1\leq i\leq n}\xi_i\sum_{j=1}^{n}a_{ij}|f_j(-c_j)|.\end{aligned} \tag{2.10}$$

Then, $\xi_0 > 0$ by (2.3) and a_0 is a positive constant by (H2). Let

$$U(0) = \left\{x \mid |x_i| < \frac{n(a_0+1)}{\xi_0}\right\}. \tag{2.11}$$

It follows from (2.11) that for any $x \in \partial(U(0))$, there exist $1 \leq i_0 \leq n$ such that

$$|x_{i_0}| = \frac{n(a_0+1)}{\xi_0}. \tag{2.12}$$

By (2.10), we can obtain that for any $\lambda \in (0,1]$,

$$\begin{aligned}\sum_{i=1}^{n}\xi_i|F_i(x,\lambda)| &\geq \lambda\sum_{i=1}^{n}\left[\xi_{i_0}d_{i_0} - \sum_{j=1}^{n}\xi_j a_{ji_0}L_{i_0}|1-b_{ji_0}|\right]|x_{i_0}| - \lambda\sum_{i=1}^{n}a_0\\ &\geq \lambda\xi_0|x_{i_0}| - \lambda n a_0\\ &= \lambda n > 0,\end{aligned} \tag{2.13}$$

which implies that $F(x,\lambda) \neq 0$ for any $x \in \partial(U(0))$ and $\lambda \in (0,1]$.

If $\lambda = 0$, from (2.6) and (H1), we have $F(x,\lambda) = g(x) \neq 0$ for any $x \in \partial(U(0))$. Hence, $F(x,\lambda) \neq 0$ for any $x \in \partial(U(0))$ and $\lambda \in [0,1]$. From (H1), it is easy to prove deg(g, $U(0),0) = 1$. From Lemma 2.2, we have

$$\deg(F,U(0),0) = \deg(g,U(0),0) = 1. \tag{2.14}$$

By topological degree theory, we can conclude that equation $h(x) = 0$ has at least a solution in $U(0)$. That is to say, system (1.4) has at least an equilibrium point x^*.

In the following, we will consider uniqueness of the equilibrium x^* of system (1.4). Suppose $y^* = (y_1^*, \dots, y_n^*)$ is also an equilibrium point of the system (1.4), then, we have

$$g_i(y_i^*) = \sum_{j=1}^{n} a_{ij} f_j[(1-b_{ij})y_j^* - c_j], \quad i = 1,2,\dots,n. \tag{2.15}$$

By (2.4) and (2.15), we have that for each $j \in \{1,2,\dots,n\}$,

$$g_i(x_i^*) - g_i(y_i^*) = \sum_{j=1}^{n} a_{ij}\{f_j[(1-b_{ij})x_j^* - c_j] - f_j[(1-b_{ij})y_j^* - c_j]\}. \tag{2.16}$$

According to (H1) and (H2), we get

$$d_i|x_i^* - y_i^*| \le \sum_{j=1}^{n} a_{ij}L_j|1-b_{ij}||x_j^* - y_j^*|, \tag{2.17}$$

and so

$$\sum_{i=1}^{n} \xi_i d_i |x_i^* - y_i^*| \le \sum_{i=1}^{n}\left[\xi_i \sum_{j=1}^{n} a_{ij}L_j|1-b_{ij}||x_j^* - y_j^*|\right], \tag{2.18}$$

namely,

$$\sum_{i=1}^{n}\left[\xi_i d_i - \sum_{j=1}^{n} \xi_j a_{ji} L_i |1-b_{ji}|\right]|x_i^* - y_i^*| \le 0. \tag{2.19}$$

In view of (2.3), we get $|x_i^* - y_i^*| = 0$, namely, $x_i^* = y_i^*$, $i = 1,2,\dots,n$. Hence, $x^* = y^*$. Therefore, system (1.4) has a unique equilibrium point x^*. The proof is complete. □

If x^* is a unique equilibrium of system (1.4), we set

$$y(t) = x(t) - x^*, \tag{2.20}$$

then, for $i = 1,2,\dots,n$, by (1.4), we have for $t \ge 0$,

$$y_i'(t) = -g_i(y_i(t) + x^*) + \sum_{j=1}^{n} a_{ij} f_j\left[y_j(t) + x_j^* - b_{ij}\int_0^{\infty} k_{ij}(s)(y_j(t-s) + x_j^*)\,ds - c_j\right]. \tag{2.21}$$

Further, by (H1), (H2), (1.2), and (2.4), we get

$$\begin{aligned} y_i'(t) &= -\left(g_i(y_i(t)+x^*) - g_i(x^*)\right) - \sum_{j=1}^{n} a_{ij} f_j\left[(1-b_{ij})x_j^* - c_j\right] \\ &\quad + \sum_{j=1}^{n} a_{ij} f_j\left[y_j(t) + (1-b_{ij})x_j^* - b_{ij}\int_0^\infty k_{ij}(s) y_j(t-s)ds - c_j\right] \\ &\le -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} L_j \left| y_j(t) - b_{ij}\int_0^\infty k_{ij}(s) y_j(t-s)ds \right| \\ &\le -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} L_j \left[|y_j(t)| + b_{ij}\int_0^\infty k_{ij}(s) |y_j(t-s)| ds \right]. \end{aligned} \tag{2.22}$$

Obviously, if x^* is a unique equilibrium point of the system (1.4), then $y(t) = 0$ is a unique equilibrium point of system (2.21), moreover, $y(t) = 0$ is the trivial solution of system (2.21). Therefore, the equilibrium x^* of system (1.4) is globally asymptotically stable and globally exponentially stable if and only if the trivial solution $y(t) = 0$ of system (2.21) is globally asymptotically stable and globally exponentially stable.

Taking $\xi_i = 1$, $i = 1,2,\ldots,n$ in condition (2.3), the following result holds.

Corollary 2.4. *Assume that (H1), (H2), and (1.2) hold and that*

$$d_i > \sum_{j=1}^{n} a_{ji} L_i |1 - b_{ji}|, \quad i = 1,2,\ldots,n. \tag{2.23}$$

Then, the system (1.4) has a unique equilibrium x^*.

3. Global stability analysis

In this section, we will consider global asymptotic stability and global exponential stability of the unique equilibrium of system (1.4).

Theorem 3.1. *Assume that (H1), (H2), (1.2), and (1.3) hold and that there exist positive constants $\xi_i > 0$ such that*

$$\xi_i d_i - \sum_{j=1}^{n} \xi_j a_{ji} L_i (1 + b_{ji}) > 0, \quad i = 1,2,\ldots,n. \tag{3.1}$$

Then, the trivial solution of the system (2.21) is globally asymptotically stable.

Proof. Since

$$\xi_i d_i - \sum_{j=1}^{n} \xi_j a_{ji} L_i |1 - b_{ji}| \ge \xi_i d_i - \sum_{j=1}^{n} \xi_j a_{ji} L_i (1 + b_{ji}) > 0, \tag{3.2}$$

the condition (2.3) of Theorem 2.3 holds. Hence, Theorem 2.3 implies that system (1.4) has a unique equilibrium x^*, and so, (2.21) holds.

Consider the Lyapunov function defined as follows:

$$V_1(t) = \sum_{i=1}^{n} \xi_i \left\{ |y_i(t)| + \sum_{j=1}^{n} a_{ij} b_{ij} L_j \int_0^{\infty} k_{ij}(s) \left(\int_{t-s}^{t} |y_j(\tau)| \, d\tau \right) ds \right\}. \tag{3.3}$$

Calculating the upper right derivative $D^+ V_1(t)$ along the solution of system (2.21), by (1.3), (2.22), and (3.1), we get

$$\begin{aligned}
& D^+ V_1(t)|_{(2.21)} \\
&\quad = \sum_{i=1}^{n} \xi_i \left\{ (\operatorname{sgn} y_i(t)) y_i'(t) + \sum_{j=1}^{n} a_{ij} b_{ij} L_j \int_0^{\infty} k_{ij}(s) [\, |y_j(t)| - |y_j(t-s)| \,] ds \right\} \\
&\quad \le \sum_{i=1}^{n} \xi_i \left\{ -d_i |y_i(t)| + \sum_{j=1}^{n} a_{ij} L_j \left[|y_j(t)| + b_{ij} \int_0^{\infty} k_{ij}(s) |y_j(t-s)| \, ds \right] \right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n} a_{ij} b_{ij} L_j \left[|y_j(t)| - \int_0^{\infty} k_{ij}(s) |y_j(t-s)| \, ds \right] \right\} \\
&\quad \le \sum_{i=1}^{n} \xi_i \left[-d_i |y_i(t)| + \sum_{j=1}^{n} a_{ij} L_j (1 + b_{ij}) |y_j(t)| \right] \\
&\quad = \sum_{i=1}^{n} \left[-\xi_i d_i + \sum_{j=1}^{n} \xi_j a_{ji} L_i (1 + b_{ji}) \right] |y_i(t)| \\
&\quad \le \alpha \sum_{i=1}^{n} |y_i(t)|,
\end{aligned} \tag{3.4}$$

where

$$\alpha = \min_{1 \le i \le n} \left\{ -\xi_i d_i + \sum_{j=1}^{n} \xi_j a_{ji} L_i (1 + b_{ji}) \right\}, \tag{3.5}$$

and $\alpha < 0$ by (3.1). Therefore, (3.4) means that the trivial solution of system (2.21) is globally asymptotically stable, and hence, the equilibrium x^* of the system (1.4) is globally asymptotically stable. The proof is complete. □

Taking $\xi_i = 1$ in (3.1), then

$$\sum_{j=1}^{n} a_{ji} L_i (1 + b_{ji}) < d_i, \quad i = 1, 2, \ldots, n. \tag{3.6}$$

In view of Theorem 2.3, we have the following result.

Corollary 3.2. *Assume that (H1), (H2), (1.2), and (1.3) hold and that*

$$\max_{1\le i\le n}\sum_{j=1}^{n} a_{ji}L_i(1+b_{ji}) < d_i. \tag{3.7}$$

Then, system (1.4) has a unique equilibrium x^* which is globally asymptotically stable.

Theorem 3.3. *Assume that (H1) and (H2) hold and f_i, $i = 1,2,\dots,n$ is bounded on $\mathbb{R}$. Then, all solutions of system (1.4) remain bounded on $(0,+\infty)$ and there exists an equilibrium for system (1.4).*

Proof. It is easy to see that all solutions of system (1.4) satisfy the following differential inequalities:

$$-d_ix_i(t)-\beta_i \le x_i'(t) \le -d_ix_i(t)+\beta_i, \quad i = 1,2,\dots,n, \tag{3.8}$$

where $\beta_i = \sum_{j=1}^{n}(a_{ij}\sup_{s\in\mathbb{R}}|f_j(s)|)$. In view of (3.8), we can obtain that all solutions of the system (1.4) remain bounded on $(0,+\infty)$.

By the well-known Brouwer's fixed point theorem, it is easy to see that the system (1.4) has an equilibrium. Since its proof is simple, it will be omitted. □

Remark 3.4. It is well known that Brouwer's fixed point theorem does not guarantee the uniqueness of the fixed point. Therefore, we will derive some criteria on the globally asymptotic stability of the equilibrium of system (1.4), which guarantee the uniqueness of the equilibrium.

Theorem 3.5. *Assume that (H1), (H2), (1.2), and (1.3) hold, and further f_i, $i = 1,2,\dots,n$ is bounded on $\mathbb{R}$ and there exists a positive diagonal matrix $\xi = \operatorname{diag}(\xi_1,\dots,\xi_n)$ such that for $i = 1,2,\dots,n$, one of the following conditions holds:*

(i) $\sum_{j=1}^{n}[a_{ij}(1+b_{ij})L_j\xi_i + a_{ji}(1+b_{ji})L_i\xi_j] < 2d_i\xi_i$;
(ii) $\sum_{j=1}^{n}[a_{ij}(L_j+b_{ij})L_j\xi_i + a_{ji}(1+L_ib_{ji})\xi_j] < 2d_i\xi_i$;
(iii) $\sum_{j=1}^{n}[a_{ij}(1+b_{ij}L_j)\xi_i + a_{ji}(L_i+b_{ji})L_i\xi_j] < 2d_i\xi_i$;
(iv) $\sum_{j=1}^{n}[a_{ij}(1+L_jb_{ij})L_j\xi_i + a_{ji}(L_i+b_{ji})\xi_j] < 2d_i\xi_i$;
(v) $\sum_{j=1}^{n}[a_{ij}(1+b_{ij})L_j^2\xi_i + a_{ji}(1+b_{ji})\xi_j] < 2d_i\xi_i$;
(vi) $\sum_{j=1}^{n}[a_{ij}(1+b_{ij}L_j^2)\xi_i + a_{ji}(L_i^2+b_{ji})\xi_j] < 2d_i\xi_i$;
(vii) $\sum_{j=1}^{n}[a_{ij}(L_j+b_{ij})\xi_i + a_{ji}(1+L_ib_{ji})L_i\xi_j] < 2d_i\xi_i$;
(viii) $\sum_{j=1}^{n}[a_{ij}(L_j^2+b_{ij})\xi_i + a_{ji}(1+b_{ji}L_i^2)\xi_j] < 2d_i\xi_i$;
(ix) $\sum_{j=1}^{n}[a_{ij}(1+b_{ij})\xi_i + a_{ji}(1+b_{ji})L_i^2\xi_j] < 2d_i\xi_i$.

Then, the trivial solution of system (2.21) is globally asymptotically stable.

Proof. Since f_i $(i = 1,2,\dots,n)$ is bounded on $\mathbb{R}$, Theorem 3.3 holds, and so (2.21) holds.

(i) Consider the Lyapunov function defined as follows:

$$V_2(t) = \sum_{i=1}^{n}\xi_i\left\{\frac{1}{2}y_i^2(t) + \frac{1}{2}\sum_{j=1}^{n}a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\left(\int_{t-s}^{t}y_j^2(\tau)d\tau\right)ds\right\}. \tag{3.9}$$

Calculating the upper right derivative of $V_2(t)$, using (1.2), (1.3), and (2.22), we have

$$\begin{aligned}
&D^+V_2(t)|_{(2.21)}\\
&\quad=\sum_{i=1}^n \xi_i\left\{y_i(t)y_i'(t)+\frac{1}{2}\sum_{j=1}^n a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\left[y_j^2(t)-y_j^2(t-s)\right]ds\right\}\\
&\quad\le\sum_{i=1}^n \xi_i\left\{-d_iy_i^2(t)+\frac{1}{2}\sum_{j=1}^n a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\int_{t-s}^t\left[y_j^2(t)-y_j^2(t-s)\right]ds\right.\\
&\qquad\qquad\left.+\sum_{j=1}^n a_{ij}L_j\left[\,|y_i(t)|\,|y_j(t)|+b_{ij}\,|y_i(t)|\int_0^\infty k_{ij}(s)\,|y_j(t-s)|\,ds\right]\right\}\\
&\quad\le\sum_{i=1}^n \xi_i\left\{-d_iy_i^2(t)+\frac{1}{2}\sum_{j=1}^n a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\int_{t-s}^t\left[y_j^2(t)-y_j^2(t-s)\right]ds\right.\\
&\qquad\qquad+\frac{1}{2}\sum_{j=1}^n a_{ij}L_j\left[y_i^2(t)+y_j^2(t)\right]\\
&\qquad\qquad\left.+\frac{1}{2}\sum_{j=1}^n a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\left[y_i^2(t)+y_j^2(t-s)\right]ds\right\}\\
&\quad=\sum_{i=1}^n \xi_i\left[-d_iy_i^2(t)+\frac{1}{2}\sum_{j=1}^n a_{ij}(1+b_{ij})L_j\left(y_i^2(t)+y_j^2(t)\right)\right]\\
&\quad=\sum_{i=1}^n\left\{-d_i\xi_i+\frac{1}{2}\sum_{j=1}^n\left[a_{ij}(1+b_{ij})L_j\xi_i+a_{ji}(1+b_{ji})L_i\xi_j\right]\right\}y_i^2(t)\\
&\quad\le\beta\sum_{i=1}^n y_i^2(t),
\end{aligned}\tag{3.10}$$

where

$$\beta=\max_{1\le i\le n}\left\{-d_i\xi_i+\frac{1}{2}\sum_{j=1}^n\left[a_{ij}(1+b_{ij})L_j\xi_i+a_{ji}(1+b_{ji})L_i\xi_j\right]\right\}\tag{3.11}$$

and $\beta<0$ by condition (i). From (3.10), we have

$$V_2(t)-\beta\int_0^t\sum_{i=1}^n y_i^2(t)dt\le V_2(0),\qquad t\ge 0.\tag{3.12}$$

It follows from (3.12) that

$$\sum_{i=1}^n y_i^2(t)\in L_1[0,+\infty).\tag{3.13}$$

In view of Theorem 3.3, we know that the solution $x_i(t)$ of system (1.4) and its derivative $x_i'(t)$ is bounded on $[0,+\infty)$, which implies boundedness of $y_i(t)$ and $y_i'(t)$, hence, $y_i(t)$ is uniformly continuous on $[0,+\infty)$, and so $\sum_{i=1}^{n} y_i^2(t)$ is also uniformly continuous on $[0,+\infty)$. From (3.10), we get

$$\sum_{i=1}^{n} y_i^2(t) \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty, \tag{3.14}$$

this means $y_i(t) \to 0$ as $t \to +\infty$ for all $i = 1,2,\ldots,n$. Thus, the trivial solution of the system (2.21) is globally asymptotically stable, and so, the equilibrium x^* of the system (1.4) is globally asymptotically stable.

The proof of (ii)–(ix) is complete similar to that of (i), with the exception of the definition of Lyapunov functions and choice of elements used to estimate the derivative $D^+V_i(t)|_{(2.21)}$ by the inequality $ab \le (1/2)(a^2+b^2)$, listed as follows.

(ii) Similar to (i), except using $L_j|y_i(t)||y_j(t)| \le (1/2)(L_j^2 y_i^2(t) + y_j^2(t))$.

(iii) Similar to (i), except using $L_j|y_i(t)||y_j(t)| \le (1/2)(y_i^2(t) + L_j^2 y_j^2(t))$.

(iv) The Lyapunov function defined by

$$V_3(t) = \sum_{i=1}^{n} \xi_i \left\{ \frac{1}{2} y_i^2(t) + \frac{1}{2} \sum_{j=1}^{n} a_{ij} b_{ij} \int_0^{\infty} k_{ij}(s) \left(\int_{t-s}^{t} y_j^2(\tau) d\tau \right) ds \right\}, \tag{3.15}$$

and $D^+V_3(t)|_{(2.21)}$ is estimated by

$$\begin{aligned} L_j |y_i(t)| |y_j(t)| &\le \frac{1}{2} L_j (y_i^2(t) + y_j^2(t)); \\ L_j |y_i(t)| |y_j(t-s)| &\le \frac{1}{2} (L_j^2 y_i^2(t) + y_j^2(t-s)). \end{aligned} \tag{3.16}$$

(v) Similar to (iv), except using $L_j|y_i(t)||y_j(t)| \le (1/2)(L_j^2 y_i^2(t) + y_j^2(t))$.

(vi) Similar to (iv), except using $L_j|y_i(t)||y_j(t)| \le (1/2)(y_i^2(t) + L_j^2 y_j^2(t))$.

(vii) The Lyapunov function defined by

$$V_4(t) = \sum_{i=1}^{n} \xi_i \left\{ \frac{1}{2} y_i^2(t) + \frac{1}{2} \sum_{j=1}^{n} a_{ij} b_{ij} L_j^2 \int_0^{\infty} k_{ij}(s) \left(\int_{t-s}^{t} y_j^2(\tau) d\tau \right) ds \right\}, \tag{3.17}$$

and $D^+V_4(t)|_{(2.21)}$ is estimated by

$$\begin{aligned} L_j |y_i(t)| |y_j(t)| &\le \frac{1}{2} L_j (y_i^2(t) + y_j^2(t)); \\ L_j |y_i(t)| |y_j(t-s)| &\le \frac{1}{2} (y_i^2(t) + L_j^2 y_j^2(t-s)). \end{aligned} \tag{3.18}$$

(viii) Similar to (vii), except using $L_j|y_i(t)||y_j(t)| \le (1/2)(L_j^2 y_i^2(t) + y_j^2(t))$.
(ix) Similar to (vi), except using $L_j|y_i(t)||y_j(t)| \le (1/2)(y_i^2(t) + L_j^2 y_j^2(t))$.
The proof is complete. □

Obviously, if we take $\xi_i = 1$ in Theorem 3.5, then we can obtain the following result.

Corollary 3.6. *Assume that (H1), (H2), (1.2), and (1.3) hold, and further f_i, $i = 1,2,\dots,n$, is bounded on $\mathbb{R}$ and one of the following conditions holds:*

(i) $\sum_{j=1}^n [a_{ij}(1+b_{ij})L_j + a_{ji}(1+b_{ji})L_i] < 2d_i$;
(ii) $\sum_{j=1}^n [a_{ij}(L_j+b_{ij})L_j + a_{ji}(1+L_i b_{ji})] < 2d_i$;
(iii) $\sum_{j=1}^n [a_{ij}(1+b_{ij}L_j) + a_{ji}(L_i+b_{ji})L_i] < 2d_i$;
(iv) $\sum_{j=1}^n [a_{ij}(1+L_j b_{ij})L_j + a_{ji}(L_i+b_{ji})] < 2d_i$;
(v) $\sum_{j=1}^n [a_{ij}(1+b_{ij})L_j^2 + a_{ji}(1+b_{ji})] < 2d_i$;
(vi) $\sum_{j=1}^n [a_{ij}(1+b_{ij}L_j^2) + a_{ji}(L_i^2+b_{ji})] < 2d_i$;
(vii) $\sum_{j=1}^n [a_{ij}(L_j+b_{ij}) + a_{ji}(1+L_i b_{ji})L_i] < 2d_i$;
(viii) $\sum_{j=1}^n [a_{ij}(L_j^2+b_{ij}) + a_{ji}(1+b_{ji}L_i^2)] < 2d_i$;
(ix) $\sum_{j=1}^n [a_{ij}(1+b_{ij}) + a_{ji}(1+b_{ji})L_i^2] < 2d_i$.

Then, the trivial solution of the system (2.21) is globally asymptotically stable.

Theorem 3.7. *Assume that (H1), (H2), and (1.2) hold and that*

(A1) $\int_0^\infty s k_j(s)e^s ds < +\infty$,
(A2) *there exists positive constant ξ_i, $i = 1,2,\dots,n$ and $\gamma > 0$ such that*

$$\xi_i(\gamma - d_i) + \sum_{j=1}^n \xi_j a_{ji}(1+b_{ji}M_{ji})L_i < 0, \quad i = 1,2,\dots,n, \tag{3.19}$$

where $M_{ij} = \int_0^\infty k_{ij}(s)e^s ds$. Then, the trivial solution of system (2.15) is globally exponentially stable.

Proof. Since $\lim_{s\to\infty} k_{ij}(s)e^s/sk_{ij}(s)e^s = 0$, by (A1), $\int_0^\infty k_{ij}(s)e^s ds < +\infty$. Further, we have $1 = \int_0^\infty k_j(s)ds < \int_0^\infty k_j(s)e^s ds$, so $M_j \ge 1$. By (A2), for $i = 1,2,\dots,n$, we get

$$\xi_i d_i - \sum_{j=1}^n \xi_j a_{ji}|1-b_{ji}|L_i \ge \xi_i(d_i-\gamma) - \sum_{j=1}^n \xi_j a_{ji}(1+b_{ji}M_{ji})L_i > 0. \tag{3.20}$$

By Theorem 2.3, we know that the system (1.4) has a unique equilibrium x^*, and so, (2.21) holds.

In the following, we only consider the case $0 < \gamma < 1$. If $\gamma \ge 1$, we can choose $0 < \delta < 1 \le \gamma$, and transform γ into δ in condition (A2), then the proof is the same as the case $0 < \gamma < 1$.

Consider the Lyapunov function defined by

$$V_5(t) = \sum_{i=1}^n \xi_i\left\{e^{\gamma t}|y_i(t)| + \sum_{j=1}^n L_j a_{ij} b_{ij}\int_0^\infty k_{ij}(s)\left(\int_{t-s}^t |y_j(\tau)|e^{\gamma(\tau+s)}d\tau\right)ds\right\}. \tag{3.21}$$

Calculating the upper right derivative $D^+V_5(t)$ along the solution of system (2.21), by (1.2), (2.22), and (A1), (A2) of Theorem 3.7, we have

$$\begin{aligned}
&D^+V_5(t)|_{(2.21)} \\
&\quad = \sum_{i=1}^{n}\xi_i\left\{\gamma e^{\gamma t}|y_i(t)| + e^{\gamma t}(\operatorname{sgn} y_i(t))y_i'(t)\right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n} a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\left[|y_j(t)|e^{\gamma(t+s)} - |y_j(t-s)|e^{\gamma t}\right]ds\right\} \\
&\quad = e^{\gamma t}\sum_{i=1}^{n}\xi_i\left\{(\gamma-d_i)|y_i(t)| + \sum_{j=1}^{n}a_{ij}L_j\left[|y_j(t)| + b_{ij}\int_0^\infty k_{ij}(s)|y_j(t-s)|ds\right]\right. \\
&\qquad\qquad \left. + \sum_{j=1}^{n} a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\left[|y_j(t)|e^{\gamma s} - |y_j(t-s)|\right]ds\right\} \\
&\quad \le e^{\gamma t}\sum_{i=1}^{n}\xi_i\left\{(\gamma-d_i)|y_i(t)| + \sum_{j=1}^{n}a_{ij}L_j\left[1 + b_{ij}\int_0^\infty k_j(s)e^{\gamma s}ds\right]|y_j(t)|\right\} \\
&\quad < e^{\gamma t}\sum_{i=1}^{n}\xi_i\left\{(\gamma-d_i)|y_i(t)| + \sum_{j=1}^{n}a_{ij}L_j(1+b_{ij}M_{ij})|y_j(t)|\right\} \\
&\quad = e^{\gamma t}\sum_{i=1}^{n}\left\{\xi_i(\gamma-d_i) + \sum_{j=1}^{n}\xi_j a_{ji}(1+b_{ji}M_{ji})L_i\right\}|y_i(t)| \\
&\quad \le \eta e^{\gamma t}\sum_{i=1}^{n}|y_i(t)|,
\end{aligned} \tag{3.22}$$

where

$$\eta = \max_{1\le i\le n}\left\{\xi_i(\gamma-d_i) + \sum_{j=1}^{n}\xi_j a_{ji}(1+b_{ji}M_{ji})L_i\right\}, \tag{3.23}$$

and $\eta < 0$ by (A2). So, we have $V_5(t) < V_5(0)$ for $t \ge 0$. Since

$$e^{\gamma t}\min_{1\le i\le n}\xi_i\sum_{i=1}^{n}|y_i(t)| \le V_5(t), \quad t \ge 0,$$

$$\begin{aligned}
V_5(0) &= \sum_{i=1}^{n}\xi_i\left\{|y_i(0)| + \sum_{j=1}^{n}a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)\left(\int_{-s}^{0}|y_j(\tau)|e^{\gamma(\tau+s)}d\tau\right)ds\right\} \\
&\le \left\{\max_{1\le i\le n}\xi_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\gamma^{-1}a_{ij}b_{ij}L_j\int_0^\infty k_{ij}(s)e^{\gamma s}ds\right\}||y(0)|| \\
&\le \left\{\max_{1\le i\le n}\xi_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\gamma^{-1}a_{ij}b_{ij}M_{ij}L_j\right\}||y(0)||,
\end{aligned} \tag{3.24}$$

where $y(0) = x^* - \phi$, then

$$e^{\gamma t} \min_{1\le i\le n} \xi_i \sum_{i=1}^{n} |\, y_i(t)\,| \le \left\{ \max_{1\le i\le n} \xi_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \gamma^{-1} a_{ij} b_{ij} M_{ij} L_j \right\} ||x^* - \phi||, \tag{3.25}$$

and so

$$\sum_{i=1}^{n} |\, y_i(t)\,| \le \theta ||x^* - \phi|| e^{-\gamma t}, \tag{3.26}$$

where

$$\theta = \frac{1}{\min_{1\le i\le n} \xi_i} \left\{ \max_{1\le i\le n} \xi_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_i \gamma^{-1} a_{ij} b_{ij} M_{ij} L_j \right\} \ge 1 \tag{3.27}$$

is a constant. From (3.26), we see that the trivial solution of system (2.21) is globally exponentially stable, and so, the equilibrium x^* of system (1.4) is globally exponentially stable. The proof is complete. □

COROLLARY 3.8. *Assume that (H1), (H2), and (1.2) hold and that*

(i) $\int_0^\infty sk(s)e^s ds < +\infty$,
(ii) $d_i > \sum_{j=1}^{n} a_{ji}(1 + b_{ji} M_{ji}) L_i$, *where* $M_{ji} = \int_0^\infty k_{ji}(s) ds \ge 1$.

Then, system (1.4) has a unique equilibrium which is globally exponentially stable.

4. Discussion and examples

It is easy to see that Theorem 3.5 concludes Theorem 3.1. However, the conditions of Corollary 3.6 are independent of the conditions of Corollary 3.2. That is to say, for any condition of (i)–(ix) of Corollary 3.6, there exists a network which satisfies it but does not satisfy the other, and also does not satisfy Corollary 3.2. The following Example 4.2 will prove the above fact. We note that Corollary 3.6 conclude Theorem 3.1 in this paper. However, the verification of Corollary 3.6 is much easier than that of Theorem 3.1.

In the following, we will give some examples to illustrate our results.

Example 4.1. Consider the following model:

$$\begin{aligned} x_1'(t) &= -3x_1(t) + \tanh\left[x_1(t) - \int_0^\infty e^{-s} x_1(t-s) ds - 1\right] \\ &\quad + \frac{3}{2}\tanh\left[x_2(t) - \frac{1}{3}\int_0^\infty e^{-s} x_2(t-s) ds - 2\right], \\ x_2'(t) &= -4x_2(t) + \frac{2}{3}\tanh\left[x_1(t) - \frac{1}{2}\int_0^\infty e^{-s} x_1(t-s) ds - 1\right] \\ &\quad + \tanh\left[x_2(t) - \frac{1}{2}\int_0^\infty e^{-s} x_2(t-s) ds - 2\right]. \end{aligned} \tag{4.1}$$

Functions $g_1(x_1(t)) = 3x_1(t)$ and $g_2(x_2(t)) = 4x_2(t)$ satisfy hypothesis (H1) with $d_1 = 3$ and $d_2 = 4$, respectively. Function $f_j(x) = \tanh x$ satisfies hypothesis (H2) with $L_j = 1$ ($j = 1,2$) and $k_{ij}(s) = e^{-s}$ satisfy (1.2) and (1.3) for $i, j = 1,2$. Then, taking $\xi_1 = 9/8$, $\xi_2 = 1$ in condition (3.1), we have

$$\begin{aligned} \xi_1 d_1 - \sum_{j=1}^{2} \xi_j a_{j1} L_1 (1 + b_{j1}) &= \frac{1}{8} > 0, \\ \xi_2 d_2 - \sum_{j=1}^{2} \xi_j a_{j2} L_2 (1 + b_{j2}) &= \frac{1}{4} > 0. \end{aligned} \tag{4.2}$$

However, for any $\xi_i > 0$ ($i = 1,2$), non of conditions (i)–(ix) of Theorem 3.5 is satisfied. Equation (4.2) implies that the condition of Theorem 3.1 holds, but Theorem 3.5 does not hold. Therefore, by Theorem 3.1, system (4.1) has a globally asymptotically stable equilibrium.

Example 4.2. Consider the following model:

$$\begin{aligned} x_1'(t) &= -\frac{36}{5} x_1(t) + 2\tanh\left[x_1(t) - \frac{1}{5}\int_0^\infty e^{-s} x_1(t-s)\,ds - 1\right] \\ &\quad + 2\tanh\frac{1}{2}\left[x_2(t) - \frac{3}{5}\int_0^\infty e^{-s} x_2(t-s)\,ds - \frac{3}{2}\right], \\ x_2'(t) &= -4x_2(t) + \frac{1}{2}\tanh\left[x_1(t) - \frac{2}{5}\int_0^\infty e^{-s} x_1(t-s)\,ds - 1\right] \\ &\quad + 4\tanh\frac{1}{2}\left[x_2(t) - \frac{2}{5}\int_0^\infty e^{-s} x_2(t-s)\,ds - \frac{3}{2}\right]. \end{aligned} \tag{4.3}$$

Functions $g_1(x_1(t)) = (36/5)x_1(t)$ and $g_2(x_2(t)) = 4x_2(t)$ satisfy hypothesis (H1) with $d_1 = 36/5$, $d_2 = 4$, respectively. Function $f_1(x) = \tanh x$, $f_2(x) = \tanh(1/2)x$ satisfies hypothesis (H2) with $L_1 = 1$ and $L_2 = 1/2$, respectively. Function $k_{ij}(s) = e^{-s}$ satisfies (1.2) and (1.3) for $i, j = 1,2$. It is easy to verify that condition (3.7) of Corollary 3.2 and conditions (ii)–(ix) of Corollary 3.8 do not hold. However, we have

$$\begin{aligned} 2d_1 - \sum_{j=1}^{2} \left[a_{1j}(1 + b_{1j})L_j + a_{j1}(1 + b_{j1})L_1\right] &= \frac{1}{10} > 0, \\ 2d_2 - \sum_{j=1}^{2} \left[a_{2j}(1 + b_{2j})L_j + a_{j2}(1 + b_{j2})L_2\right] &= \frac{1}{10} > 0. \end{aligned} \tag{4.4}$$

Namely, (4.4) implies that condition (i) of Corollary 3.6 holds. Therefore, by Corollary 3.6, system (4.3) has a globally asymptotically stable equilibrium.

Example 4.3. Consider the following model:

$$\begin{aligned} x_1'(t) &= -2x_1(t) + \frac{2}{3}\tanh\left[x_1(t) - \frac{1}{4}\int_0^\infty 2e^{-2s}x_1(t-s)ds - 1\right] \\ &\quad + \tanh\left[x_2(t) - \frac{1}{6}\int_0^\infty 2e^{-2s}x_2(t-s)ds - \frac{3}{2}\right], \\ x_2'(t) &= -5x_2(t) + \frac{5}{6}\tanh\left[x_1(t) - \frac{1}{6}\int_0^\infty 2e^{-2s}x_1(t-s)ds - 1\right] \\ &\quad + \tanh\left[x_2(t) - \frac{1}{4}\int_0^\infty 2e^{-2s}x_2(t-s)ds - \frac{3}{2}\right]. \end{aligned} \tag{4.5}$$

Functions $g_1(x_1(t)) = 2x_1(t)$ and $g_2(x_2(t)) = 5x_2(t)$ satisfy hypothesis (H1) with $d_1 = 2$ and $d_2 = 5$, respectively. Function $f_j(x) = \tanh x$ satisfies hypothesis (H2) with $L_j = 1$ for $j = 1,2$. Function $k_{ij}(s) = 2e^{-2s}$ satisfies (1.2) and condition (A1), $M_{ij} = \int_0^\infty k_{ij}(s)e^s ds = 2$ for $i, j = 1,2$. Then, taking $\xi_1 = 1$, $\xi_2 = 1/2$, we have

$$\begin{aligned} \xi_1(\gamma - d_1) + \sum_{j=1}^{2}\xi_j a_{j1}(1 + b_{j1}M_{j1})L_1 &= -\frac{1}{12} < 0, \\ \xi_2(\gamma - d_2) + \sum_{j=1}^{2}\xi_j a_{j2}(1 + b_{j2}M_{j2})L_2 &= -\frac{25}{36} < 0. \end{aligned} \tag{4.6}$$

Equation (4.6) implies that condition (A2) of Theorem 3.7 holds. Therefore, by Theorem 3.7, system (4.5) has a globally exponentially stable equilibrium.

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References

[1] J. D. Cao, *On exponential stability and periodic solutions of CNNs with delays*, Phys. Lett. A **267** (2000), no. 5-6, 312–318.

[2] T. Chen, *Global exponential stability of delayed Hopfield neural networks*, Neural Networks **14** (2001), no. 8, 977–980.

[3] J. Cronin, *Fixed Points and Topological Degree in Nonlinear Analysis*, Mathematical Surveys, no. 11, American Mathematical Society, Rhode Island, 1964.

[4] W. Y. Cui, *Global stability for a class of neural network models under dynamical thresholds with delays*, J. Biomath. **15** (2000), no. 4, 420–424 (Chinese).

[5] K. Gopalsamy and X.-Z. He, *Delay-independent stability in bidirectional associative memory networks*, IEEE Trans. Neural Networks **5** (1994), no. 6, 998–1002.

[6] K. Gopalsamy and I. K. C. Leung, *Convergence under dynamical thresholds with delays*, IEEE Trans. Neural Networks **8** (1997), no. 2, 341–348.

[7] Z.-H. Guan and G. R. Chen, *On delayed impulsive Hopfield neural networks*, Neural Networks **12** (1999), no. 2, 273–280.

[8] M. W. Hirsch, *Convergent activation dynamics in continuous time networks*, Neural Networks **2** (1989), no. 5, 331–349.

[9] J. J. Hopfield, *Neural networks and physical systems with emergent collective computational abilities*, Proc. Nat. Acad. Sci. U.S.A. **79** (1982), no. 8, 2554–2558.

[10] ———, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 10, 3088–3092.

[11] J. J. Hopfield and D. W. Tank, *"Neural" computation of decisions in optimization problems*, Biol. Cybernet. **52** (1985), no. 3, 141–152.

[12] H. Huang and J. D. Cao, *On global asymptotic stability of recurrent neural networks with time-varying delays*, Appl. Math. Comput. **142** (2003), no. 1, 143–154.

[13] X. Z. Liu and R. Dickson, *Stability analysis of Hopfield neural networks with uncertainty*, Math. Comput. Modelling **34** (2001), no. 3-4, 353–363.

[14] H. T. Lu, *On stability of nonlinear continuous-time neural networks with delays*, Neural Networks **13** (2000), no. 10, 1135–1143.

[15] C. M. Marcus and R. M. Westervelt, *Stability of analog neural networks with delay*, Phys. Rev. A (3) **39** (1989), no. 1, 347–359.

[16] S. Mohamad, *Convergence dynamics of delayed Hopfield-type neural networks under almost periodic stimuli*, Acta Appl. Math. **76** (2003), no. 2, 117–135.

[17] J. Zhang and X. Jin, *Global stability analysis in delayed Hopfield neural network models*, Neural Networks **13** (2000), no. 7, 745–753.

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