# ON MONOTONE SOLUTIONS OF SOME CLASSES OF DIFFERENCE EQUATIONS

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We describe a method for finding monotone solutions of some classes of difference equations converging to the corresponding equilibria. The method enables us to confirm three conjectures posed by the present author in a talk, which are extensions of three conjectures by M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations. With Open Problems and Conjectures*. Chapman and Hall/CRC, 2002. It is interesting that the method, in some cases, can be applied also when the parameters are variable.



## 1. Introduction

Recently there has been a great interest in studying nonlinear difference equations of order greater than one. Many of these equations stem from mathematical biology, economy, population dynamics , and so forth (see, e.g., [5, 7–9, 11, 14] and the references therein). An interesting problem in the theory of difference equations is finding monotone solutions. This paper is devoted to this problem.

Motivated by [8, Conjectures 5.4.6 and 6.10.3] in a talk (see, [16]) we posed the following three conjectures. The first one concerns a generalization of (1.2).

Conjecture 1.1. *Show that for every $p > -1$, the following equation:*

$$x_{n+1} = p + \frac{x_{n-k}}{\sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

*where $k \in \mathbb{N}$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a positive solution which remains above the equilibrium $\bar{x}_1 = p + 1$ for all $n \geq -k$.*

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*In [6] DeVault et al. investigate the behavior of the positive solutions of the difference equation*

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.2}$$

*where $p > 0$ and $k \in \mathbb{N}$ is fixed. Among other things they have proved that all nonoscillatory solutions of (1.2) converge to the positive equilibrium $\bar{x} = p + 1$.*

Based on this observation they have posed the following open problem.

*Open problem 1.2.* Do there exist nonoscillatory solutions of (1.2)?

The following conjectures are generalizations of [8, Conjectures 5.4.6 and 6.10.3].

Conjecture 1.3. *Show that the following equation:*

$$x_{n+1} = \frac{1 + x_{n-k}}{\sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.3}$$

*where $k \in \mathbb{N}$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a nontrivial positive solution which decreases to the equilibrium $x_2 = (1+\sqrt{5})/2$.*

Conjecture 1.4. *Show that the following equation:*

$$x_{n+1} = \frac{\alpha + x_{n-k}}{1 + \sum_{i=0}^{k-1} \alpha_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{1.4}$$

*where $k \in \mathbb{N}$, $\alpha > 0$, $\alpha_i \geq 0$, $i = 0, \ldots, k-1$, and $\sum_{i=0}^{k-1} \alpha_i = 1$, has a positive solution which decreases to the equilibrium $x_3 = \sqrt{\alpha}$.*

Our aim in this paper is to confirm the above mentioned conjectures.

The linearized equation for (1.1), respectively, (1.3) and (1.4), about the corresponding positive equilibrium $\bar{x}_i$, $i \in \{1, 2, 3\}$, is

$$(p+1)y_{n+1} + \alpha_0 y_n + \cdots + \alpha_{k-1} y_{n-k+1} - y_{n-k} = 0, \tag{1.5}$$

$$\bar{x}_2 \left( y_{n+1} + \alpha_0 y_n + \cdots + \alpha_{k-1} y_{n-k+1} \right) - y_{n-k} = 0, \tag{1.6}$$

$$(1+\sqrt{\alpha}) y_{n+1} + \sqrt{\alpha} \left( \alpha_0 y_n + \cdots + \alpha_{k-1} y_{n-k+1} \right) - y_{n-k} = 0. \tag{1.7}$$

The characteristic polynomial associated with (1.5), respectively, (1.6) and (1.7), is

$$p_1(t) = (p+1)t^{k+1} + \alpha_0 t^k + \cdots + \alpha_{k-1} t - 1 = 0, \tag{1.8}$$

$$p_2(t) = \bar{x}_2 \left( t^{k+1} + \alpha_0 t^k + \cdots + \alpha_{k-1} t \right) - 1 = 0, \tag{1.9}$$

$$p_3(t) = (1+\sqrt{\alpha}) t^{k+1} + \sqrt{\alpha} \left( \alpha_0 t^k + \cdots + \alpha_{k-1} t \right) - 1 = 0. \tag{1.10}$$

Since $p_1(0) = -1 < 0$, $p_1(1) = p + 1$, and $p_1'(t) = (p+1)(k+1)t^k + \alpha_0 k t^{k-1} + \cdots + \alpha_{k-1} > 0$ for $t \in (0, 1]$, it follows that for each $p > -1$, there is a unique positive root $t_1$ of the polynomial (1.8) belonging to the interval $(0, 1)$.

Similarly, it can be shown that (1.9) and (1.10) have also a unique positive roots $t_2$ and $t_3$ in the interval $(0,1)$.

This fact motivated us to believe that there are solutions of (1.1), (1.3), and (1.4) which have the following asymptotics:

$$x_n = \bar{x} + at_i^n + o(t_i^n), \tag{1.11}$$

where $a \in \mathbb{R}$ and $t_i$, $i \in \{1,2,3\}$, are the above mentioned roots of polynomials (1.5), (1.6), and (1.7), respectively .

We solve the open problem, showing that such solutions exist, developing Berg's idea in [2] which are based on asymptotics. Asymptotics for solutions of difference equations has been investigated for a long time by L. Berg and S. Stević, see, for example, [1–4, 10–15] and the reference therein. We solve it by constructing two appropriate sequences $y_n$ and $z_n$ with

$$y_n \le x_n \le z_n \tag{1.12}$$

for sufficiently large $n$. In [1, 2], some methods can be found for the construction of these bounds, see, also [3, 4].

From (1.11) and results in Berg's paper [2], we expect that for $k \ge 2$ such solutions have the first four members in their asymptotics in the following form:

$$\varphi_n = \bar{x} + at^n + bt^{2n} + ct^{3n}. \tag{1.13}$$

## 2. The inclusion theorem

We need the following result in the proof of the main theorem. The proof of the result is similar to that of [2, Theorem 1].

Theorem 2.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $(y_n)$ and $(z_n)$ be sequences with $y_n < z_n$ for $n \ge n_0$ and such that*

$$y_{n-k} \le f(n, y_{n-k+1}, \dots, y_{n+1}), \quad f(n, z_{n-k+1}, \dots, z_{n+1}) \le z_{n-k}, \tag{2.1}$$

*for $n > n_0 + k - 1$.*

*Then there is a solution of the following difference equation:*

$$x_{n-k} = f(n, x_{n-k+1}, \dots, x_{n+1}), \tag{2.2}$$

*with property (1.12) for $n \ge n_0$.*

*Proof.* Let $N$ be an arbitrary integer such that $N > n_0 + k - 1$. The solution $(x_n)$ of (2.2) with given initial values $x_N, x_{N+1}, \dots, x_{N+k}$ satisfying (1.12) for $n \in \{N, N+1, \dots, N+k\}$ can be continued by (2.2) to all $n < N$. Inequalities (2.1) and the monotonic character of $f$ imply that (1.12) holds for all $n \in \{n_0, \dots, N+k\}$. Let $A_N$ be the set of all $(k+1)$-tuples $(x_{n_0}, \dots, x_{n_0+k})$ such that there exist solutions $(x_n)$ of (2.2) with these initial values satisfying (1.12) for all $n \in \{n_0, \dots, N+k\}$. It is clear that $A_N$ is a closed nonempty set for every $N > n_0 + k - 1$, and that $A_{N+1} \subset A_N$. It follows that the set $A = \cap_{n=n_0+k}^{\infty} A_N$ is a nonempty subset of $\mathbb{R}^{k+1}$ and that if $(x_{n_0},\dots,x_{n_0+k}) \in A$, then the corresponding solutions of (2.2) satisfy (1.12) for all $n \geq n_0$, as desired. □

## 3. The main result

In this section we prove the main result of this paper, which confirms Conjectures 1.1, 1.3, and 1.4.

THEOREM 3.1. *The following statements are true:*

(a) *let* $\alpha_i \geq 0$, $i = 0,\dots,k-1$, $\sum_{i=0}^{k-1} \alpha_i = 1$, *and* $p > -1$. *Then (1.1) has a positive solution which remains above the equilibrium* $\bar{x}_1 = p+1$;

(b) *let* $\alpha_i \geq 0$, $i = 0,\dots,k-1$, $\sum_{i=0}^{k-1} \alpha_i = 1$. *Then (1.3) has a nontrivial positive solution which decreases to the equilibrium* $\bar{x}_2$;

(c) *let* $\alpha > 0$, $\alpha_i \geq 0$, $i = 0,\dots,k-1$, $\sum_{i=0}^{k-1} \alpha_i = 1$. *Then (1.4) has a nontrivial positive solution which decreases to the equilibrium* $\bar{x}_3 = \sqrt{\alpha}$.

*Proof.* (a) Note that (1.2) can be written in the following equivalent form:

$$F(x_{n-k},\dots,x_n,x_{n+1}) = (x_{n+1} - p)(\alpha_0 x_n + \cdots + \alpha_{k-1} x_{n-k+1}) - x_{n-k} = 0. \tag{3.1}$$

We expect that solutions of (1.2) have asymptotic approximation (1.13). Thus, we calculate $F(\varphi_{n-k},\dots,\varphi_n,\varphi_{n+1})$. We have

$$\begin{aligned} F &= \left(1 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)}\right) \\ &\quad \times \left(p+1+a\alpha_0 t^n + \cdots + a\alpha_{k-1}t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1}t^{2(n-k+1)} + \mathbb{O}(t^{3n})\right) \\ &\quad - \left(p+1+at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)}\right) \\ &= at^n\left((p+1)t + \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} - t^{-k}\right) \\ &\quad + t^{2n}\left(b\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}}\right) + a^2 t\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) + b(p+1)t^2 - bt^{-2k}\right) + \mathbb{O}(t^{3n}). \end{aligned} \tag{3.2}$$

Let

$$D_1(t) = (p+1)t + \alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}} - \frac{1}{t^k}. \tag{3.3}$$

Choose $t \in (0,1)$ such that $D_1(t) = 0$, and $a,b \in \mathbb{R}$, $a \neq 0$, such that the coefficients in (3.2) are equal to zero. $D_1(t) = 0$ implies that $t = t_1$ (see, Section 1). Further we obtain

$$b = -\frac{a^2 t_1 (\alpha_0 + \cdots + \alpha_{k-1} t_1^{-k+1})}{(p+1)t_1^2 + \alpha_0 + \cdots + (\alpha_{k-1})/t_1^{2(k-1)} - t_1^{-2k}} = -\frac{a^2 t_1 (\alpha_0 + \cdots + \alpha_{k-1} t_1^{-k+1})}{D_1(t_1^2)}. \tag{3.4}$$

If $\widehat{\varphi}_n = p+1+at_1^n+qt_1^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-k},\dots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1}) \sim \left(qD_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1})\right)t_1^{2n}. \tag{3.5}$$

Let

$$H_{t_1}(q) = qD_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1}). \tag{3.6}$$

We have

$$D_1'(t) = p+1+\frac{k}{t^{k+1}}-\frac{\alpha_1}{t^2}-\cdots-\frac{(k-1)\alpha_{k-1}}{t^k}. \tag{3.7}$$

Hence, when $t \in (0,1)$, it follows that

$$\begin{aligned} D_1'(t) &> p+1+\frac{k}{t^{k+1}}-\frac{\alpha_1+\cdots+(k-1)\alpha_{k-1}}{t^{k+1}} \\ &> p+1+\frac{k}{t^{k+1}}-\frac{(k-1)\sum_{i=1}^{k_1}\alpha_i}{t^{k+1}} > p+1+\frac{1}{t^{k+1}} > 0. \end{aligned} \tag{3.8}$$

From this, since $D_1(t_1)=0$, and $t_1^2 < t_1$, we have that $D_1(t_1^2) < 0$. Thus, we obtain that there are $q_1 < b$ and $q_2 > b$ such that $H_{t_1}(q_1) > 0$ and $H_{t_2}(q_2) < 0$.

With the notations

$$y_n = p+1+at_1^n+q_1t_1^{2n}, \qquad z_n = p+1+at_1^n+q_2t_1^{2n}, \tag{3.9}$$

we get

$$\begin{aligned} F(y_{n-k},\dots,y_n,y_{n+1}) &\sim \left(q_1D_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1})\right)t_1^{2n} > 0, \\ F(z_{n-k},\dots,z_n,z_{n+1}) &\sim \left(q_2D_1(t_1^2)+a^2t_1(\alpha_0+\cdots+\alpha_{k-1}t_1^{-k+1})\right)t_1^{2n} < 0. \end{aligned} \tag{3.10}$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large $n$, where $f = F + x_{n-k}$ and $F$ is given by (3.1). Applying Theorem 2.1 it follows that there is a solution of (1.1) with the asymptotics $x_n = \widehat{\varphi}_n + o(t_1^{2n})$, in particular, the solution of (1.1) converges monotonically to the positive equilibrium $\bar{x}_1 = p+1$, when $p > -1$ and $n \geq n_0$. Hence, the solution $x_{n+n_0+k}$ converges monotonically for $n \geq -k$.

(b) Equation (1.3) can be written in the following equivalent form:

$$F(x_{n-k},\dots,x_n,x_{n+1}) = x_{n+1}(\alpha_0x_n+\cdots+\alpha_{k-1}x_{n-k+1})-(1+x_{n-k}) = 0. \tag{3.11}$$

Now we calculate $F(\varphi_{n-k},\dots,\varphi_n,\varphi_{n+1})$. We have

$$\begin{aligned}F &= \left(\bar{x}_2+at^{n+1}+bt^{2(n+1)}+ct^{3(n+1)}\right)\\ &\quad\times\left(\bar{x}_2+a\alpha_0t^n+\cdots+a\alpha_{k-1}t^{n-k+1}+b\alpha_0t^{2n}+\cdots+b\alpha_{k-1}t^{2(n-k+1)}+\mathbb{O}(t^{3n})\right)\\ &\quad-\left(1+\bar{x}_2+at^{n-k}+bt^{2(n-k)}+ct^{3(n-k)}\right)\\ &= at^n\left(\bar{x}_2t+\bar{x}_2\left(\alpha_0+\cdots+\frac{\alpha_{k-1}}{t^{k-1}}\right)-\frac{1}{t^k}\right)\\ &\quad+t^{2n}\left(\bar{x}_2b\left(\alpha_0+\cdots+\frac{\alpha_{k-1}}{t^{2(k-1)}}\right)+a^2t\left(\alpha_0+\cdots+\frac{\alpha_{k-1}}{t^{k-1}}\right)+\bar{x}_2bt^2-bt^{-2k}\right)+\mathbb{O}(t^{3n}).\end{aligned}\tag{3.12}$$

Let

$$D_2(t)=\bar{x}_2t+\bar{x}_2\left(\alpha_0+\cdots+\frac{\alpha_{k-1}}{t^{k-1}}\right)-t^{-k}=\frac{p_2(t)}{t^k}.\tag{3.13}$$

Choose $t\in(0,1)$ such that $D_2(t)=0$, and $a,b\in\mathbb{R},a\neq 0$, such that the coefficients in (3.12) are equal to zero. Since $D_2(t)=0$ is equivalent to $p_2(t)=0$, we have that $t=t_2$, and consequently

$$b=-\frac{a^2t_2\left(\alpha_0+\cdots+\alpha_{k-1}t_2^{-k+1}\right)}{\bar{x}_2t_2^2+\bar{x}_2\left(\alpha_0+\cdots+(\alpha_{k-1})/t_2^{2(k-1)}\right)-t_2^{-2k}}=-\frac{a^2t_2\left(\alpha_0+\cdots+\alpha_{k-1}t_2^{-k+1}\right)}{D_2(t_2^2)}.\tag{3.14}$$

If $\widehat{\varphi}_n=\bar{x}_2+at_2^n+qt_2^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-k},\dots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1})\sim\left(qD_2(t_2^2)+a^2t_2\left(\alpha_0+\cdots+\alpha_{k-1}t_2^{-k+1}\right)\right)t_2^{2n}.\tag{3.15}$$

Let

$$H_{t_2}(q)=qD_2(t_2^2)+a^2t_2\left(\alpha_0+\cdots+\alpha_{k-1}t_2^{-k+1}\right).\tag{3.16}$$

Since

$$p_2'(t)=\bar{x}_2\left((k+1)t^k+k\alpha_0t^{k-1}+\cdots+\alpha_{k-1}\right)>0,\tag{3.17}$$

when $t\in(0,1)$, and since $p_2(t_2)=0$, and $t_2^2<t_2$, we have that $p_2(t_2^2)<0$, which implies $D_2(t_2^2)<0$. Thus, we obtain that there are $q_3<b$ and $q_4>b$ such that $H_{t_2}(q_3)>0$ and $H_{t_2}(q_4)<0$.

With the notations

$$y_n=\bar{x}_2+at_2^n+q_3t_2^{2n},\qquad z_n=\bar{x}_2+at_2^n+q_4t_2^{2n},\tag{3.18}$$

we get

$$\begin{aligned}F(y_{n-k},\dots,y_n,y_{n+1})&\sim H_{t_2}(q_3)t_2^{2n}>0,\\ F(z_{n-k},\dots,z_n,z_{n+1})&\sim H_{t_2}(q_4)t_2^{2n}<0.\end{aligned}\tag{3.19}$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large $n$, where $f = F + x_{n-k}$ and $F$ is given by (3.11). Applying Theorem 2.1 it follows that there is a solution of (1.3) with the asymptotics $x_n = \widehat{\varphi}_n + o(t_2^{2n})$. This solution obviously converges monotonically to the positive equilibrium $\bar{x}_2 = (\sqrt{5}+1)/2$, for $n \geq n_1$. A suitable shift of $x_n$ is decreasing for all $n \geq -k$.

(c) Equation (1.4) can be written in the following equivalent form:

$$F(x_{n-k},\dots,x_n,x_{n+1}) = x_{n+1}(1+\alpha_0 x_n + \cdots + \alpha_{k-1}x_{n-k+1}) - (\alpha + x_{n-k}) = 0. \tag{3.20}$$

We have

$$\begin{aligned} F &= (\bar{x}_3 + at^{n+1} + bt^{2(n+1)} + ct^{3(n+1)}) \\ &\quad \times (1+\bar{x}_3 + a\alpha_0 t^n + \cdots + a\alpha_{k-1}t^{n-k+1} + b\alpha_0 t^{2n} + \cdots + b\alpha_{k-1}t^{2(n-k+1)} + \mathcal{O}(t^{3n})) \\ &\quad - (\alpha + \bar{x}_3 + at^{n-k} + bt^{2(n-k)} + ct^{3(n-k)}) \\ &= at^n\left((1+\bar{x}_3)t + \bar{x}_3\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) - t^{-k}\right) \\ &\quad + t^{2n}\left(\bar{x}_3 b\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{2(k-1)}}\right) + a^2 t\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) + (1+\bar{x}_3)bt^2 - bt^{-2k}\right) \\ &\quad + \mathcal{O}(t^{3n}). \end{aligned} \tag{3.21}$$

Let

$$D_3(t) = (1+\bar{x}_3)t + \bar{x}_3\left(\alpha_0 + \cdots + \frac{\alpha_{k-1}}{t^{k-1}}\right) - t^{-k} = \frac{p_3(t)}{t^k}. \tag{3.22}$$

Choose $t \in (0,1)$ such that $D_3(t) = 0$, and $a,b \in \mathbb{R}$, $a \neq 0$, such that the coefficients in (3.21) are equal to zero.

Since

$$p_3'(t) = (1+\sqrt{\alpha})(k+1)t^k + \sqrt{\alpha}(k\alpha_0 t^{k-1} + \cdots + \alpha_{k-1}) > 0, \tag{3.23}$$

when $t \in (0,1]$, and $D_3(t) = 0$ is equivalent to $p_3(t) = 0$, we have that $t = t_3$. From this and (3.21) it follows that

$$b = -\frac{a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1})}{(1+\sqrt{\alpha})t_3^2 + \sqrt{\alpha}(\alpha_0 + \cdots + (\alpha_{k-1})/t_3^{2(k-1)}) - t_3^{-2k}} = -\frac{a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1})}{D_3(t_3^2)}. \tag{3.24}$$

If $\widehat{\varphi}_n = \sqrt{\alpha} + at_3^n + qt_3^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-k},\dots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1}) \sim (qD_3(t_3^2) + a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1}))t_3^{2n}. \tag{3.25}$$

Let

$$H_{t_3}(q) = qD_3(t_3^2) + a^2 t_3(\alpha_0 + \cdots + \alpha_{k-1}t_3^{-k+1}). \tag{3.26}$$

Since $D_3(t_3) = 0$, and $t_3^2 < t_3$, we have that $D_3(t_3^2) < 0$. Thus, we obtain that there are $q_5 < b$ and $q_6 > b$ such that $H_{t_3}(q_5) > 0$ and $H_{t_3}(q_6) < 0$.

With the notations

$$y_n = \sqrt{\alpha} + at_3^n + q_5 t_3^{2n}, \qquad z_n = \sqrt{\alpha} + at_3^n + q_6 t_3^{2n}, \tag{3.27}$$

we get

$$F(y_{n-k},\dots,y_n,y_{n+1}) \sim H_{t_3}(q_5)t_3^{2n} > 0, \qquad F(z_{n-k},\dots,z_n,z_{n+1}) \sim H_{t_3}(q_6)t_3^{2n} < 0. \tag{3.28}$$

These relations show that the inequalities in (1.12) are satisfied for sufficiently large $n$, where $f = F + x_{n-k}$ and $F$ is given by (3.20). Hence, there is a solution of (1.4) with the asymptotics $x_n = \hat{\varphi}_n + o(t_3^{2n})$. The result follows similarly to the above mentioned cases. □

From Theorem 3.1(a) with $\alpha_0 = 1$ and $\alpha_i = 0$, $i \neq 0$, we get the following corollary.

Corollary 3.2. *There is a nonoscillatory solution of (1.2).*

*Remark 3.3.* Since $a \in \mathbb{R} \setminus \{0\}$ is an arbitrary parameter, by Theorem 3.1 we find a set of nonoscillatory solutions of (1.1), (1.3), and (1.4) converging to the corresponding positive equilibria.

*Remark 3.4.* Note that using (1.13) better asymptotics for these solutions can be obtained, that is, $x_n = \varphi_n + o(t_i^{3n})$, $i \in \{1,2,3\}$, where $b$ is given by (3.4), (3.14), or (3.24), and $c$ can be found equating to zero the coefficient nearby $t^{3n}$.

*Remark 3.5.* From the proof of Theorem 3.1, we see that we can assume that the parameter $p$ in (1.1) can be replaced by a nondecreasing sequence with the following asymptotics: $p_n = p + o(t_1^{2n})$.

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## References

[1] L. Berg, *Asymptotische Darstellungen und Entwicklungen*, Hochschulbücher für Mathematik, vol. 66, VEB Deutscher Verlag der Wissenschaften, Berlin, 1968.

[2] ———, *On the asymptotics of nonlinear difference equations*, Zeitschrift für Analysis und ihre Anwendungen **21** (2002), no. 4, 1061–1074.

[3] ———, *Inclusion theorems for non-linear difference equations with applications*, Journal of Difference Equations and Applications **10** (2004), no. 4, 399–408.

[4] ———, *Corrections to: "Inclusion theorems for non-linear difference equations with applications" from [3]*, Journal of Difference Equations and Applications **11** (2005), no. 2, 181–182.

[5] J. M. Cushing and Y. Kuang, *Global stability in a nonlinear difference delay equation model of flour beetle population growth*, Journal of Difference Equations and Applications **2** (1996), no. 1, 31–37.

[6] R. DeVault, C. Kent, and W. Kosmala, *On the recursive sequence* $x_{n+1} = p + \frac{x_{n-k}}{x_n}$, Journal of Difference Equations and Applications **9** (2003), no. 8, 721–730.

[7] M. E. Fisher and B. S. Goh, *Stability results for delayed-recruitment models in population dynamics*, Journal of Mathematical Biology **19** (1984), no. 1, 147–156.

[8] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations. With Open Problems and Conjectures*, Chapman and Hall/CRC, Florida, 2002.

[9] E. C. Pielou, *Population and Community Ecology*, Gordon and Breach, New York, 1974.

[10] S. Stević, *Asymptotic behaviour of a sequence defined by iteration*, Matematički Vesnik **48** (1996), no. 3-4, 99–105.

[11] ______, *Behavior of the positive solutions of the generalized Beddington-Holt equation*, Panamerican Mathematical Journal **10** (2000), no. 4, 77–85.

[12] ______, *Asymptotic behavior of a sequence defined by iteration with applications*, Colloquium Mathematicum **93** (2002), no. 2, 267–276.

[13] ______, *Asymptotic behaviour of a sequence defined by a recurrence formula. II*, The Australian Mathematical Society Gazette **29** (2002), no. 4, 209–215.

[14] ______, *Asymptotic behavior of a nonlinear difference equation*, Indian Journal of Pure and Applied Mathematics **34** (2003), no. 12, 1681–1687.

[15] ______, *On the recursive sequence* $x_{n+1} = x_n + \frac{x_n^\alpha}{n^\beta}$, Bulletin of the Calcutta Mathematical Society **95** (2003), no. 1, 39–46.

[16] ______, *Some open problems and conjectures on difference equations*, April 29, 2004, http://www.mi.sanu.ac.yu.

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