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Research Article

Several Existence Theorems of Nonlinear *m*-Point BVP for an Increasing Homeomorphism and Homomorphism on Time Scales

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Several existence theorems of positive solutions are established for nonlinear m-point boundary value problem for the following dynamic equations on time scales $(\phi(u^{\Delta}))^{\nabla} + a(t)f(t,u(t)) = 0$, $t \in (0,T)$, $\phi(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \phi(u^{\Delta}(\xi_i))$, $u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. As an application, an example to demonstrate our results is given.

1. Introduction

In this paper, we study the existence of positive solutions of the following dynamic equations on time scales:

$$\begin{gathered}
\left(\phi\left(u^{\Delta}\right)\right)^{\nabla} + a(t) f\left(t, u(t)\right) = 0, \quad t \in (0,T), \\
\phi\left(u^{\Delta}(0)\right) = \sum_{i=1}^{m-2} a_i \phi\left(u^{\Delta}\left(\xi_i\right)\right), \qquad u(T) = \sum_{i=1}^{m-2} b_i u\left(\xi_i\right),
\end{gathered} \tag{1.1}$$

where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$.

A projection $\phi : R \to R$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:

(i) if $x \le y$, then $\phi(x) \le \phi(y)$, for all $x, y \in R$;

(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;

(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in R$.

We will assume that the following conditions are satisfied throughout this paper:

(H_1) $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, $a_i, b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$, $T\sum_{i=1}^{m-2} b_i \ge \sum_{i=1}^{m-2} b_i \xi_i$;

(H_2) $a(t) \in C_{\mathrm{ld}}((0,T), [0,+\infty))$ and there exists $t_0 \in (\xi_{m-2}, T)$, such that $a(t_0) > 0$;

(H_3) $f \in C([0,T] \times [0,+\infty), [0,+\infty))$.

Recently, there is much attention focused on the existence of positive solutions for second-order, three-point boundary value problem on time scales. On the other hand, three-point and m-point boundary value problems with p-Laplacian operators on time scales have also been studied extensively, for details, see [1–11] and references therein. But with an increasing homeomorphism and homomorphism, few works were done as far as we know.

A time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that $0, T$ are points in $\mathbf{T}$. By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is, $(0, T) \cap \mathbf{T}$.

We would like to mention some results of Anderson et al. [2], He [4, 5], Sun and Li [9], Ma et al. [12], Wang and Hou [13], Wang and Ge [14], which motivate us to consider our problem.

In [2], Anderson et al. considered the following problem:

$$\begin{gathered}\left(\phi_p(u^\Delta)\right)^\nabla + a(t) f(u(t)) = 0, \quad t \in (a,b), \\ u(a) - B_0(u^\Delta(v)) = 0, \qquad u^\Delta(b) = 0,\end{gathered} \tag{1.2}$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $v \in (a,b)$, $f \in C_{\mathrm{ld}}([0,+\infty), [0,+\infty))$, $a(t) \in C_{\mathrm{ld}}((a,b), [0,+\infty))$, and $K_m x \le B_0 x \le K_M x$ for some positive constants K_m, K_M. They established the existence results of at least one positive solution by using a fixed point theorem of cone expansion and compression of functional type.

In [4, 5], He considered the existence of positive solutions of the p-Laplacian dynamic equations on time scales:

$$\left(\phi_p(u^\Delta)\right)^\nabla + a(t) f(u(t)) = 0, \quad t \in (0,T) \tag{1.3}$$

satisfying the boundary conditions

$$u(0) - B_0(u^\Delta(\eta)) = 0, \qquad u^\Delta(T) = 0 \tag{1.4}$$

or

$$u^\Delta(0) = 0, \qquad u(T) - B_1(u^\Delta(\eta)) = 0, \tag{1.5}$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $\eta \in (0, \rho(T))$, $a(t) \in C_{\mathrm{ld}}((0,T), [0,+\infty))$, $f \in C([0,+\infty), [0,+\infty))$, and $Ax \le B_i x \le Bx$ $(i = 0,1)$ for some positive constants A, B. He obtained the existence of at least double and triple positive solutions of the problem (1.3), (1.4), and (1.5) by using a new double fixed point theorem and triple fixed point theorem, respectively.

In recent papers, Ma et al. [12] have obtained the existence of monotone positive solutions for the following BVP:

$$\begin{gathered} (\phi_p(u'))' + a(t) f(t, u(t)) = 0, \quad t \in (0,1), \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \end{gathered} \tag{1.6}$$

The main tool is the monotone iterative technique.

In [9], Sun and Li studied the following p-Laplacian, m-point BVP on time scales:

$$\begin{gathered} (\varphi_p(u^{\Delta}))^{\nabla} + a(t) f(t, u(t)) = 0, \quad t \in (0,T), \\ u(0) = 0, \qquad \varphi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} a_i \varphi_p(u^{\Delta}(\xi_i)), \end{gathered} \tag{1.7}$$

where $\varphi_p(u) = |u|^{p-2}u$, $p > 1$, $a_i \ge 0$ for $i = 1, \ldots, m-2$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, $\sum_{i=1}^{m-2} a_i < 1$, $a(t) \in C_{\mathrm{ld}}((0,T), [0,+\infty))$, $f \in C_{\mathrm{ld}}((0,T) \times [0,+\infty), [0,+\infty))$. Some new results are obtained for the existence of at least twin or triple positive solutions of the problem (1.7) by applying Avery-Henderson and Leggett-Williams fixed point theorems, respectively.

In [15], Sang and xi investigated the existence of positive solutions of the p-Laplacian dynamic equations on time scales:

$$\begin{gathered} (\phi_p(u^{\Delta}))^{\nabla} + a(t) f(t, u(t)) = 0, \quad t \in (0,T), \\ \phi_p(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \phi_p(u^{\Delta}(\xi_i)), \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{gathered} \tag{1.8}$$

where $\phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$. Let

$$\begin{gathered} f_{\gamma\rho}^{\rho} = \min\left\{ \min_{\xi_{m-2} \le t \le T} \frac{f(t,u)}{\phi_p(\rho)} : u \in [\gamma\rho, \rho] \right\}, \\ f_0^{\rho} = \max\left\{ \max_{0 \le t \le T} \frac{f(t,u)}{\phi_p(\rho)} : u \in [0, \rho] \right\}, \qquad \gamma = \frac{\sum_{i=1}^{m-2} b_i (T - \xi_i)}{T - \sum_{i=1}^{m-2} b_i \xi_i}, \\ f^{\alpha} = \limsup_{u \to \alpha} \max_{0 \le t \le T} \frac{f(t,u)}{\phi_p(u)}, \quad f_{\alpha} = \liminf_{u \to \alpha} \max_{\xi_{m-2} \le t \le T} \frac{f(t,u)}{\phi_p(u)}, \quad (\alpha := \infty \text{ or } 0^+), \\ m = \left\{ \frac{1}{1 - \sum_{i=1}^{m-2} b_i} \int_0^T \phi_q\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i} \right] \Delta s \right\}^{-1}, \\ M = \left\{ \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i \xi_i}{T(1 - \sum_{i=1}^{m-2} b_i)} \int_0^T \phi_q\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i} \right] \Delta s \right\}^{-1}, \end{gathered} \tag{1.9}$$

they mainly obtained the following results.

Theorem 1.1. *Assume* (H_1), (H_2), *and* (H_3) *hold, and assume that one of the following conditions holds:*

(H_4) *there exist* $\rho_1, \rho_2 \in (0, +\infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that*

$$f_0^{\rho_1} \leq \phi_p(m), \qquad f_{\gamma\rho_2}^{\rho_2} \geq \phi_p(M\gamma); \tag{1.10}$$

(H_5) *there exist* $\rho_1, \rho_2 \in (0, +\infty)$ *with* $\rho_1 < \rho_2$ *such that*

$$f_0^{\rho_2} \leq \phi_p(m), \qquad f_{\gamma\rho_1}^{\rho_1} \geq \phi_p(M\gamma). \tag{1.11}$$

Then, (1.8) *have a positive solution.*

In this paper, we will establish two new theorems of positive solution of (1.8), our work concentrates on the case when the nonlinear term does not satisfy the conditions of Theorem 1.1. At the end of the paper, we will give an example which illustrates that our work is true.

2. Preliminaries and some lemmas

For convenience, we list the following definitions which can be found in [16–19].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \\ \rho(r) &= \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}, \end{aligned} \tag{2.1}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered; and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense; and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise, set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise, set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$, (provided it exists), with the property that for each $\epsilon > 0$; there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \epsilon \left| \sigma(t) - s \right| \tag{2.2}$$

for all $s \in U$.

For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t is the number $f^{\nabla}(t)$, (provided it exists), with the property that for each $\epsilon > 0$; there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \leq \epsilon \left| \rho(t) - s \right| \tag{2.3}$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e., ld continuous) if f is continuous at each left-dense point in $\mathbf{T}$, and its right-sided limit exists at each right-dense point in $\mathbf{T}$.

Definition 2.4. If $G^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a). \tag{2.4}$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a). \tag{2.5}$$

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear BVP:

$$\begin{gathered}
\left(\phi(u^{\Delta})\right)^{\nabla} + h(t) = 0, \quad t \in (0,T), \\
\phi(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \phi(u^{\Delta}(\xi_i)), \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i).
\end{gathered} \tag{2.6}$$

We can prove the following lemmas by the methods of [15].

Lemma 2.5. *For $h \in C_{\mathrm{ld}}[0,T]$, the* BVP (2.6) *has the unique solution:*

$$u(t) = -\int_0^t \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - A\right)\Delta s + B, \tag{2.7}$$

where

$$\begin{aligned}
A &= -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}, \\
B &= \frac{\int_0^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - A\right)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - A\right)\Delta s}{1 - \sum_{i=1}^{m-2} b_i}.
\end{aligned} \tag{2.8}$$

Lemma 2.6. *Assume (H_1) holds, For $h \in C_{\mathrm{ld}}[0,T]$ and $h \geq 0$, then the unique solution u of* (2.6) *satisfies*

$$u(t) \geq 0, \quad \text{for } t \in [0,T]. \tag{2.9}$$

Lemma 2.7. *Assume (H_1) holds, if $h \in C_{\mathrm{ld}}[0,T]$ and $h \geq 0$, then the unique solution u of* (2.6) *satisfies*

$$\inf_{t\in[0,T]} u(t) \geq \gamma \|u\|, \tag{2.10}$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} b_i (T - \xi_i)}{T - \sum_{i=1}^{m-2} b_i \xi_i}, \qquad \|u\| = \max_{t\in[0,T]} |u(t)|. \tag{2.11}$$

Let the norm on $C_{\mathrm{ld}}[0,T]$ be the maximum norm. Then, the $C_{\mathrm{ld}}[0,T]$ is a Banach space. It is easy to see that the BVP (1.1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator equation:

$$(Au)(t) = -\int_0^t \phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s + \widetilde{B}, \tag{2.12}$$

where

$$\widetilde{A} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i},$$
$$\widetilde{B} = \frac{\int_0^T \phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s}{1 - \sum_{i=1}^{m-2} b_i}. \tag{2.13}$$

Throughout this paper, we will assume that $0 \le \mu \le \nu \le T$. Denote

$$K = \left\{ u \mid u \in C_{\mathrm{ld}}[0,T],\ u(t) \ge 0,\ \inf_{t \in [\mu,\nu]} u(t) \ge \gamma \|u\| \right\}, \tag{2.14}$$

where γ is the same as in Lemma 2.7. It is obvious that K is a cone in $C_{\mathrm{ld}}[0,T]$. By Lemma 2.7, $A(K) \subset K$. So by applying Arzela-Ascoli theorem on time scales [20], we can obtain that $A(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [21], it is easy to prove that A is continuous. Hence, $A : K \to K$ is completely continuous.

Lemma 2.8. *Let*

$$\varphi(s) = \phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau - \widetilde{A}\right), \tag{2.15}$$

for ξ_i $(i = 1, \ldots, m-2)$, *then*

$$\int_0^{\xi_i} \varphi(s)\Delta s \le \frac{\xi_i}{T}\int_0^T \varphi(s)\Delta s. \tag{2.16}$$

Lemma 2.9 ([22]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subset of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$F : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \longrightarrow K \tag{2.17}$$

be a completely continuous operator such that

(i) $\|Fu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Fu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Fu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Fu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then, F has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Now, we introduce the following notations. Let

$$
\begin{aligned}
A_0 &= \left\{ \frac{1}{1-\sum_{i=1}^{m-2} b_i} \int_0^T \phi^{-1} \left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i} \right] \Delta s \right\}^{-1}, \\
B_0 &= \left\{ \frac{T\sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i \xi_i}{T\left(1-\sum_{i=1}^{m-2} b_i\right)} \int_\mu^\nu \phi^{-1} \left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i} \right] \Delta s \right\}^{-1}.
\end{aligned}
\tag{2.18}
$$

For $l > 0$, $\Omega_l = \{u \in K : \|u\| < l\}$, $\partial\Omega_l = \{u \in K : \|u\| = l\}$,

$$
\alpha(l) = \sup\left\{\|Au\| : u \in \partial\Omega_l\right\}, \qquad \beta(l) = \inf\left\{\|Au\| : u \in \partial\Omega_l\right\}. \tag{2.19}
$$

By Lemma 2.6, α and β are well defined.

3. Existence theorems of positive solution

Theorem 3.1. *Assume (H_1), (H_2), (H_3) hold, and assume that the following conditions hold:*

(A_1) $p_i \in C([0,+\infty),[0,+\infty))$, $i = 1,2$, *and*

$$
\overline{\lim_{l\to 0}} \frac{p_1(l)}{\phi(l)} < \phi(A_0), \qquad \underline{\lim_{l\to\infty}} \frac{p_2(l)}{\phi(l)} > \phi\left(\frac{B_0}{\gamma}\right); \tag{3.1}
$$

(A_2) $k_1 \in L^1([0,T],[0,+\infty))$, $k_2 \in L^1([0,T],[0,+\infty))$;

(A_3) *there exist* $0 < c_1 \le c_2$, $0 \le \lambda_2 < 1 < \lambda_1$, *such that*

$$
\begin{aligned}
f(t,l) &\le p_1(l) + k_1(t)\left[\phi(l)\right]^{\lambda_1}, \quad (t,l) \in [0,T] \times [0,c_1], \\
f(t,l) &\ge p_2(l) - k_2(t)\left[\phi(l)\right]^{\lambda_2}, \quad (t,l) \in [\mu,\nu] \times [c_2,+\infty).
\end{aligned}
\tag{3.2}
$$

Then, the problems (1.1) *have at least one positive solution.*

Theorem 3.2. *Assume (H_1), (H_2), (H_3) hold, and assume that the following conditions hold:*

(B_1) $p_i \in C([0,+\infty),[0,+\infty))$, $i = 3,4$, *and*

$$
\overline{\lim_{l\to \infty}} \frac{p_3(l)}{\phi(l)} < \phi(A_0), \qquad \underline{\lim_{l\to 0^+}} \frac{p_4(l)}{\phi(l)} > \phi\left(\frac{B_0}{\gamma}\right); \tag{3.3}
$$

(B_2) $k_3 \in L^1([0,T],[0,+\infty))$, $k_4 \in L^1([0,T],[0,+\infty))$;

(B_3) *there exist* $0 < c_3 \le c_4$, $0 \le \lambda_4 < 1 < \lambda_3$, *such that*

$$
\begin{aligned}
f(t,l) &\le p_3(l) + k_3(t)\left[\phi(l)\right]^{\lambda_4}, \quad (t,l) \in [0,T] \times [c_4,+\infty), \\
f(t,l) &\ge p_4(l) - k_4(t)\left[\phi(l)\right]^{\lambda_3}, \quad (t,l) \in [\mu,\nu] \times [0,c_3].
\end{aligned}
\tag{3.4}
$$

Then, the problems (1.1) *have at least one positive solution.*

Proof of Theorem 3.1. By $\overline{\lim}_{l\to 0}(p_1(l)/\phi(l)) < \phi(A_0)$, then we can get that there exist $0 < \overline{a}_1 \leq c_1$, $0 < \epsilon < \phi(A_0)$ such that

$$p_1(l) \leq (\phi(A_0) - \epsilon)\phi(l), \quad 0 \leq l \leq \overline{a}_1. \tag{3.5}$$

If $0 \leq l \leq \overline{a}_1$, $u \in \partial\Omega_l$, then $0 \leq u(t) \leq l$, $0 \leq t \leq T$. By condition (A_3), we have

$$\begin{aligned} f(t,u(t)) &\leq p_1(u(t)) + k_1(t)\left[\phi(u(t))\right]^{\lambda_1} \\ &\leq (\phi(A_0) - \epsilon)\phi(u(t)) + k_1(t)\left[\phi(u(t))\right]^{\lambda_1} \\ &\leq (\phi(A_0) - \epsilon)\phi(\|u\|) + k_1(t)\left[\phi(\|u\|)\right]^{\lambda_1} \\ &= (\phi(A_0) - \epsilon)\phi(l) + k_1(t)\left[\phi(l)\right]^{\lambda_1}. \end{aligned} \tag{3.6}$$

Let $M(l,\tau) = (\phi(A_0) - \epsilon)\phi(l) + k_1(\tau)[\phi(l)]^{\lambda_1}$ so that

$$\begin{aligned} \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau - \widetilde{A} &= \int_0^s a(\tau) f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) f(\tau,u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i} \\ &\leq \int_0^s a(\tau) M(l,\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) M(l,\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}. \end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned} \|Au\| \leq \widetilde{B} &= \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\left(\int_0^T \varphi(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi(s)\Delta s\right) \\ &\leq \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \varphi(s)\Delta s \\ &\leq \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau) M(l,\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau) M(l,\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s. \end{aligned} \tag{3.8}$$

It follows that

$$\frac{\alpha(l)}{l} \leq \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)\frac{M(l,\tau)}{\phi(l)}\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\left(M(l,\tau)/\phi(l)\right)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s. \tag{3.9}$$

Noticing $\lambda_1 > 1$, we have

$$\begin{aligned} \overline{\lim_{l\to 0^+}}\frac{\alpha(l)}{l} &\leq \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)(\phi(A_0) - \epsilon)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)(\phi(A_0) - \epsilon)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s \\ &= \frac{\phi^{-1}\left[\phi(A_0) - \epsilon\right]}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s \\ &\leq \phi^{-1}\left[\phi(A_0)\right]A_0^{-1} = A_0 \cdot A_0^{-1} = 1. \end{aligned} \tag{3.10}$$

Therefore, there exist $0 < a_1 < \overline{a}_1$ such that $\alpha(a_1) < a_1$. It implies that $\|Au\| < \|u\|$, $u \in \partial\Omega_{a_1}$.

On the other hand, by $\underline{\lim}_{l\to\infty}(p_2(l)/\phi(l)) > \phi(B_0/\gamma)$, we can get that there exist $\epsilon' > 0$, and $c_2 \le d < +\infty$ such that

$$p_2(l) \ge \left[\phi\left(\frac{B_0}{\gamma}\right) + \epsilon'\right]\phi(l), \quad d \le l \le +\infty, \text{ for } d \le \gamma l < +\infty\,,\ u \in \partial\Omega_l, \tag{3.11}$$

then $d \le \gamma l \le u(t) \le l,\ \mu \le T \le \nu$.

Let $m(l,\tau) = [\phi(B_0/\gamma) + \epsilon']\phi(\gamma l) - k_2(\tau)[\phi(l)]^{\lambda_2}$. By Lemma 2.8 and condition (A$_3$), applying Lemma 2.8, it follows that

$$\begin{aligned}
\|Au\| = \max_{0\le t\le T}(Au)(t) &\ge -\int_0^T \varphi(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2}b_i}\left(\int_0^T \varphi(s)\Delta s - \sum_{i=1}^{m-2}b_i\int_0^{\xi_i}\varphi(s)\Delta s\right)\\
&= \frac{\sum_{i=1}^{m-2}b_i}{1-\sum_{i=1}^{m-2}b_i}\int_0^T \varphi(s)\Delta s - \frac{\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}\varphi(s)\Delta s}{1-\sum_{i=1}^{m-2}b_i}\\
&\ge \frac{\sum_{i=1}^{m-2}b_i}{1-\sum_{i=1}^{m-2}b_i}\int_0^T \varphi(s)\Delta s - \frac{\sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\int_0^T \varphi(s)\Delta s\\
&= \frac{T\sum_{i=1}^{m-2}b_i - \sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\int_0^T \varphi(s)\Delta s\\
&\ge \frac{T\sum_{i=1}^{m-2}b_i - \sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\int_\mu^\nu \phi^{-1}\left[\int_0^s a(\tau)m(l,\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)m(l,\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\Delta s.
\end{aligned}\tag{3.12}$$

It follows that

$$\frac{\beta(l)}{l} \ge \frac{T\sum_{i=1}^{m-2}b_i - \sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\int_\mu^\nu \phi^{-1}\left[\int_0^s a(\tau)\frac{m(l,\tau)}{\phi(l)}\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)\left(m(l,\tau)/\phi(l)\right)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\Delta s. \tag{3.13}$$

Noticing $0 \le \lambda_2 < 1$, we get

$$\begin{aligned}
\lim_{l\to\infty}\frac{\beta(l)}{l} &\ge \frac{T\sum_{i=1}^{m-2}b_i - \sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\\
&\quad\times\int_\mu^\nu \phi^{-1}\left[\int_0^s a(\tau)\frac{\phi(B_0/\gamma)\phi(\gamma l)}{\phi(l)}\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)\left(\left(\phi(B_0/\gamma)\phi(\gamma l)\right)/\phi(l)\right)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\Delta s\\
&= \phi^{-1}\phi(B_0)B_0^{-1} = B_0B_0^{-1} = 1.
\end{aligned}\tag{3.14}$$

Therefore, there exists b with $b > a_1 > 0$ such that $\beta(b) > b$. It implies that $\|Au\| > \|u\|$ for $u \in \partial\Omega_b$.

By Lemma 2.9, we assert that the operator A has one fixed point $u^* \in K$ such that $a_1 \le \|u^*\| \le b$. Therefore, u^* is positive solution of the problems (1.1). □

Proof of Theorem 3.2. By $\overline{\lim}_{l\to\infty}(p_3(l)/\phi(l)) < \phi(A_0)$, then we can get that there exist $d \geq \lambda_4$, $0 < \epsilon < \phi(A_0)$ such that

$$p_3(l) \leq (\phi(A_0) - \epsilon)\phi(l), \quad l \geq d. \tag{3.15}$$

For $l \geq d$, $u \in \partial\Omega_l$, then $0 \leq u(t) \leq l$, $0 \leq t \leq T$. By condition (B_3), we have

$$\begin{aligned} f(t,u(t)) &\leq p_3(u(t)) + k_3(t)[\phi(u(t))]^{\lambda_4} \\ &\leq (\phi(A_0) - \epsilon)\phi(u(t)) + k_3(t)[\phi(u(t))]^{\lambda_4} \\ &\leq (\phi(A_0) - \epsilon)\phi(\|u\|) + k_3(t)[\phi(\|u\|)]^{\lambda_4} \\ &= (\phi(A_0) - \epsilon)\phi(l) + k_3(t)[\phi(l)]^{\lambda_4}. \end{aligned} \tag{3.16}$$

Let $M_1(l,\tau) = (\phi(A_0) - \epsilon)\phi(l) + k_3(\tau)[\phi(l)]^{\lambda_4}$ so that

$$\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A} \leq \int_0^s a(\tau)M_1(l,\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)M_1(l,\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}. \tag{3.17}$$

Denote

$$m_1 = \max\left\{ f(\tau,u(\tau)) : \tau \in [0,s] \bigcup [0,\xi_i],\ u(\tau) \leq d \right\} \tag{3.18}$$

satisfying

$$m_1 + \phi^{-1}[\phi(A_0) - \epsilon] \leq A_0. \tag{3.19}$$

Therefore,

$$\|Au\| \leq m_1 A_0^{-1} + \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)M_1(l,\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)M_1(l,\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s. \tag{3.20}$$

It follows that

$$\frac{\alpha(l)}{l} \leq m_1 A_0^{-1} + \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)\frac{M_1(l,\tau)}{\phi(l)}\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)(M_1(l,\tau)/\phi(l))\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s. \tag{3.21}$$

Noticing $0 \leq \lambda_4 < 1$, we have

$$\begin{aligned} &\overline{\lim_{l\to\infty}}\frac{\alpha(l)}{l} \\ &\quad\leq m_1 A_0^{-1} + \frac{1}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)(\phi(A_0) - \epsilon)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)(\phi(A_0) - \epsilon)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s \\ &\quad= m_1 A_0^{-1} + \frac{\phi^{-1}[\phi(A_0) - \epsilon]}{1 - \sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left[\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right]\Delta s \\ &\quad= m_1 A_0^{-1} + \phi^{-1}[\phi(A_0) - \epsilon]A_0^{-1} \leq A_0 \cdot A_0^{-1} = 1. \end{aligned} \tag{3.22}$$

Therefore, there exists $a_1 \geq d$ such that $\alpha(a_1) < a_1$. It implies that $\|Au\| < \|u\|$, $u \in \partial\Omega_{a_1}$.

On the other hand, by $\underline{\lim}_{l\to 0^+}(p_4(l)/\phi(l)) > \phi(B_0/\gamma)$, we can get that there exist $\epsilon' > 0$, and $0 < e < c_3$ such that

$$p_4(l) \geq \left[\phi\left(\frac{B_0}{\gamma}\right) + \epsilon'\right]\phi(l), \qquad \text{for } 0 \leq l \leq e,\ u \in \partial\Omega_l, \tag{3.23}$$

then $\gamma l \leq u(t) \leq l,\ \mu \leq T \leq \nu$.

Let $m_1(l,\tau) = [\phi(B_0/\gamma) + \epsilon']\phi(\gamma l) - k_4(\tau)[\phi(l)]^{\lambda_3}$. By Lemma 2.8 and condition (B_3), applying Lemma 2.8, it follows that

$$\|Au\| \geq \frac{T\sum_{i=1}^{m-2}b_i - \sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\int_\mu^\nu \phi^{-1}\left[\int_0^s a(\tau)m_1(l,\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)m_1(l,\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\Delta s. \tag{3.24}$$

It follows that

$$\frac{\beta(l)}{l} \geq \frac{T\sum_{i=1}^{m-2}b_i - \sum_{i=1}^{m-2}b_i\xi_i}{T\left(1-\sum_{i=1}^{m-2}b_i\right)}\int_\mu^\nu \phi^{-1}\left[\int_0^s a(\tau)\frac{m_1(l,\tau)}{\phi(l)}\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)\left(m_1(l,\tau)/\phi(l)\right)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right]\Delta s. \tag{3.25}$$

Noticing $\lambda_3 > 1$, we can get

$$\underline{\lim}_{l\to 0^+}\frac{\beta(l)}{l} \geq 1. \tag{3.26}$$

Therefore, there exists b with $0 < b < a_1$ such that $\beta(b) > b$. It implies that $\|Au\| > \|u\|$ for $u \in \partial\Omega_b$.

By Lemma 2.9, we assert that the operator A has one fixed point $u^* \in K$ such that $b \leq \|u^*\| \leq a_1$. Therefore, u^* is positive solution of the problems (1.1). □

4. Example

In this section, we present a simple example to explain our results.

Let $f(t,0) \equiv 0$, $\mathbf{T} = R$, $T = 1$. Consider the following BVP:

$$\begin{gathered}\left(\phi\left(u^\Delta\right)\right)^\nabla + f(t,u(t)) = 0, \quad t \in (0,T),\\ \phi\left(u^\Delta(0)\right) = \frac{1}{4}\phi\left(u^\Delta\left(\frac{1}{3}\right)\right), \qquad u(T) = \frac{1}{2}u\left(\frac{1}{3}\right),\end{gathered} \tag{4.1}$$

where

$$\phi(u) = \begin{cases} -u^2, & u \leq 0,\\ u^2, & u > 0,\end{cases}$$

$$f(t,u) = \begin{cases} 45u^3 + \dfrac{5\sqrt{2}}{2}\min\left\{\dfrac{1}{\sqrt{t(1-t)}}, \dfrac{4}{u}\right\}\sqrt{u^5}, & (t,u) \in [0,1]\times[0,2],\\ 400, & (t,u) \in [0,1]\times[2,5],\\ \dfrac{17}{5}u^3 - \dfrac{\sqrt{5}}{2}\min\left\{\dfrac{1}{\sqrt{t(1-t)}}, \dfrac{2u}{5}\right\}\sqrt{u^3}, & (t,u) \in \left[\dfrac{2}{5}, \dfrac{1}{2}\right]\times[5,+\infty).\end{cases} \tag{4.2}$$

It is easy to check that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. In this case, $a(t) \equiv 1$, $m = 3$, $a_1 = 1/4$, $b_1 = 1/2$, $\xi_1 = 1/3$, it follows from a direct calculation that

$$\begin{aligned}
A_0 &= \left[\frac{1}{1-b_1}\int_0^T \phi^{-1}\left(s + \frac{a_1\xi_1}{1-a_1}\right)\Delta s\right]^{-1} = \left[\frac{1}{1-(1/2)}\int_0^1 \left(s + \frac{1/4\cdot 1/3}{1-(1/4)}\right)^{1/2}\Delta s\right]^{-1} = \frac{3\cdot 27}{4(10\sqrt{10}-1)},\\
\gamma &= \frac{b_1(T-\xi_1)}{T-b_1\xi_1} = \frac{1/2(1-(1/3))}{1-(1/2\cdot 1/3)} = \frac{2}{5},\\
B_0 &= \left[\frac{Tb_1 - b_1\xi_1}{T(1-b_1)}\int_{2/5}^{1/2} \phi^{-1}\left(s + \frac{a_1\xi_1}{1-a_1}\right)\Delta s\right]^{-1} = \left[\frac{1/2-(1/2\cdot 1/3)}{1-(1/2)}\int_{2/5}^{1/2} \left(s + \frac{1/4\cdot 1/3}{1-(1/4)}\right)^{1/2}\Delta s\right]^{-1}\\
&= \left[\frac{4}{9}\,\frac{33\times 25\sqrt{22} - 69\times 4\sqrt{115}}{8100}\right]^{-1}.
\end{aligned} \tag{4.3}$$

Choose $c_1 = 2$, $c_2 = 5$, $b = 5$, $\lambda_1 = 5/4$, $\lambda_2 = 3/4$, $p_1(u) = 45u^3$, $p_2(u) = (17/5)u^3$, $k_1(t) = (5\sqrt{2}/2)(1/\sqrt{t(1-t)})$, $k_2(t) = (\sqrt{5}/2)(1/\sqrt{t(1-t)})$, it is easy to check that

$$\begin{aligned}
&f(t,u) \le p_1(u) + k_1(t)\left(u^2\right)^{5/4}, \quad (t,u) \in [0,1]\times[0,2],\\
&f(t,u) \ge p_2(u) - k_2(t)\left(u^2\right)^{3/4}, \quad (t,u) \in \left[\frac{2}{5},\frac{1}{2}\right]\times[5,+\infty),\\
&\overline{\lim_{u\to 0}}\,\frac{p_1(u)}{\phi(u)} = \overline{\lim_{u\to 0}}\,\frac{45u^3}{u^2} = 0 < A_0^2 = \left[\frac{3\cdot 27}{4(10\sqrt{10}-1)}\right]^2,\\
&\overline{\lim_{u\to\infty}}\,\frac{p_2(u)}{\phi(u)} = \overline{\lim_{u\to\infty}}\,\frac{(17/5)u^3}{u^2} = +\infty > \phi\left(\frac{B_0}{\gamma}\right).
\end{aligned} \tag{4.4}$$

It follows that f satisfies the conditions (A_1)–(A_3) of Theorem 3.1, then problems (1.1) have at least one positive solution.

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