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Research Article

Global Behavior of Four Competitive Rational Systems of Difference Equations in the Plane

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We investigate the global dynamics of solutions of four distinct competitive rational systems of difference equations in the plane. We show that the basins of attractions of different locally asymptotically stable equilibrium points are separated by the global stable manifolds of either saddle points or nonhyperbolic equilibrium points. Our results give complete answer to Open Problem 2 posed recently by Camouzis et al. (2009).

1. Introduction and Preliminaries

We consider the following open problem (see [1, Open Problem 2]).

Foreach of the following four distinct systems

$$(14, 21), (15, 21), (21, 21), (21, 38), \tag{1.1}$$

determine the following:

(i) *the boundedness character of its solutions,*

(ii) *the local stability of its equilibrium points,*

(iii) *the existence of prime period-two solutions,*

(iv) *the global character of the systems.*

Equation (3.4) is of the form

$$x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n}, \quad n = 0, 1, \ldots; \tag{1.2}$$

equation (3.5) is of the form

$$x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad n = 0, 1, \ldots; \tag{1.3}$$

equation (3.16) is either of the form

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 x_n}{y_n}, \quad n = 0, 1, \ldots \tag{1.4}$$

or the form

$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n}, \quad n = 0, 1, \ldots \tag{1.5}$$

depending on whether it appears as first or second equation in the system; equation (38) is of the form

$$y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}, \quad n = 0, 1, \ldots. \tag{1.6}$$

The typical results are the following theorems. The first theorem is a combination of Theorems 2.3 and 2.5 and the second theorem is Theorem 3.3.

Theorem 1.1. *Consider system (14, 21) and assume that $\gamma_2 A_1 \neq \alpha_2$. If $\beta_1 > A_1$, then there exists a set $\mathcal{C} \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of E. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates $\mathcal{R}$ into two connected and invariant components, namely,*

$$\begin{aligned} \mathcal{W}_- &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \\ \mathcal{W}_+ &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \end{aligned} \tag{1.7}$$

which satisfy

$$\begin{aligned} \lim_{n \to \infty} (x_n, y_n) &= (0, \infty) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_-, \\ \lim_{n \to \infty} (x_n, y_n) &= (\infty, 0) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_+. \end{aligned} \tag{1.8}$$

Assume that $\beta_1 \le A_1$. Every solution $\{(x_n, y_n)\}$ of system (14, 21), with $x_0 > 0,\ y_0 \ge 0$, satisfies

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty. \tag{1.9}$$

Theorem 1.2. *Consider system (21, 21). There exists a set $\mathcal{C} \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of the unique equilibrium E. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates $\mathcal{R}$ into two connected and invariant components, namely,*

$$\begin{aligned}
\mathcal{W}_- &:= \left\{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{\text{se}} y\right\}, \\
\mathcal{W}_+ &:= \left\{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{\text{se}} x\right\}.
\end{aligned} \tag{1.10}$$

which satisfy

$$\begin{aligned}
&\lim_{n\to\infty} (x_n, y_n) = (0, \infty) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_-, \\
&\lim_{n\to\infty} (x_n, y_n) = (\infty, 0) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_+.
\end{aligned} \tag{1.11}$$

All considered systems are competitive systems, which we discuss next.

A first-order system of difference equations

$$\begin{aligned}
x_{n+1} &= f(x_n, y_n) \\
y_{n+1} &= g(x_n, y_n)
\end{aligned} \qquad n = 0, 1, 2, \ldots, \ (x_{-1}, x_0) \in \mathcal{R}, \tag{1.12}$$

where $\mathcal{R} \subset \mathbb{R}^2$, $(f, g) : \mathcal{R} \to \mathcal{R}$, f, g are continuous functions, is *competitive* if $f(x, y)$ is nondecreasing in x and nonincreasing in y, and $g(x, y)$ is nonincreasing in x and nondecreasing in y. If both f and g are nondecreasing in x and y, the system (1.12) is *cooperative*. A map T that corresponds to the system (1.12) is defined as $T(x, y) = (f(x, y), g(x, y))$. Competitive and cooperative maps, which are called monotone maps, are defined similarly. *Strongly competitive* systems of difference equations or maps are those for which the functions f and g are coordinatewise strictly monotone.

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $Q_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, that is, $Q_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \ge u, y \ge v\}$, $Q_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \le u, y \ge v\}$, and so on. Define the *South-East* partial order $\preceq_{\text{se}}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{\text{se}} (s, t)$ if and only if $x \le s$ and $y \ge t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \le s$ and $y \le t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, define the *distance from x to $\mathcal{A}$* as $\operatorname{dist}(x, \mathcal{A}) := \inf\{\|x - y\| : y \in \mathcal{A}\}$. By $\operatorname{int}\mathcal{A}$ we denote the interior a set $\mathcal{A}$.

It is easy to show that a map F is competitive if it is nondecreasing with respect to the South-East partial order, that is, if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{\mathrm{se}} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Longrightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{\mathrm{se}} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \quad (1.13)$$

Competitive systems were studied by many authors; see [2–17], and others. All known results, with the exception of [2, 3, 18], deal with hyperbolic dynamics. The results presented here are results that hold in both the hyperbolic and the nonhyperbolic case.

We now state three results for competitive maps in the plane. The following definition is from [17].

Definition 1.3. Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $(O+)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition $(O-)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.

The following theorem was proved by DeMottoni-Schiaffino for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [14, 15].

Theorem 1.4. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which $(O+)$ holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead $(O-)$ holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [17], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 1.5. *Let $\mathcal{R} \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a C^1 (continuously differentiable) competitive map. If T is injective and* $\det J_T(x) > 0$ *for all $x \in \mathcal{R}$, then T satisfies $(O+)$. If T is injective and* $\det J_T(x) < 0$ *for all $x \in \mathcal{R}$, then T satisfies $(O-)$.*

The next results are the modifications of [8, Theorem 4]. See [18].

Theorem 1.6. *Let T be a monotone map on a closed and bounded rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Suppose that T has a unique fixed point $\overline{\mathbf{e}}$ in $\mathcal{R}$. Then $\overline{\mathbf{e}}$ is a global attractor T on $\mathcal{R}$.*

The following four results were proved by Kulenović and Merino [18] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or nonhyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. These results are useful for determining basins of attraction of fixed points of competitive maps.

Our first result gives conditions for the existence of a global invariant curve through a fixed point (hyperbolic or not) of a competitive map that is differentiable in a neighborhood of the fixed point, when at least one of two nonzero eigenvalues of the Jacobian matrix of the map at the fixed point has absolute value less than one. A region $\mathcal{R} \subset \mathbb{R}^2$ is *rectangular* if it is the cartesian product of two intervals in $\mathbb{R}$.

Theorem 1.7. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}(Q_1(\overline{x}) \cup Q_3(\overline{x}))$ is nonempty (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

(a) *The map T has a C^1 extension to a neighborhood of $\overline{x}$.*

(b) *The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{x}$ that is invariant and a subset of the basin of attraction of $\overline{x}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{x}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Corollary 1.8. *If T has no fixed point nor periodic points of minimal period-two in Δ, then the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$.*

For maps that are strongly competitive near the fixed point, hypothesis (b) of Theorem 1.7 reduces just to $|\lambda| < 1$. This follows from a change of variables [17] that allows the Perron-Frobenius Theorem to be applied to give that at any point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

The following result gives a description of the global stable and unstable manifolds of a saddle point of a competitive map. The result is a modification of [8, Theorem 5].

Theorem 1.9. *In addition to the hypotheses of Theorem 1.7, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 1.7 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the global stable manifold $\mathcal{W}^s(\overline{x})$ of $\overline{x}$, and the global unstable manifold $\mathcal{W}^u(\overline{x})$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{x})$ in $\mathcal{R}$ are fixed points of T.*

The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 1.10. *Assume the hypotheses of Theorem 1.7, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 1.7. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely,*

$$\begin{aligned}\mathcal{W}_- &:= \{x \in \mathcal{R}? \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\},\\ \mathcal{W}_+ &:= \{x \in \mathcal{R}? \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\},\end{aligned} \tag{1.14}$$

such that the following statements are true.

(i) *$\mathcal{W}_-$ is invariant, and $\operatorname{dist}(T^n(x), Q_2(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.*

(ii) *$\mathcal{W}_+$ is invariant, and $\operatorname{dist}(T^n(x), Q_4(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.*

If, in addition, $\overline{x}$ *is an interior point of* $\mathcal{R}$ *and* T *is* C^2 *and strongly competitive in a neighborhood of* $\overline{x}$*, then* T *has no periodic points in the boundary of* $Q_1(\overline{x}) \cup Q_3(\overline{x})$ *except for* $\overline{x}$*, and the following statements are true.*

(i) *For every* $x \in \mathcal{W}^-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in \operatorname{int} Q_2(\overline{x})$ *for* $n \geq n_0$*.*

(ii) *For every* $x \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in \operatorname{int} Q_4(\overline{x})$ *for* $n \geq n_0$*.*

In this paper we study the global dynamics of four rational systems of difference equations mentioned earlier, where all parameters are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers. Two of these systems have a nonhyperbolic semistable equilibrium point. In general all four systems share the common feature that the global stable manifolds of either saddle points or nonhyperbolic equilibrium points serve as boundaries of basins of attraction of different local attractors or points at infinities. The techniques used here can be applied to treat number of competitive systems which appear in applications, such as Leslie-Gower competition model, see [19], or Leslie-Gower competition model with stocking, see [20], or genetic model, see [13]. An important new feature of our techniques is that they are applicable to nonhyperbolic case as well, which was shown for the first time in [18] where we have completed analysis of basic Leslie-Gower competition model from [19]. Furthermore, system (21, 38) can be considered as a variant of Leslie-Gower competition model, where the first equation has been replaced by another equation, which does not allow extinction of both species. In fact, all four considered competitive systems share common feature that they do not allow the extinction of both species.

2. System (14,21)

Now we consider the following system of difference equations:

$$x_{n+1} = \frac{\beta_1 x_n}{A_1 + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n}, \quad n = 0, 1, \ldots, \tag{2.1}$$

where the parameters A_1, β_1, α_2, and γ_2 are positive numbers and initial conditions $x_0 > 0$, $y_0 \geq 0$.

System (2.1) was considered in [1, Example 1], where it was shown that the associated map $T(x, y) = (\beta_1 x/(A_1 + y), (\alpha_2 + \gamma_2 y)/x)$ is injective and

$$\det J_T(x, y) = \frac{\beta_1}{(A_1 + y)^2 x} \cdot (\gamma_2 A_1 - \alpha_2). \tag{2.2}$$

When $\gamma_2 A_1 > \alpha_2$, $\det J_T(x, y) > 0$. Therefore, in view of Theorems 1.4 and 1.5 every solution of system (2.1) is eventually componentwise monotonic. If $\gamma_2 A_1 < \alpha_2$, then $\det J_T(x, y) < 0$, and four subsequences

$$\{x_{2n}\}, \ \{x_{2n+1}\}, \ \{y_{2n}\}, \ \{y_{2n+1}\} \tag{2.3}$$

of every solution $\{(x_n, y_n)\}$ of system (2.1) are eventually monotonic.

Thus, if $\gamma_2 A_1 \neq \alpha_2$, the Jacobian matrix of T in (x, y) is invertible.

The Jacobian matrix of the corresponding map $T(x,y)$ is of the form

$$J_T(x,y) = \begin{bmatrix} \dfrac{\beta_1}{A_1+y} & -\dfrac{\beta_1 x}{(A_1+y)^2} \\ -\dfrac{\alpha_2+\gamma_2 y}{x^2} & \dfrac{\gamma_2}{x} \end{bmatrix}. \tag{2.4}$$

2.1. Linearized Stability Analysis

The equilibrium points $(\overline{x},\overline{y})$ of system (2.1) are solutions of the system of equations

$$\overline{x} = \frac{\beta_1 \overline{x}}{A_1+\overline{y}}, \qquad \overline{y} = \frac{\alpha_2+\gamma_2\overline{y}}{\overline{x}}, \tag{2.5}$$

from which we obtain

$$\overline{y} = \beta_1 - A_1, \qquad \overline{x} = \frac{\alpha_2}{\beta_1 - A_1} + \gamma_2. \tag{2.6}$$

Lemma 2.1. (i) *If $\beta_1 > A_1$, then system* (2.1) *has a unique equilibrium point:*

$$E = \left(\frac{\alpha_2}{\beta_1 - A_1} + \gamma_2, \beta_1 - A_1\right), \tag{2.7}$$

which is a saddle point.

(ii) *If $\beta_1 \leq A_1$, then system* (2.1) *has no equilibrium points.*

Proof. By (2.6) and (2.4) the Jacobian matrix evaluated at the equilibrium point E has the form

$$J_T(E) = \begin{bmatrix} 1 & -\dfrac{\overline{x}}{\beta_1} \\ -\dfrac{\overline{y}}{\overline{x}} & \dfrac{\gamma_2}{\overline{x}} \end{bmatrix}. \tag{2.8}$$

The corresponding characteristic equation evaluated at the equilibrium point E is

$$\lambda^2 - p\lambda + q = 0, \tag{2.9}$$

where

$$\begin{aligned} p &= \operatorname{Tr} J_T(\overline{x},\overline{y}) = 1 + \frac{\gamma_2}{\overline{x}} > 0, \\ q &= \operatorname{Det} J_T(\overline{x},\overline{y}) = \frac{\gamma_2}{\overline{x}} - \frac{\overline{y}}{\beta_1}. \end{aligned} \tag{2.10}$$

Notice that in view of (2.6) $\overline{y}/\beta_1 = 1 - A_1/\beta_1$ and so

$$1 + q = 1 + \frac{\gamma_2}{\overline{x}} - \frac{\overline{y}}{\beta_1} = 1 + \frac{\gamma_2}{\overline{x}} - 1 + \frac{A_1}{\beta_1} > 0. \tag{2.11}$$

Since $p > 0$ and $1 + q > 0$, we need to show

(I) $p > 1 + q$,

(II) $p^2 - 4q > 0$.

Indeed,

(I) $p > 1 + q \Leftrightarrow 1 + \gamma_2/\overline{x} > 1 + \gamma_2/\overline{x} - \overline{y}/\beta_1 \Leftrightarrow 0 > -\overline{y}/\beta_1$,

which is satisfied (because $\beta_1 > 0$ and $\overline{y} > 0$). Furthermore

(II) $p^2 - 4q > 0 \Leftrightarrow (1 + \gamma_2/\overline{x})^2 - 4(\gamma_2/\overline{x}) + 4(\overline{y}/\beta_1) > 0 \Leftrightarrow (1 - \gamma_2/\overline{x})^2 + 4(\overline{y}/\beta_1) > 0$,

which is satisfied. □

2.2. Global Results

2.2.1. Case $\beta_1 > A_1$

Theorem 2.2. *System* (2.1) *has no prime period-two solutions.*

Proof. System (2.1) can be reduced to the following second-order difference equation:

$$y_{n+2} = \frac{y_{n+1}(A_1 + y_n)(\alpha_2 + \gamma_2 y_{n+1})}{\beta_1(\alpha_2 + \gamma_2 y_n)}, \tag{2.12}$$

or to the following second-order difference equation:

$$x_{n+2} = \frac{\beta_1 x_n x_{n+1}^2}{(A_1 x_n + \alpha_2 - \gamma_2 A_1)x_{n+1} + \gamma_2 \beta_1 x_n}. \tag{2.13}$$

Now it is sufficient to prove that both of the difference equations (2.12) and (2.13) have no prime period-two solutions. Assume that this is not true for (2.12), that is, that

$$\phi, \psi, \phi, \psi, \ldots, \quad (\phi \neq \psi) \tag{2.14}$$

is a prime period-two solution of (2.12). Then we have

$$\phi = \frac{\psi(A_1 + \phi)(\alpha_2 + \gamma_2\psi)}{\beta_1(\alpha_2 + \gamma_2\phi)}, \qquad \psi = \frac{\phi(A_1 + \psi)(\alpha_2 + \gamma_2\phi)}{\beta_1(\alpha_2 + \gamma_2\psi)}. \tag{2.15}$$

This implies

$$\begin{aligned} \beta_1\phi(\alpha_2+\gamma_2\phi) &= \psi(A_1+\phi)(\alpha_2+\gamma_2\psi), \\ \beta_1\psi(\alpha_2+\gamma_2\psi) &= \phi(A_1+\psi)(\alpha_2+\gamma_2\phi). \end{aligned} \tag{2.16}$$

By subtraction, we obtain

$$\beta_1(\phi-\psi)[\alpha_2+\gamma_2(\phi+\psi)] = A_1\left[\alpha_2(\psi-\phi)+\gamma_2\left(\psi^2-\phi^2\right)\right]+\phi\psi\gamma_2(\psi-\phi), \tag{2.17}$$

that is,

$$(\phi-\psi)[\beta_1(\alpha_2+\gamma_2(\phi+\psi))+A_1(\alpha_2+\gamma_2(\psi+\phi))+\phi\psi\gamma_2]=0, \tag{2.18}$$

and this implies that $\phi=\psi$, which is a contradiction.

Now assume that

$$\chi,\varphi,\chi,\varphi,\ldots,\quad (\chi\neq\varphi) \tag{2.19}$$

is a prime period-two solution of (2.13). Then we have

$$\chi=\frac{\beta_1\chi\varphi^2}{(A_1\chi+\alpha_2-\gamma_2A_1)\varphi+\gamma_2\beta_1\chi},\qquad \varphi=\frac{\beta_1\varphi\chi^2}{(A_1\varphi+\alpha_2-\gamma_2A_1)\chi+\gamma_2\beta_1\varphi}, \tag{2.20}$$

from which

$$(\chi-\varphi)[A_1\chi\varphi+\gamma_2\beta_1(\chi+\varphi)+\beta_1\chi\varphi]=0, \tag{2.21}$$

and this implies that $\chi=\varphi$, which is a contradiction. □

Theorem 2.3. *Consider system* (2.1) *and assume that* $\beta_1>A_1$ *and* $\gamma_2A_1\neq\alpha_2$*. Then there exists a set* $\mathcal{C}\subset\mathcal{R}$ *which is invariant and a subset of the basin of attraction of E. The set* $\mathcal{C}$ *is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates* $\mathcal{R}$ *into two connected and invariant components, namely,*

$$\begin{aligned} \mathcal{W}_- &:= \{x\in\mathcal{R}\setminus\mathcal{C} : \exists y\in\mathcal{C} \text{ with } x\preceq_{se}y\}, \\ \mathcal{W}_+ &:= \{x\in\mathcal{R}\setminus\mathcal{C} : \exists y\in\mathcal{C} \text{ with } y\preceq_{se}x\}, \end{aligned} \tag{2.22}$$

which satisfy

$$\begin{aligned} &\lim_{n\to\infty}(x_n,y_n)=(0,\infty)\quad \text{for every } (x_0,y_0)\in\mathcal{W}_-, \\ &\lim_{n\to\infty}(x_n,y_n)=(\infty,0)\quad \text{for every } (x_0,y_0)\in\mathcal{W}_+. \end{aligned} \tag{2.23}$$

Proof. Clearly, system (2.1) is strongly competitive on $(0,\infty)\times[0,\infty)$. In view of Theorem 2.2 we see that all conditions of Theorems 1.7, 1.9, and 1.10 and Corollary 1.8 are satisfied with $\mathcal{R}=(0,\infty)\times[0,\infty)$ and so the conclusion follows. □

Remark 2.4 (see [1]). If $\gamma_2 A_1=\alpha_2$, then system (2.1) can be decoupled as follows:

$$x_{n+1}=\frac{\beta_1 x_n^2}{A_1 x_n+\beta_1\gamma_2},\quad y_{n+1}=\frac{1}{\beta_1}y_n(A_1+y_n),\quad n=0,1,\ldots \tag{2.24}$$

and every solution of this system (depending of the choice of the initial condition (x_0,y_0)) is either bounded and converges to an equilibrium point or increases monotonically to infinity.

2.2.2. Case $\beta_1\le A_1$

In this case system (2.1) has no equilibrium points. Now we have the following.

Theorem 2.5. *Assume that* $\beta_1\le A_1$ *and* $\gamma_2 A_1\neq\alpha_2$. *Every solution* $\{(x_n,y_n)\}$ *of system* (2.1), *with* $x_0>0,\ y_0\ge 0$, *satisfies*

$$\lim_{n\to\infty}x_n=0,\qquad \lim_{n\to\infty}y_n=\infty. \tag{2.25}$$

Proof. If $\beta_1<A_1$, then

$$x_{n+1}<\frac{\beta_1}{A_1}x_n\Longrightarrow x_n<\left(\frac{\beta_1}{A_1}\right)^n x_0\longrightarrow 0\quad(n\longrightarrow\infty), \tag{2.26}$$

which implies $\lim_{n\to\infty}x_n=0$.

On the other hand, if $\beta_1=A_1$, then $x_{n+1}=A_1x_n/(A_1+y_n)<x_n$, and we obtain that the sequence $\{x_n\}_{n=0}^{\infty}$ is strictly decreasing. Because $x_n>0$ for all n, we see that $\{x_n\}_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty}x_n=0$, since otherwise, that is, $\lim_{n\to\infty}x_n=a>0$, the first equation of system (2.1) implies $\lim_{n\to\infty}y_n=\beta_1-A_1=0$ or the second equation of system (2.1) implies $\lim_{n\to\infty}y_n=\alpha_2/(a-\gamma_2)\neq 0$, which is a contradiction, since otherwise system (2.1) would have an equilibrium point in the first quadrant.

We see that if $\beta_1\le A_1$, then every solution $\{(x_n,y_n)\}$ of system (2.1) satisfies $\lim_{n\to\infty}x_n=0$.

But then the denominator in

$$y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{x_n} \tag{2.27}$$

is, for all large n, strictly less than a constant $\eta<\gamma_2$, which in turn implies

$$y_{n+1}>\frac{\alpha_2}{\eta}+\frac{\gamma_2}{\eta}y_n,\quad n\ge N. \tag{2.28}$$

Iterating this inequality we obtain

$$y_n > \frac{\alpha_2}{\eta} + \frac{\gamma_2}{\eta}\left(\frac{\alpha_2}{\eta}\right)^{n-N} + \left(\frac{\gamma_2}{\eta}\right)^{n-1-N} y_N, \quad n \geq N, \tag{2.29}$$

and this forces y_n to infinity. □

The obtained results lead to the following characterization of the boundedness of solutions of system (2.1).

Corollary 2.6. *Consider system* (2.1) *subject to the condition* $\alpha_2 \neq A_1\gamma_2$*. If* $\beta_1 > A_1$*, then all bounded solutions converge to the unique equilibrium with the corresponding initial conditions belonging to the graph of a continuous increasing function* C *in the plane of initial conditions. All solutions that start in the complement of* C *are asymptotic to either* $(\infty, 0)$ *or* $(0, \infty)$*. If* $\beta_1 \leq A_1$*, then all solutions are unbounded in the sense that* $\{x_n\}$ *is bounded and* $\{y_n\}$ *approaches* ∞*.*

3. System (21,21)

Now we consider the following system of difference equations:

$$x_{n+1} = \frac{\beta_1 x_n + \alpha_1}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where the parameters $\alpha_1, \beta_1, \alpha_2$, and γ_2 are positive numbers and initial conditions $x_0 > 0$, $y_0 > 0$.

System (3.1) was considered in [1, Example 3], where it was shown that the associated map T is injective and

$$\det J_T(x, y) = \frac{-\alpha_1(\alpha_2 + \gamma_2 y) - \beta_1\alpha_2 x}{x^2y^2} < 0, \tag{3.2}$$

that is, the Jacobian matrix of T in (x, y) is invertible. Therefore, in view of Theorems 1.4 and 1.5, four subsequences

$$\{x_{2n}\}, \ \{x_{2n+1}\}, \ \{y_{2n}\}, \ \{y_{2n+1}\} \tag{3.3}$$

of every solution $\{(x_n, y_n)\}$ of system (3.1) are eventually monotonic.

3.1. Linearized Stability Analysis

Equilibrium points of system (3.1) are solutions of the system

$$\overline{x} = \frac{\beta_1\overline{x} + \alpha_1}{\overline{y}}, \qquad \overline{y} = \frac{\alpha_2 + \gamma_2\overline{y}}{\overline{x}}. \tag{3.4}$$

Since $\overline{x} \neq 0$ and $\overline{y} \neq 0$, we have

$$\overline{y}_{\pm} = \frac{1}{2\gamma_2}\left[-(\alpha_2 - \alpha_1 - \beta_1\gamma_2) \pm \sqrt{D_1}\right], \tag{3.5}$$

where

$$D_1 = (\alpha_2 - \alpha_1 - \beta_1\gamma_2)^2 + 4\beta_1\gamma_2\alpha_2. \tag{3.6}$$

Since $\overline{y}_{-} < 0$ and $\overline{y}_{+} > 0$, system (3.1) has a unique positive equilibrium $E = (\overline{x}_{+}, \overline{y}_{+})$, where

$$\overline{x}_{+} = \frac{1}{2\beta_1}\left(\alpha_2 - \alpha_1 + \beta_1\gamma_2 + \sqrt{D_2}\right), \tag{3.7}$$

where $D_2 = (\alpha_2 - \alpha_1 + \beta_1\gamma_2)^2 + 4\beta_1\gamma_2\alpha_1$.

Lemma 3.1. *System* (3.1) *has a unique positive equilibrium point:*

$$E = \left(\frac{1}{2\beta_1}\left(\alpha_2 - \alpha_1 + \beta_1\gamma_2 + \sqrt{D_2}\right), \frac{1}{2\gamma_2}\left(-(\alpha_2 - \alpha_1 - \beta_1\gamma_2) + \sqrt{D_1}\right)\right), \tag{3.8}$$

which is a saddle point.

Proof. The Jacobian matrix of the corresponding map $T(x, y) = ((\beta_1 x + \alpha_1)/y, (\alpha_2 + \gamma_2 y)/x)$ is of the form

$$J_T(x, y) = \begin{bmatrix} \dfrac{\beta_1}{y} & -\dfrac{\beta_1 x + \alpha_1}{y^2} \\ -\dfrac{\alpha_2 + \gamma_2 y}{x^2} & \dfrac{\gamma_2}{x} \end{bmatrix}. \tag{3.9}$$

By using (3.4) we obtain

$$J_T(E) = \begin{bmatrix} \dfrac{\beta_1}{\overline{y}} & -\dfrac{\overline{x}}{\overline{y}} \\ -\dfrac{\overline{y}}{\overline{x}} & \dfrac{\gamma_2}{\overline{x}} \end{bmatrix}. \tag{3.10}$$

The corresponding characteristic equation evaluated at the equilibrium point E of system (3.1) is

$$\lambda^2 - p\lambda + q = 0, \tag{3.11}$$

where

$$p = \operatorname{Tr} J_T(\overline{x}, \overline{y}) = \frac{\beta_1}{\overline{y}} + \frac{\gamma_2}{\overline{x}} > 0,$$
$$q = \operatorname{Det} J_T(\overline{x}, \overline{y}) = \frac{\beta_1 \gamma_2}{\overline{xy}} - 1. \tag{3.12}$$

Notice that

$$1 + q = 1 + \frac{\beta_1 \gamma_2}{\overline{xy}} - 1 = \frac{\beta_1 \gamma_2}{\overline{xy}} > 0. \tag{3.13}$$

Since $p > 0$ and $1 + q > 0$, we need to show

(I) $p > 1 + q$,

(II) $p^2 - 4q > 0$.

Now, we get

(I) $p > 1 + q \Leftrightarrow \beta_1/\overline{y} + \gamma_2/\overline{x} > \beta_1\gamma_2/\overline{xy} \Leftrightarrow \overline{xy} - \alpha_1 + \overline{xy} - \alpha_2 > \beta_1\gamma_2$.

By using (3.4), (3.5), and (3.7) we obtain

$$\beta_1 \overline{x} + \gamma_2 \overline{y} = 2\overline{xy} - \alpha_1 - \alpha_2 = \beta_1 \gamma_2 + \frac{1}{2}\sqrt{D_1} + \frac{1}{2}\sqrt{D_2} > \beta_1 \gamma_2. \tag{3.14}$$

Furthermore

(II) $p^2 - 4q > 0 \Leftrightarrow (\beta_1/\overline{y} + \gamma_2/\overline{x})^2 - 4(\beta_1\gamma_2/\overline{xy} - 1) > 0 \Leftrightarrow (\beta_1/\overline{y} - \gamma_2/\overline{x})^2 + 4 > 0$,

which is satisfied. □

3.2. Global Results

Theorem 3.2. *System* (3.1) *has no prime period-two solutions.*

Proof. System (3.1) can be reduced to the following second-order difference equation:

$$y_{n+2} = \frac{y_n y_{n+1} (\alpha_2 + \gamma_2 y_{n+1})}{\alpha_1 y_{n+1} + \beta_1 (\alpha_2 + \gamma_2 y_n)}, \tag{3.15}$$

or to the following second-order difference equation:

$$x_{n+2} = \frac{(\beta_1 x_{n+1} + \alpha_1) x_n x_{n+1}}{\alpha_2 x_{n+1} + \gamma_2 (\beta_1 x_n + \alpha_1)}. \tag{3.16}$$

Now it is sufficient to prove that both of the difference equations (3.15) and (3.16) have no prime period-two solutions. Assume that this is not true for (3.15), that is, that

$$\phi, \psi, \phi, \psi, \dots, \quad (\phi \neq \psi) \tag{3.17}$$

is a prime period-two solution of (3.15). Then we have

$$\phi = \frac{\phi\psi(\alpha_2 + \gamma_2\psi)}{\alpha_1\psi + \beta_1(\alpha_2 + \gamma_2\phi)}, \qquad \psi = \frac{\phi\psi(\alpha_2 + \gamma_2\phi)}{\alpha_1\phi + \beta_1(\alpha_2 + \gamma_2\psi)}, \tag{3.18}$$

that is,

$$\frac{\psi(\alpha_2 + \gamma_2\psi)}{\alpha_1\psi + \beta_1(\alpha_2 + \gamma_2\phi)} = \frac{\phi(\alpha_2 + \gamma_2\phi)}{\alpha_1\phi + \beta_1(\alpha_2 + \gamma_2\psi)}, \tag{3.19}$$

from which

$$(\psi - \phi)\left\{\alpha_1\phi\psi\gamma_2 + \beta_1\left[\alpha_2^2 + 2\alpha_2\gamma_2(\phi + \psi) + \gamma_2^2\left(\phi^2 + \phi\psi + \psi^2\right)\right]\right\} = 0, \tag{3.20}$$

and this implies that $\phi = \psi$, which is a contradiction.

Now assume that

$$\chi, \varphi, \chi, \varphi, \dots, \quad (\chi \neq \varphi) \tag{3.21}$$

is a prime period-two solution of (3.16). Then we have

$$\chi = \frac{(\beta_1\varphi + \alpha_1)\chi\varphi}{\alpha_2\varphi + \gamma_2(\beta_1\chi + \alpha_1)}, \qquad \varphi = \frac{(\beta_1\chi + \alpha_1)\chi\varphi}{\alpha_2\chi + \gamma_2(\beta_1\varphi + \alpha_1)}, \tag{3.22}$$

from which

$$(\chi - \varphi)\left[\gamma_2\beta_1(\chi + \varphi) + \gamma_2\alpha_1 + \beta_1\chi\varphi\right] = 0, \tag{3.23}$$

and this implies that $\chi = \varphi$, which is a contradiction. □

The global behavior system (3.1) is described by the following result.

Theorem 3.3. *Consider system* (3.1). *There exists a set $\mathcal{C} \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of E. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates $\mathcal{R}$ into two connected and invariant components, namely,*

$$\begin{aligned} \mathcal{W}_- &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \\ \mathcal{W}_+ &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \end{aligned} \tag{3.24}$$

which satisfy

$$\lim_{n\to\infty}(x_n, y_n) = (0,\infty) \quad \textit{for every } (x_0, y_0) \in \mathcal{W}_-,$$
$$\lim_{n\to\infty}(x_n, y_n) = (\infty, 0) \quad \textit{for every } (x_0, y_0) \in \mathcal{W}_+. \tag{3.25}$$

Proof. In view of Theorem 3.2 and the injectivity of the map T we see that all conditions of Theorems 1.7, 1.9, and 1.10 and Corollary 1.8 are satisfied with $\mathcal{R} = (0,\infty) \times [0,\infty)$ and so the conclusion follows. □

The obtained result leads to the following characterization of the boundedness of solutions of system (3.1).

Corollary 3.4. *All bounded solutions of system* (3.1) *converge to the unique equilibrium with the corresponding initial conditions which belong to the graph of a continuous increasing function* $\mathcal{C}$ *in the plane of initial conditions. All solutions that start in the complement of* $\mathcal{C}$ *are asymptotic to either* $(\infty, 0)$ *or* $(0,\infty)$.

4. System (15,21)

Now we consider the following system of difference equations:

$$x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n}, \quad n = 0, 1, \ldots, \tag{4.1}$$

where the parameters β_1, B_1, α_2, and γ_2 are positive numbers and initial conditions $x_0 > 0$, $y_0 \geq 0$. The Jacobian matrix of the corresponding map $T(x, y) = (\beta_1 x/(B_1 x + y), (\alpha_2 + \gamma_2 y)/x)$ is of the form

$$J_T(x, y) = \begin{bmatrix} \dfrac{\beta_1 y}{(B_1 x + y)^2} & -\dfrac{\beta_1 x}{(B_1 x + y)^2} \\ -\dfrac{\alpha_2 + \gamma_2 y}{x^2} & \dfrac{\gamma_2}{x} \end{bmatrix}. \tag{4.2}$$

System (4.1) was considered in [1, Example 2], where it was shown that the corresponding map T is injective and

$$\det J_T(x, y) = -\frac{\beta_1 \alpha_2}{x(B_1 x + y)^2} < 0, \tag{4.3}$$

that is, the Jacobian matrix of T in (x, y) is invertible. Therefore, in view of Theorems 1.4 and 1.5, four subsequences

$$\{x_{2n}\}, \ \{x_{2n+1}\}, \ \{y_{2n}\}, \ \{y_{2n+1}\} \tag{4.4}$$

of every solution $\{(x_n, y_n)\}$ of system (4.1) are eventually monotonic.

4.1. Linearized Stability Analysis

Equilibrium points of system (4.1) are solutions of the system

$$\overline{x} = \frac{\beta_1 \overline{x}}{B_1 \overline{x} + \overline{y}}, \qquad \overline{y} = \frac{\alpha_2 + \gamma_2 \overline{y}}{\overline{x}}. \tag{4.5}$$

Since $\overline{x} \neq 0$, we obtain

$$\overline{x}_{\pm} = \frac{\beta_1 + \gamma_2 B_1 \pm \sqrt{D_3}}{2B_1}, \tag{4.6}$$

where $D_3 = (\beta_1 - B_1\gamma_2)^2 - 4B_1\alpha_2 \geq 0$.

This implies that we have the following three cases for the equilibrium points.

(i) If $\beta_1 - B_1\gamma_2 > 2\sqrt{B_1\alpha_2}$, then there exist two equilibrium points of system (4.1):

$$\begin{aligned} E_+ &= \left(\frac{\beta_1 + \gamma_2 B_1 + \sqrt{D_3}}{2B_1}, \frac{\beta_1 - \gamma_2 B_1 - \sqrt{D_3}}{2} \right), \\ E_- &= \left(\frac{\beta_1 + \gamma_2 B_1 - \sqrt{D_3}}{2B_1}, \frac{\beta_1 - \gamma_2 B_1 + \sqrt{D_3}}{2} \right). \end{aligned} \tag{4.7}$$

(ii) If $\beta_1 - B_1\gamma_2 = 2\sqrt{B_1\alpha_2}$, then system (4.1) has a unique equilibrium point:

$$E = \left(\frac{B_1\gamma_2 + \beta_1}{2B_1}, \frac{\beta_1 - B_1\gamma_2}{2} \right). \tag{4.8}$$

(iii) If $\beta_1 - B_1\gamma_2 \leq 0$ or $0 < \beta_1 - B_1\gamma_2 < 2\sqrt{B_1\alpha_2}$, then system (4.1) has no equilibrium points.

Next, by using (4.5) we have

$$J_T(\overline{x}, \overline{y}) = \begin{bmatrix} 1 - \dfrac{B_1}{\beta_1}\overline{x} & -\dfrac{\overline{x}}{\beta_1} \\ B_1 - \dfrac{\beta_1}{\overline{x}} & \dfrac{\gamma_2}{\overline{x}} \end{bmatrix}. \tag{4.9}$$

The corresponding characteristic equation evaluated at the equilibrium point $E = (\overline{x}, \overline{y})$ is

$$\lambda^2 - p\lambda + q = 0, \tag{4.10}$$

where

$$p = \operatorname{Tr} J_T(\overline{x}, \overline{y}) = \frac{\overline{y}}{\beta_1} + \frac{\gamma_2}{\overline{x}} = 1 - \frac{B_1\overline{x}}{\beta_1} + \frac{\gamma_2}{\overline{x}},$$
$$q = \operatorname{Det} J_T(\overline{x}, \overline{y}) = \frac{\gamma_2}{\overline{x}} - \frac{B_1\gamma_2}{\beta_1} + \frac{B_1\overline{x}}{\beta_1} - 1 = \frac{\gamma_2\overline{y}}{\beta_1\overline{x}} - \frac{\overline{y}}{\beta_1}. \tag{4.11}$$

Notice that $p > 0$.

Lemma 4.1. *If $\beta_1 - B_1\gamma_2 > 2\sqrt{B_1\alpha_2}$, then the equilibrium point E_+ of system* (4.1) *is locally asymptotically stable and the equilibrium point E_- is a saddle point.*

If $\beta_1 - B_1\gamma_2 = 2\sqrt{B_1\alpha_2}$, then the equilibrium point E of system (4.1) *is nonhyperbolic.*

Proof. First, assume $\beta_1 - B_1\gamma_2 > 2\sqrt{B_1\alpha_2}$. For the equilibrium point E_+ we need to prove that

$$|p| < 1 + q < 2, \tag{4.12}$$

or equivalently (because $p > 0$):

(I) $p < 1 + q$,

(II) $q < 1$.

Indeed,

(I) we have

$$p < 1 + q \Longleftrightarrow 1 - \frac{B_1\overline{x}}{\beta_1} + \frac{\gamma_2}{\overline{x}} < 1 + \frac{\gamma_2}{\overline{x}} - \frac{B_1\gamma_2}{\beta_1} + \frac{B_1\overline{x}}{\beta_1} - 1$$
$$\Longleftrightarrow 1 + \frac{B_1\gamma_2}{\beta_1} < 2\frac{B_1\overline{x}}{\beta_1} \Longleftrightarrow 1 + \frac{B_1\gamma_2}{\beta_1} < \frac{\beta_1 + \gamma_2 B_1 + \sqrt{D_3}}{\beta_1} \Longleftrightarrow 0 < \sqrt{D_3}, \tag{4.13}$$

which is true. Furthermore

(II) we have

$$q < 1 \Longleftrightarrow \frac{\gamma_2}{\overline{x}} - \frac{B_1\gamma_2}{\beta_1} + \frac{B_1\overline{x}}{\beta_1} - 1 < 1 \Longleftrightarrow \gamma_2\left(\frac{1}{\overline{x}} - \frac{B_1}{\beta_1}\right) + \frac{B_1\overline{x} - \beta_1}{\beta_1} < 1$$
$$\Longleftrightarrow \gamma_2\left(\frac{\beta_1 - B_1\overline{x}}{\beta_1\overline{x}}\right) - \frac{\beta_1 - B_1\overline{x}}{\beta_1} < 1 \Longleftrightarrow \frac{\beta_1 - B_1\overline{x}}{\beta_1}\left(\frac{\gamma_2}{\overline{x}} - 1\right) < 1 \tag{4.14}$$
$$\Longleftrightarrow \frac{\overline{y}}{\beta_1}\left(1 - \frac{\alpha_2}{\overline{x}\overline{y}} - 1\right) < 1 \Longleftrightarrow -\frac{\alpha_2}{\beta_1\overline{x}} < 1,$$

which is true.

For the equilibrium point E_- we need to prove that

$$|p| > |1+q|, \qquad p^2 - 4q > 0, \tag{4.15}$$

that is (because $p > 0$ and $1+q > 0$)

(I) $p > 1+q$,

(II) $p^2 - 4q > 0$.

Indeed,

$$\begin{aligned} 1+q > 0 &\Longleftrightarrow 1 + \frac{\gamma_2}{\overline{x}} - \frac{B_1\gamma_2}{\beta_1} + \frac{B_1\overline{x}}{\beta_1} - 1 > 0 \\ &\Longleftrightarrow \gamma_2 \frac{\beta_1 - B_1\overline{x}}{\overline{x}\beta_1} + \frac{B_1\overline{x}}{\beta_1} > 0 \Longleftrightarrow \gamma_2 \frac{\overline{y}}{\overline{x}\beta_1} + \frac{B_1\overline{x}}{\beta_1} > 0. \end{aligned} \tag{4.16}$$

Now

(I) we have

$$\begin{aligned} p > 1+q &\Longleftrightarrow 1 - \frac{B_1\overline{x}}{\beta_1} + \frac{\gamma_2}{\overline{x}} > 1 + \frac{\gamma_2}{\overline{x}} - \frac{B_1\gamma_2}{\beta_1} + \frac{B_1\overline{x}}{\beta_1} - 1 \\ &\Longleftrightarrow 1 + \frac{B_1\gamma_2}{\beta_1} > \frac{2B_1\overline{x}}{\beta_1} \Longleftrightarrow 1 + \frac{B_1\gamma_2}{\beta_1} > \frac{\beta_1 + \gamma_2 B_1 - \sqrt{D_3}}{\beta_1} \\ &\Longleftrightarrow 0 > -\sqrt{D_3}, \end{aligned} \tag{4.17}$$

which is true.

Similarly

(II) we have

$$\begin{aligned} p^2 - 4q > 0 &\Longleftrightarrow \left(\frac{\overline{y}}{\beta_1} + \frac{\gamma_2}{\overline{x}}\right)^2 - 4\left(\frac{\gamma_2\overline{y}}{\beta_1\overline{x}} - \frac{\overline{y}}{\beta_1}\right) > 0 \\ &\Longleftrightarrow \frac{\overline{y}^2}{\beta_1^2} - 2\frac{\gamma_2\overline{y}}{\beta_1\overline{x}} + \frac{\gamma_2^2}{\overline{x}^2} + 4\frac{\overline{y}}{\beta_1} > 0 \Longleftrightarrow \left(\frac{\overline{y}}{\beta_1} - \frac{\gamma_2}{\overline{x}}\right)^2 + 4\frac{\overline{y}}{\beta_1} > 0, \end{aligned} \tag{4.18}$$

which is satisfied.

Assume that $\beta_1 - B_1\gamma_2 = 2\sqrt{B_1\alpha_2}$.

We need to prove that

$$|1+q| = |p|, \tag{4.19}$$

that is (because $p > 0$ and, $1 + q > 0$),

$$1 + q = p. \tag{4.20}$$

We have

$$1 + q = p \Longleftrightarrow 1 + \frac{B_1\gamma_2}{\beta_1} = \frac{2B_1\overline{x}}{\beta_1} \Longleftrightarrow 1 + \frac{B_1\gamma_2}{\beta_1} = \frac{2B_1}{\beta_1}\frac{B_1\gamma_2 + \beta_1}{2B_1}. \tag{4.21}$$

□

4.2. Global Results

Theorem 4.2. *System* (4.1) *has no prime period-two solutions.*

Proof. The second iterate of map T is

$$\begin{aligned} T^2(x, y) &= T\left(\frac{\beta_1 x}{B_1 x + y}, \frac{\alpha_2 + \gamma_2 y}{x}\right) \\ &= \left(\frac{\beta_1(\beta_1 x/(B_1 x + y))}{B_1(\beta_1 x/(B_1 x + y)) + (\alpha_2 + \gamma_2 y)/x}, \frac{\alpha_2 + \gamma_2((\alpha_2 + \gamma_2 y)/x)}{\beta_1 x/(B_1 x + y)}\right) \\ &= \left(\frac{\beta_1^2 x^2}{B_1\beta_1 x^2 + (\alpha_2 + \gamma_2 y)(B_1 x + y)}, \frac{(B_1 x + y)(\alpha_2 x + \gamma_2\alpha_2 + \gamma_2^2 y)}{\beta_1 x^2}\right). \end{aligned} \tag{4.22}$$

Period-two solution satisfies

$$\begin{aligned} &\frac{\beta_1^2 x^2}{B_1\beta_1 x^2 + (\alpha_2 + \gamma_2 y)(B_1 x + y)} - x = 0, \\ &\frac{(B_1 x + y)(\alpha_2 x + \gamma_2\alpha_2 + \gamma_2^2 y)}{\beta_1 x^2} - y = 0, \\ &\beta_1^2 x = B_1\beta_1 x^2 + (\alpha_2 + \gamma_2 y)(B_1 x + y), \\ &\beta_1 y x^2 = (B_1 x + y)\left(\alpha_2 x + \alpha_2\gamma_2 + \gamma_2^2 y\right). \end{aligned} \tag{4.23}$$

From this system we have

(i) $x = (1/2B_1)(\beta_1 + \gamma_2 B_1 + \sqrt{(\beta_1 - B_1\gamma_2)^2 - 4B_1\alpha_2})$, $y = (1/2B_1)(\beta_1 - \gamma_2 B_1 - \sqrt{(\beta_1 - B_1\gamma_2)^2 - 4B_1\alpha_2})$,

(ii) $x = (1/2B_1)(\beta_1 + \gamma_2 B_1 - \sqrt{(\beta_1 - B_1\gamma_2)^2 - 4B_1\alpha_2})$, $y = (1/2B_1)(\beta_1 - \gamma_2 B_1 + \sqrt{(\beta_1 - B_1\gamma_2)^2 - 4B_1\alpha_2})$,

(iii) $x = -\gamma_2 + (1/2\beta_1^2)(\alpha_2\beta_1 + \beta_1^2\gamma_2 - B_1\alpha_2\gamma_2 + B_1\beta_1\gamma_2^2 - \sqrt{\Delta_1})$, $y = -(1/2\beta_1\gamma_2)(\alpha_2\beta_1 + \beta_1^2\gamma_2 - B_1\alpha_2\gamma_2 + B_1\beta_1\gamma_2^2 - \sqrt{\Delta_1})$,

(iv) $x = -\gamma_2 + (1/2\beta_1^2)(\alpha_2\beta_1 + \beta_1^2\gamma_2 - B_1\alpha_2\gamma_2 + B_1\beta_1\gamma_2^2 + \sqrt{\Delta_1})$, $y = -(1/2\beta_1\gamma_2)(\alpha_2\beta_1 + \beta_1^2\gamma_2 - B_1\alpha_2\gamma_2 + B_1\beta_1\gamma_2^2 + \sqrt{\Delta_1})$,

(v) $x = 0$, $y = -\alpha/\gamma$,

where $\Delta_1 = (\alpha - \beta\gamma)(\alpha\beta^2 + 3\beta^3\gamma + B^2\alpha\gamma^2 + 2B\beta^2\gamma^2 - B^2\beta\gamma^3 - 2B\alpha\beta\gamma)$.

In cases (i) and (ii) solutions (x, y) are equilibrium points E_+ and E_-, and in case (v) solution (x, y) is not in the first quadrant in the plane. It is sufficient to prove that solutions (x, y) in cases (iii) and (iv) are not in the first quadrant in the plane. Namely, if $\Delta_1 < 0$, x and y are not real. Supose that $\Delta_1 \geq 0$. If $\alpha_2\beta_1 + \beta_1^2\gamma_2 - B_1\alpha_2\gamma_2 + B_1\beta_1\gamma_2^2 - \sqrt{\Delta_1} \geq 0$, then $y \leq 0$. If $\alpha_2\beta_1 + \beta_1^2\gamma_2 - B_1\alpha_2\gamma_2 + B_1\beta_1\gamma_2^2 - \sqrt{\Delta_1} < 0$, then $x < 0$ for solution in case (iii). By analogous reasoning we have that the same conclusion for case (iv) holds. □

Our linearized stability analysis indicates that there are three cases with different asymptotic behavior, depending on the values of parameters β_1, B_1, α_2, and γ_2.

Case 1. $\beta_1 - B_1\gamma_2 > 2\sqrt{B_1\alpha_2}$.

Case 2. $\beta_1 - B_1\gamma_2 = 2\sqrt{B_1\alpha_2}$.

Case 3. $\beta_1 - B_1\gamma_2 \leq 0$ or $0 < \beta_1 - B_1\gamma_2 < 2\sqrt{B_1\alpha_2}$.

4.2.1. Global Results—Case 1

Theorem 4.3. *Consider system* (4.1) *and assume that* $\beta_1 - B_1\gamma_2 > 2\sqrt{B_1\alpha_2}$. *Then there exists a set* $\mathcal{C} \subset \mathcal{R}$ *which is invariant and a subset of the basin of attraction of* E_-. *The set* $\mathcal{C}$ *is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates* $\mathcal{R}$ *into two connected and invariant components, namely,*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{\text{se}} y\}$$
$$\mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{\text{se}} x\}, \tag{4.24}$$

which satisfy

$$\lim_{n\to\infty}(x_n, y_n) = (0, \infty) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_-,$$
$$\lim_{n\to\infty}(x_n, y_n) = \left(\frac{\beta_1 + \gamma_2 B_1 + \sqrt{D_3}}{2B_1}, \frac{\beta_1 - \gamma_2 B_1 - \sqrt{D_3}}{2}\right) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_+ \tag{4.25}$$

Proof. Clearly, system (4.1) is strongly competitive on $\mathcal{R} = (0, \infty) \times [0, \infty)$. In view of injectivity of T, invertibility of J_T, and Theorem 4.2, we see that all conditions of Theorems 1.7, 1.9, and 1.10 and Corollary 1.8 are satisfied and the conclusion of the theorem follows. □

4.2.2. Global Results—Case 2

Theorem 4.4. *Consider system* (4.1) *and assume that* $\beta_1 - B_1\gamma_2 = 2\sqrt{B_1\alpha_2}$. *Then there exists a set* $\mathcal{C} \subset \mathcal{R}$ *which is invariant and a subset of the basin of attraction of* E. *The set* $\mathcal{C}$ *is a graph of a strictly* *increasing continuous function of the first variable on an interval (and so is a manifold) and separates $\mathcal{R}$ into two connected and invariant components, namely,*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\},$$
$$\mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{4.26}$$

which satisfy

$$\lim_{n\to\infty} (x_n, y_n) = (0, \infty) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_-,$$
$$\lim_{n\to\infty} (x_n, y_n) = E = \left(\frac{B_1\gamma_2 + \beta_1}{2B_1}, \frac{\beta_1 - B_1\gamma_2}{2}\right) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_+. \tag{4.27}$$

Proof. In this case system (4.1) has a unique equilibrium point $E = ((B_1\gamma_2 + \beta_1)/2B_1, (\beta_1 - B_1\gamma_2)/2)$ which is nonhyperbolic. For $p = q + 1$, the corresponding characteristic equation is of the form

$$\lambda^2 - p\lambda + p - 1 = 0. \tag{4.28}$$

This implies

$$\lambda_1 = p - 1, \qquad \lambda_2 = 1, \tag{4.29}$$

and $|\lambda_1| < 1 \Leftrightarrow |p - 1| < 1 \Leftrightarrow 0 < p < 2$.

It is obvious that $p > 0$. We will show that $p < 1$. Indeed

$$\begin{aligned} p < 1 &\Longleftrightarrow \frac{\sqrt{B_1\alpha_2}}{\beta_1} + \frac{\gamma_2 B_1}{B_1\gamma_2 + \sqrt{B_1\alpha_2}} < 1 \\ &\Longleftrightarrow B_1\gamma_2\sqrt{B_1\alpha_2} + B_1\alpha_2 + \beta_1\gamma_2 B_1 < \beta_1\gamma_2 B_1 + \beta_1\sqrt{B_1\alpha_2} \\ &\Longleftrightarrow B_1\gamma_2 + \sqrt{B_1\alpha_2} < \beta_1 = B_1\gamma_2 + 2\sqrt{B_1\alpha_2} \end{aligned} \tag{4.30}$$

which is satisfied. Thus, $\lambda_1 \in (-1, 0)$.

The eigenvector corresponding to $\lambda_1 = p - 1$ is

$$\begin{pmatrix} 1 \\ \dfrac{2B_1\beta_1(\beta_1 - B_1\gamma_2)}{(B_1\gamma_2 + \beta_1)^2} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{4B_1\beta_1\sqrt{B_1\alpha_2}}{(B_1\gamma_2 + \beta_1)^2} \end{pmatrix}. \tag{4.31}$$

It means that all conditions of Theorems 1.7 and 1.10 are satisfied with $\mathcal{R} = (0, \infty) \times [0, \infty)$.

Assume that $(x_0, y_0) \in \mathcal{W}_+$. Then $(x_n, y_n) \in \mathcal{W}_+$ for all n, and sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$, and $\{y_{2n+1}\}$ are monotone and bounded since $x_n \leq \beta_1/B_1$. Thus these sequences are convergent, which in view of Theorem 4.2 shows that they converge to the equilibrium point. Since E is the unique equilibrium point in $\mathcal{W}_+$ the statement for $\mathcal{W}_+$ follows. The same conclusion is obtained by using Theorem 1.6.

If (x_0, y_0) is in $\mathcal{W}_-$, by Theorem 1.10 the orbit of (x_0, y_0) eventually enters $Q_2(E)$. Assume (without loss of generality) that $(x_0, y_0) \in \operatorname{int} Q_2(E)$. An eigenvector associated with the nonhyperbolic eigenvalue $\lambda_2 = 1$ is $v = (-1, B_1)$. Choose a value of t small enough so that $E + tv \in Q_2(E)$ and $(x_0, y_0) \preceq E + tv$. Let us show that $T(E + tv) \preceq E + tv$. Indeed

$$
\begin{aligned}
T(E+tv) &= \left(\frac{B_1\gamma_2 + \beta_1}{2B_1} - t, \frac{B_1(2\alpha_2 + \beta_1\gamma_2 - B_1\gamma_2^2 + 2B_1\gamma_2 t)}{B_1\gamma_2 + \beta_1 + 2B_1 t} \right) \\
&\preceq \left(\frac{B_1\gamma_2 + \beta_1}{2B_1} - t, \frac{\beta_1 - B_1\gamma_2}{2} + B_1 t \right)
\end{aligned} \tag{4.32}
$$

because

$$
\frac{B_1(2\alpha_2 + \beta_1\gamma_2 - B_1\gamma_2^2 + 2B_1\gamma_2 t)}{B_1\gamma_2 + \beta_1 + 2B_1 t} \geq \frac{\beta_1 - B_1\gamma_2}{2} + B_1 t \tag{4.33}
$$

reduces to

$$
4B_1^2 t^2 + 4B_1\alpha_2 + 2B_1\beta_1\gamma_2 \geq \beta_1^2 + B_1^2\gamma_2^2 = 4B_1\alpha_2 + 2B_1\beta_1\gamma_2, \tag{4.34}
$$

where the last equality follows from the condition $\beta_1 - B_1\gamma_2 = 2\sqrt{B_1\alpha_2}$.

Since $T(E + tv) \preceq E + tv$, it follows that $\{T^n(E + tv)\}$ is a monotonically decreasing sequence in $Q_2(E)$ which is bounded above by E. Since $\{T^n(E + tv)\}$ is coordinatewise monotone and it does not converge (if it did it would have to converge to E, which is impossible), we have that $T^n(E + tv)$ has second coordinate which is monotone and unbounded. But $(x_n, y_n) := T^n(x_0, y_0) \preceq T^n(E + tv)$, which implies that $y_n \to \infty$. From (4.1) it follows that $x_n \to 0$. □

4.2.3. *Global Results—Case 3*

Theorem 4.5. *Consider system* (4.1) *and assume that* $\beta_1 - B_1\gamma_2 \leq 0$ *or* $0 < \beta_1 - B_1\gamma_2 < 2\sqrt{B_1\alpha_2}$. *Then every solution* $\{(x_n, y_n)\}$ *of system* (4.1) *satisfies*

$$
\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \infty. \tag{4.35}
$$

Proof. In this case system (4.1) has no equilibrium points. Consider now the following system satisfied by subsequences of the solution of system (4.1):

$$x_{2k+1} = \frac{\beta_1 x_{2k}}{B_1 x_{2k} + y_{2k}}, \qquad x_{2k+2} = \frac{\beta_1 x_{2k+1}}{B_1 x_{2k+1} + y_{2k+1}},$$
$$y_{2k+1} = \frac{\alpha_2 + \gamma_2 y_{2k}}{x_{2k}}, \qquad y_{2k+2} = \frac{\alpha_2 + \gamma_2 y_{2k+1}}{x_{2k+1}}. \tag{4.36}$$

We know that each of the four subsequences

$$\{x_{2k}\},\ \{x_{2k+1}\},\ \{y_{2k}\},\ \{y_{2k+1}\} \tag{4.37}$$

of every solution $\{(x_n, y_n)\}$ of system (4.1) is eventually monotonic. The subsequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ are bounded (by β_1/B_1), which implies that they are convergent. Suppose that (a) $\lim_{k\to\infty} x_{2k} = x_E$ and (b) $\lim_{k\to\infty} x_{2k+1} = x_O$. For the other two subsequences the following four cases are possible: (1) $\lim_{k\to\infty} y_{2k} = y_E$, (2) $\lim_{k\to\infty} y_{2k} = \infty$, (3) $\lim_{k\to\infty} y_{2k+1} = y_O$, or (4) $\lim_{k\to\infty} y_{2k+1} = \infty$.

Case 1 and Case 3 imply

$$x_O = \frac{\beta_1 x_E}{B_1 x_E + y_E}, \qquad x_E = \frac{\beta_1 x_O}{B_1 x_O + y_O},$$
$$y_O = \frac{\alpha_2 + \gamma_2 y_E}{x_E}, \qquad y_E = \frac{\alpha_2 + \gamma_2 y_O}{x_O} \tag{4.38}$$

that is, system (4.1) has a period-two solution, which is a contradiction by Theorem 4.2.

Case 1 and Case 4 imply

$$\lim_{k\to\infty} x_{2k} = x_E = 0 \Longrightarrow \lim_{k\to\infty} x_{2k+1} = x_O = 0 \Longrightarrow \lim_{k\to\infty} y_{2k+2} = \infty, \tag{4.39}$$

which is a contradiction by Case 1.

Case 2 and Case 3 imply

$$\lim_{k\to\infty} x_{2k+1} = x_O = 0 \Longrightarrow \lim_{k\to\infty} x_{2k+2} = x_E = 0 \Longrightarrow \lim_{k\to\infty} y_{2k+1} = \infty, \tag{4.40}$$

which is a contradiction by Case 3.

Case 2 and Case 4 imply

$$\lim_{k\to\infty} x_{2k} = x_E = \lim_{k\to\infty} x_{2k+1} = x_O = 0. \tag{4.41}$$

□

The obtained results lead to the following characterization of the boundedness of solutions of system (3.1).

Corollary 4.6. *Consider system* (4.1) *and assume that* $\beta_1 - B_1\gamma_2 \geq 2\sqrt{B_1\alpha_2}$. *All bounded solutions of system* (4.1) *converge to the unique equilibrium with the corresponding initial conditions which belong to region below and on the graph of a continuous increasing function* $\mathcal{C}$ *in the plane of initial conditions. All solutions that start above* $\mathcal{C}$ *are asymptotic to* $(0,\infty)$.

Consider system (4.1) *and assume that either* $\beta_1 - B_1\gamma_2 \leq 0$ *or* $\beta_1 - B_1\gamma_2 < 2\sqrt{B_1\alpha_2}$. *Then every solution of* (4.1) *is asymptotic to* $(0,\infty)$.

5. System (21,38)

Now we consider the following system of difference equations:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}, \quad n = 0, 1, \ldots, \tag{5.1}$$

where the parameters $\alpha_1, \beta_1, A_2, B_2$, and γ_2 are positive numbers and initial conditions $x_0 \geq 0,\ y_0 > 0$.

The Jacobian matrix of the corresponding map $T(x,y) = ((\alpha_1+\beta_1 x)/y, \gamma_2 y/(A_2+B_2x+y))$ is of the form

$$J_T(x,y) = \begin{bmatrix} \dfrac{\beta_1}{y} & -\dfrac{\alpha_1+\beta_1 x}{y^2} \\ -\dfrac{\gamma_2 B_2 y}{(A_2+B_2x+y)^2} & \dfrac{\gamma_2(A_2+B_2x)}{(A_2+B_2x+y)^2} \end{bmatrix}. \tag{5.2}$$

System (5.1) was considered in [1, Example 4], where it was shown that the map T is injective. In addition, when

$$\beta_1 A_2 > \alpha_1 B_2, \tag{5.3}$$

we see that

$$\det J_T(x,y) = \frac{(\beta_1 A_2 - \alpha_1 B_2)}{(A_2+B_2x+y)^2 y} > 0. \tag{5.4}$$

Therefore, when (5.3) holds, the Jacobian matrix of T in (x,y) is invertible and in view of Theorems 1.4 and 1.5 every solution of system (5.1) is eventually componentwise monotonic.

When

$$\beta_1 A_2 < \alpha_1 B_2, \tag{5.5}$$

we see that

$$\det J_T(x,y) = \frac{(\beta_1 A_2 - \alpha_1 B_2)}{(A_2+B_2x+y)^2 y} < 0, \tag{5.6}$$

and the Jacobian matrix of T in (x, y) is invertible. Therefore, in view of Theorems 1.4 and 1.5, four subsequences

$$\{x_{2n}\},\ \{x_{2n+1}\},\ \{y_{2n}\},\ \{y_{2n+1}\} \tag{5.7}$$

of every solution $\{(x_n, y_n)\}$ of system (5.1) are eventually monotonic.

5.1. Linearized Stability Analysis

Equilibrium points of system (5.1) are solutions of the system

$$\overline{x} = \frac{\alpha_1 + \beta_1 \overline{x}}{\overline{y}}, \qquad \overline{y} = \frac{\gamma_2 \overline{y}}{A_2 + B_2 \overline{x} + \overline{y}}. \tag{5.8}$$

Since $\overline{y} \neq 0$, we have

$$\overline{x}_{\pm} = \frac{1}{2B_2}\left(\gamma_2 - A_2 - \beta_1 \pm \sqrt{D_4}\right), \tag{5.9}$$

where

$$D_4 = \left(\gamma_2 - A_2 - \beta_1\right)^2 - 4\alpha_1 B_2. \tag{5.10}$$

It is easy to prove that the following result holds.

Lemma 5.1. (i) *If* $\gamma_2 - A_2 - \beta_1 > 2\sqrt{\alpha_1 B_2}$, *then system* (5.1) *has two equilibrium points:*

$$\begin{aligned} E_+ &= \left(\frac{\gamma_2 - A_2 - \beta_1 + \sqrt{D_4}}{2B_2}, \frac{\gamma_2 - A_2 + \beta_1 - \sqrt{D_4}}{2}\right), \\ E_- &= \left(\frac{\gamma_2 - A_2 - \beta_1 - \sqrt{D_4}}{2B_2}, \frac{\gamma_2 - A_2 + \beta_1 + \sqrt{D_4}}{2}\right). \end{aligned} \tag{5.11}$$

(ii) *If* $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$, *then system* (5.1) *has a unique equilibrium point:*

$$E = \left(\frac{\gamma_2 - A_2 - \beta_1}{2B_2}, \frac{\gamma_2 - A_2 + \beta_1}{2}\right). \tag{5.12}$$

(iii) *If* $\gamma_2 \leq A_2 + \beta_1$ *or* $0 < \gamma_2 - A_2 - \beta_1 < 2\sqrt{\alpha_1 B_2}$, *then system* (5.1) *has no equilibrium points.*

Lemma 5.2. *If* $\gamma_2 - A_2 - \beta_1 > 2\sqrt{\alpha_1 B_2}$, *then the equilibrium point* E_+ *of system* (5.1) *is a saddle point and* E_- *is locally asymptotically stable.*

If $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$, *then the equilibrium point* E *of system* (5.1) *is nonhyperbolic.*

Proof. By using (5.8) we have

$$J_T(\overline{x},\overline{y}) = \begin{bmatrix} \dfrac{\beta_1}{\overline{y}} & -\dfrac{\overline{x}}{\overline{y}} \\ -\dfrac{B_2\overline{y}}{\gamma_2} & 1-\dfrac{\overline{y}}{\gamma_2} \end{bmatrix}. \tag{5.13}$$

The corresponding characteristic equation evaluated at the equilibrium point $E(\overline{x},\overline{y})$ is

$$\lambda^2 - p\lambda + q = 0, \tag{5.14}$$

where

$$\begin{aligned} p &= \operatorname{Tr} J_T(\overline{x},\overline{y}) = \frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2}, \\ q &= \operatorname{Det} J_T(\overline{x},\overline{y}) = \frac{\beta_1}{\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2}. \end{aligned} \tag{5.15}$$

For the equilibrium point E_+ we need to prove that

$$|p| > |1+q|, \qquad p^2 - 4q > 0, \tag{5.16}$$

that is (because $p > 0$ and $1+q > 0$),

(I) $p > 1+q$,

(II) $p^2 - 4q > 0$.

Indeed

$$p = \frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2} = \frac{1}{\gamma_2}\left(\frac{\beta_1\gamma_2}{\overline{y}} + \gamma_2 - \overline{y}\right) = \frac{1}{\gamma_2}\left(\frac{\beta_1\gamma_2}{\overline{y}} + A_2 + B_2\overline{x}\right) > 0, \tag{5.17}$$

which is always true, and in view of (5.8) $\gamma_2 - B_2\overline{x} = \overline{y} + A_2$ we obtain

$$\begin{aligned} 1+q > 0 &\Longleftrightarrow 1 + \frac{\beta_1}{\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2} > 0 \\ &\Longleftrightarrow \gamma_2 + \frac{\beta_1\gamma_2}{\overline{y}} - \beta_1 - B_2\overline{x} > 0 \Longleftrightarrow \overline{y} + A_2 + \beta_1\frac{\gamma_2 - \overline{y}}{\overline{y}} > 0, \end{aligned} \tag{5.18}$$

which is true because $\overline{y} < \gamma_2$.

Next, in view of (5.8) $\gamma_2 - B_2\overline{x} = \overline{y} + A_2$,

(I) we have

$$p > 1 + q \Longleftrightarrow \frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2} > 1 + \frac{\beta_1}{\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2}$$
$$\Longleftrightarrow \overline{y} < \beta_1 + B_2\overline{x} \Longleftrightarrow \gamma_2 - A_2 - B_2\overline{x} < \beta_1 + B_2\overline{x} \Longleftrightarrow 0 < \sqrt{D_4}, \tag{5.19}$$

which is true.

Similarly,

(II) we have

$$p^2 - 4q > 0 \Longleftrightarrow \left(\frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2}\right)^2 - 4\frac{\beta_1}{\overline{y}} + 4\frac{\beta_1}{\gamma_2} + 4\frac{B_2\overline{x}}{\gamma_2} > 0$$
$$\Longleftrightarrow \left(\frac{\beta_1}{\overline{y}}\right)^2 + 2\frac{\beta_1}{\overline{y}} + 1 - 2\frac{\overline{y}}{\gamma_2}\frac{\beta_1}{\overline{y}} - 2\frac{\overline{y}}{\gamma_2} + \left(\frac{\overline{y}}{\gamma_2}\right)^2 - 4\frac{\beta_1}{\overline{y}} + 4\frac{\beta_1}{\gamma_2} + 4\frac{B_2\overline{x}}{\gamma_2} > 0 \tag{5.20}$$
$$\Longleftrightarrow \left(\frac{\beta_1}{\overline{y}} - 1 + \frac{\overline{y}}{\gamma_2}\right)^2 + 4\frac{B_2\overline{x}}{\gamma_2} > 0,$$

which is true.

For the equilibrium point E_- we need to prove that

$$p < 1 + q < 2, \tag{5.21}$$

or equivalently

(I) $p < 1 + q$,

(II) $q < 1$.

Indeed,

(I) we have

$$p < 1 + q \Longleftrightarrow \frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2} < 1 + \frac{\beta_1}{\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2}$$
$$\Longleftrightarrow \overline{y} > \beta_1 + B_2\overline{x} \Longleftrightarrow \gamma_2 - A_2 - B_2\overline{x} > \beta_1 + B_2\overline{x} \Longleftrightarrow 0 > -\sqrt{D_4}, \tag{5.22}$$

which is true, and in view of (5.8) $\beta_1/\overline{y} = 1 - \alpha_1/\overline{x}\overline{y}$

(II) we obtain

$$q < 1 \Longleftrightarrow \frac{\beta_1}{\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2} < 1 \Longleftrightarrow 1 - \frac{\alpha_1}{\overline{x}\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2} < 1$$
$$\Longleftrightarrow -\frac{\alpha_1}{\overline{x}\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2} < 0, \tag{5.23}$$

which is true.

Assume that $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$.

Let us prove that $1 + q = p$. We have

$$1 + q = p \Longleftrightarrow 1 + \frac{\beta_1}{\overline{y}} - \frac{\beta_1}{\gamma_2} - \frac{B_2\overline{x}}{\gamma_2} = \frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2}$$
$$\Longleftrightarrow \overline{y} = \beta_1 + B_2\overline{x} \Longleftrightarrow 2\overline{y} = \beta_1 + \gamma_2 - A_2, \tag{5.24}$$

which is true. □

5.2. Global Results

Theorem 5.3. *System* (5.1) *has no prime period-two solutions.*

Proof. The second iterate of the map T is

$$T^2(x,y) = T\left(\frac{\alpha_1 + \beta_1 x}{y}, \frac{\gamma_2 y}{A_2 + B_2 x + y}\right)$$
$$= \left(\frac{\alpha_1 + \beta_1((\alpha_1 + \beta_1 x)/y)}{\gamma_2 y/(A_2 + B_2 x + y)}, \frac{\gamma_2(\gamma_2 y/(A_2 + B_2 x + y))}{A_2 + B_2((\alpha_1 + \beta_1 x)/y) + \gamma_2 y/(A_2 + B_2 x + y)}\right), \tag{5.25}$$

that is,

$$T^2(x,y) = \begin{pmatrix} \dfrac{(A_2 + B_2 x + y)(\alpha_1 y + \alpha_1\beta_1 + \beta_1^2 x)}{\gamma_2 y^2} \\[2ex] \dfrac{\gamma_2^2 y^2}{y\left[(A_2 + B_2 x + y)(A_2 y + B_2\alpha_1 + B_2\beta_1 x) + \gamma_2 y^2\right]} \end{pmatrix} \tag{5.26}$$

Period-two solutions satisfy

$$\frac{(A_2+B_2x+y)(\alpha_1y+\alpha_1\beta_1+\beta_1^2x)}{\gamma_2y^2}-x=0,$$
$$\frac{\gamma_2^2y^2}{y\left[(A_2+B_2x+y)(A_2y+B_2\alpha_1+B_2\beta_1x)+\gamma_2y^2\right]}-y=0. \tag{5.27}$$

From this system we have

(i) $x=(\gamma_2-A_2-\beta_1+\sqrt{D_4})/2B_2,\ y=(\gamma_2-A_2+\beta_1-\sqrt{D_4})/2,$

(ii) $x=(\gamma_2-A_2-\beta_1-\sqrt{D_4})/2B_2,\ y=(\gamma_2-A_2+\beta_1+\sqrt{D_4})/2,$

(iii) $x=-A_2/B_2,\ y=0,$

(iv) $x=-\alpha_1/\beta_1,\ y=0,$

(v) $x=-(1/B_2)(A_2+\gamma_2)-(1/2B_2\beta_1\gamma_2)(\Lambda+\sqrt{\Delta_2}),\ y=(1/2\gamma_2(A_2+\gamma_2))(\Lambda+\sqrt{\Delta_2}),$

(vi) $x=-(1/B)(A+\gamma)-(1/2B_2\beta_1\gamma_2)(\Lambda-\sqrt{\Delta_2}),\ y=(1/2\gamma_2(A_2+\gamma_2))(\Lambda-\sqrt{\Delta_2}),$

Where

$$\Lambda=A_2\beta_1^2-A_2^2\beta_1-\beta_1\gamma_2^2+\beta_1^2\gamma_2+A_2B_2\alpha_1-2A_2\beta_1\gamma_2-B_2\alpha_1\beta_1+B_2\alpha_1\gamma_2,$$
$$\begin{aligned}\Delta_2=-(A_2\beta_1-B_2\alpha_1+\beta_1\gamma_2)\Big(&2A_2^2\beta_1^2+2\beta_1^2\gamma_2^2-A_2\beta_1^3-A_2^3\beta_1+3\beta_1\gamma_2^3-\beta_1^3\gamma_2\\&+A_2^2B_2\alpha_1+A_2\beta_1\gamma_2^2+4A_2\beta_1^2\gamma_2+B_2\alpha_1\beta_1^2-3A_2^2\beta_1\gamma_2\\&+B_2\alpha_1\gamma_2^2-2A_2B_2\alpha_1\beta_1+2A_2B_2\alpha_1\gamma_2-2B_2\alpha_1\beta_1\gamma_2\Big).\end{aligned} \tag{5.28}$$

In cases (i) and (ii) solutions (x,y) are the equilibrium points E_+ and E_-, and in cases (iii) and (iv) solution(x,y) is not in the first quadrant in the plane. It is sufficient to prove that solutions (x,y) in cases (v) and (vi) are not in the first quadrant in the plane. Namely, $\Delta_2<0$ implies that x and y are not real. Suppose that $\Delta_2\geq 0$. If

$$A_2\beta_1^2-A_2^2\beta_1-\beta_1\gamma_2^2+\beta_1^2\gamma_2+A_2B_2\alpha_1-2A_2\beta_1\gamma_2-B_2\alpha_1\beta_1+B_2\alpha_1\gamma_2+\sqrt{\Delta_2}\geq 0, \tag{5.29}$$

then $y\geq 0$ and $x<0$. If

$$A_2\beta_1^2-A_2^2\beta_1-\beta_1\gamma_2^2+\beta_1^2\gamma_2+A_2B_2\alpha_1-2A_2\beta_1\gamma_2-B_2\alpha_1\beta_1+B_2\alpha_1\gamma_2+\sqrt{\Delta_2}<0, \tag{5.30}$$

then $y<0$ for solution in the case (v). By analogous reasoning we have that the same conclusion for case (vi) holds. □

Remark 5.4 (see [1]). When

$$\beta_1A_2=\alpha_1B_2, \tag{5.31}$$

we see that

$$y_{n+1} = \frac{\beta_1\gamma_2}{B_2 x_{n+1} + \beta_1}, \quad n = 0, 1, \ldots, \tag{5.32}$$

and so system (5.1) can be decoupled as follows:

$$x_{n+1} = \frac{(\alpha_1 + \beta_1 x_n)(\beta_1 + B_2 x_n)}{\beta_1\gamma_2}, \quad y_{n+1} = \frac{\gamma_2 y_n^2}{y_n^2 + (A_2 - \beta_1)y_n + \beta_1\gamma_2}, \quad n = 0, 1, \ldots. \tag{5.33}$$

The solutions of first equation (depending of the choice of the initial condition x_0) are either bounded and converge to a finite limit or increase monotonically to infinity. Using this and (5.32) we find the behavior of solutions of second equation.

Our linearized stability analysis indicates that there are three cases with different asymptotic behavior, depending on the values of parameters β_1, B_1, α_2, and γ_2:

Case 1. $\gamma_2 - A_2 - \beta_1 > 2\sqrt{\alpha_1 B_2}$.

Case 2. $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$.

Case 3. $\gamma_2 \le A_2 + \beta_1$ or $0 < \gamma_2 - A_2 - \beta_1 < 2\sqrt{\alpha_1 B_2}$.

5.2.1. *Case* $\gamma_2 - A_2 - \beta_1 > 2\sqrt{\alpha_1 B_2}$

Theorem 5.5. *Consider system* (5.1) *and assume that* $\gamma_2 - A_2 - \beta_1 > 2\sqrt{\alpha_1 B_2}$ *and* $\beta_1 A_2 \ne \alpha_1 B_2$. *Then there exists a set* $\mathcal{C} \subset \mathcal{R}$ *which is invariant and a subset of the basin of attraction of* E_+. *The set* $\mathcal{C}$ *is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates* $\mathcal{R}$ *into two connected and invariant components, namely,*

$$\begin{aligned} \mathcal{W}_- &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{\text{se}} y\}, \\ \mathcal{W}_+ &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{\text{se}} x\}. \end{aligned} \tag{5.34}$$

which satisfy

$$\begin{aligned} \lim_{n\to\infty}(x_n, y_n) &= \left(\frac{\gamma_2 - A_2 - \beta_1 - \sqrt{D_4}}{2B_2}, \frac{\gamma_2 - A_2 + \beta_1 + \sqrt{D_4}}{2}\right) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_-, \\ \lim_{n\to\infty}(x_n, y_n) &= (\infty, 0) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_+. \end{aligned} \tag{5.35}$$

Proof. Clearly, system (5.1) is strongly competitive on $\mathcal{R} = [0,\infty) \times (0,\infty)$. In view of the injectivity of T, the invertibility of J_T and Theorem 5.3, we see that all conditions of Theorems 1.7, 1.9, and 1.10 and Corollary 1.8 are satisfied and the conclusion of the theorem follows. □

5.2.2. Case $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$

Theorem 5.6. *Consider system* (5.1) *and assume that $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$ and $\beta_1 A_2 \neq \alpha_1 B_2$. Then there exists a set $\mathcal{C} \subset \mathcal{R}$ which is invariant and a subset of the basin of attraction of E. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates $\mathcal{R}$ into two connected and invariant components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\},$$
$$\mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{5.36}$$

which satisfy

$$\lim_{n\to\infty}(x_n, y_n) = E = \left(\frac{\gamma_2 - A_2 - \beta_1}{2B_2}, \frac{\gamma_2 - A_2 + \beta_1}{2}\right) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_-,$$
$$\lim_{n\to\infty}(x_n, y_n) = (\infty, 0) \quad \text{for every } (x_0, y_0) \in \mathcal{W}_+. \tag{5.37}$$

Proof. In this case system (5.1) has a unique equilibrium point $E = ((\gamma_2 - A_2 - \beta_1)/2B_2, (\gamma_2 - A_2 + \beta_1)/2)$ which is nonhyperbolic. By $p = q + 1$, the corresponding characteristic equation is of the form

$$\lambda^2 - p\lambda + p - 1 = 0. \tag{5.38}$$

This implies

$$\lambda_1 = p - 1, \qquad \lambda_2 = 1, \tag{5.39}$$

and $|\lambda_1| < 1 \Leftrightarrow |p - 1| < 1 \Leftrightarrow 0 < p < 2$.

It is obvious that $p > 0$. In view of $\beta_1/\overline{y} = 1 - \alpha_1/\overline{xy}$, we have

$$p < 2 \Longleftrightarrow \frac{\beta_1}{\overline{y}} + 1 - \frac{\overline{y}}{\gamma_2} < 2 \Longleftrightarrow 1 - \frac{\alpha_1}{\overline{xy}} + 1 - \frac{\overline{y}}{\gamma_2} < 2, \tag{5.40}$$

which is satisfied. Thus, $|\lambda_1| < 1$.

The eigenvector corresponding to $\lambda_1 = p - 1$ is

$$\begin{pmatrix} 1 \\ \dfrac{B_2(\gamma_2 - A_2 + \beta_1)^2}{2\gamma_2(\gamma_2 - A_2 - \beta_1)} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{B_2\left(\sqrt{\alpha_1 B_2} + \beta_1\right)^2}{\gamma_2\sqrt{\alpha_1 B_2}} \end{pmatrix}. \tag{5.41}$$

It means that all conditions of Theorems 1.7 and 1.10 are satisfied with $\mathcal{R} = [0, \infty) \times (0, \infty)$. In view of the fact that $y_n \leq \gamma_2$ we obtain the conclusion of the theorem in the case $(x_0, y_0) \in \mathcal{W}_-$. The same conclusion is obtained by using Theorem 1.6.

Next, assume that $(x_0, y_0) \in \mathcal{W}_+$. By Theorem 1.10 the orbit of (x_0, y_0) eventually enters $Q_4(E)$. Assume (without loss of generality) that $(x_0, y_0) \in \operatorname{int} Q_4(E)$. An eigenvector associated with the nonhyperbolic eigenvalue $\lambda_2 = 1$ is $v = (1, -B_2)$. Choose a value of t small enough so that $E + tv \in Q_4(E)$ and $E + tv \preceq (x_0, y_0)$. Let us show that $E + tv \preceq T(E + tv)$. Indeed

$$\begin{aligned} T(E+tv) &= \left(\frac{2\alpha_1 B_2 + \beta_1(\gamma_2 - A_2 - \beta_1 + 2B_2 t)}{B_2(\gamma_2 - A_2 + \beta_1 - 2B_2 t)}, \frac{\gamma_2 - A_2 + \beta_1}{2} - B_2 t\right) \\ &\succeq \left(\frac{\gamma_2 - A_2 - \beta_1}{2B_2} + t, \frac{\gamma_2 - A_2 + \beta_1}{2} - B_2 t\right) = E + tv \end{aligned} \tag{5.42}$$

because

$$\frac{2\alpha_1 B_2 + \beta_1(\gamma_2 - A_2 - \beta_1 + 2B_2 t)}{B_2(\gamma_2 - A_2 + \beta_1 - 2B_2 t)} \geq \frac{\gamma_2 - A_2 - \beta_1}{2B_2} + t \tag{5.43}$$

reduces to

$$(\gamma_2 - A_2 - \beta_1)^2 \leq 4B_2 t^2 + 4\alpha_1 B_2 = 4B_2 t^2 + (\gamma_2 - A_2 - \beta_1)^2, \tag{5.44}$$

where the last equality follows from the condition $\gamma_2 - A_2 - \beta_1 = 2\sqrt{\alpha_1 B_2}$.

Since $E + tv \preceq T(E + tv)$, it follows that $\{T^n(E + tv)\}$ is a monotonically increasing sequence in $Q_4(E)$ which is bounded below by E. Since $\{T^n(E + tv)\}$ is coordinatewise monotone and it does not converge (if it did it would have to converge to E, which is impossible), we have that $T^n(E+tv)$ has a first coordinate which is monotone and unbounded. But $T^n(E + tv) \preceq (x_n, y_n) := T^n(x_0, y_0)$, which implies that $x_n \to \infty$. From (5.1) it follows that $y_n \to 0$. □

5.2.3. *Case* $\gamma_2 \leq A_2 + \beta_1$ *or* $0 < \gamma_2 - A_2 - \beta_1 < 2\sqrt{\alpha_1 B_2}$

In this case system (5.1) has no equilibrium points.

Theorem 5.7. *Consider system* (5.1) *and assume that* $\gamma_2 \leq A_2 + \beta_1$ *or* $0 < \gamma_2 - A_2 - \beta_1 < 2\sqrt{\alpha_1 B_2}$ *and* $\beta_1 A_2 \neq \alpha_1 B_2$. *Then every solution* $\{(x_n, y_n)\}$ *of system* (4.1) *satisfies*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = 0. \tag{5.45}$$

Proof. (1) Assume that $\beta_1 A_2 > \alpha_1 B_2$. Then every solution of system (5.1) is eventually componentwise monotonic. The sequence $\{y_n\}$ is bounded (by γ_2), which implies that it converges, that is, $\lim_{n\to\infty} y_n = Y$. For the sequence $\{x_n\}$ the following two cases are possible: (a) $\lim_{n\to\infty} x_n = X$, (b) $\lim_{n\to\infty} x_n = \infty$.

If $\lim_{n\to\infty} x_n = X$, then we obtain

$$X = \frac{\alpha_1 + \beta_1 X}{Y}, \qquad Y = \frac{\gamma_2 Y}{A_2 + B_2 X + Y}; \tag{5.46}$$

that is, (X, Y) is an equilibrium point of system (5.1), which is a contradiction.

If $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} y_n = Y = 0$.

(2) Assume that $\beta_1 A_2 < \alpha_1 B_2$. Consider now the following system satisfied by subsequences of the solution of system (5.1):

$$\begin{aligned} x_{2k+1} &= \frac{\alpha_1 + \beta_1 x_{2k}}{y_{2k}}, \qquad & x_{2k+2} &= \frac{\alpha_1 + \beta_1 x_{2k+1}}{y_{2k+1}}, \\ y_{2k+1} &= \frac{\gamma_2 y_{2k}}{A_2 + B_2 x_{2k} + y_{2k}}, \qquad & y_{2k+2} &= \frac{\gamma_2 y_{2k+1}}{A_2 + B_2 x_{2k+1} + y_{2k+1}}. \end{aligned} \tag{5.47}$$

We know that each of the four subsequences

$$\{x_{2k}\}, \ \{x_{2k+1}\}, \ \{y_{2k}\}, \ \{y_{2k+1}\} \tag{5.48}$$

of every solution $\{x_k, y_k\}$ of system (5.1) is eventually monotonic. The subsequences $\{y_{2k}\}$ and $\{y_{2k+1}\}$ are bounded by γ_2, which implies that they are convergent. Suppose that (a) $\lim_{k\to\infty} y_{2k} = y_E$ and (b) $\lim_{k\to\infty} y_{2k+1} = y_O$. For the other two subsequences the following four cases are possible: (1) $\lim_{k\to\infty} x_{2k} = x_E$, (2) $\lim_{k\to\infty} x_{2k} = \infty$, (3) $\lim_{k\to\infty} x_{2k+1} = x_O$, or (4) $\lim_{k\to\infty} x_{2k+1} = \infty$.

By similar reasoning as in the proof of Theorem 4.5, we obtain

$$\lim_{k\to\infty} y_{2k} = y_E = \lim_{k\to\infty} y_{2k+1} = y_O = 0. \tag{5.49}$$

□

The obtained results lead to the following characterization of the boundedness of solutions of system (5.1).

Corollary 5.8. *Consider system* (5.1) *and assume that* $\beta_1 A_2 \neq \alpha_1 B_2$. *If* $\gamma_2 - A_2 - \beta_1 \geq 2\sqrt{B_2 \alpha_1}$, *then all bounded solutions of system* (5.1) *converge to the unique equilibrium with the corresponding initial conditions which belong to the region above and on the graph of a continuous increasing function* $\mathcal{C}$ *in the plane of initial conditions. All solutions that start below* $\mathcal{C}$ *are asymptotic to* $(\infty, 0)$.

Consider system (5.1) *and assume that either* $\gamma_2 \leq A_2 + \beta_1$ *or* $0 < \gamma_2 - \beta_1 - A_2\gamma_2 < 2\sqrt{B_2\alpha_1}$. *Then every solution of* (5.1) *is asymptotic to* $(\infty, 0)$.

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