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Research Article

A Recurrence Formula for D Numbers $D_{2n}^{(2n-1)}$

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we establish a recurrence formula for D numbers $D_{2n}^{(2n-1)}$. A generating function for D numbers $D_{2n}^{(2n-1)}$ is also presented.

1. Introduction and Results

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k, for any integer k, may be defined by (see [1–4])

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.1}$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k, $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [2, 5]). By (1.1), we can get (see [4, page 145])

$$\frac{d}{dx}B_n^{(k)}(x) = nB_{n-1}^{(k)}(x), \tag{1.2}$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k}B_n^{(k)}(x) + (x-k)\frac{n}{k}B_{n-1}^{(k)}(x), \tag{1.3}$$

$$B_n^{(k+1)}(x+1) = \frac{nx}{k}B_{n-1}^{(k)}(x) - \frac{n-k}{k}B_n^{(k)}(x), \tag{1.4}$$

where $n \in \mathbb{N}$, with $\mathbb{N}$ being the set of positive integers.

The numbers $B_n^{(n)}$ are called the Nörlund numbers (see [2, 4, 6]). A generating function for the Nörlund numbers $B_n^{(n)}$ is (see [4, page 150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}. \tag{1.5}$$

The D numbers $D_{2n}^{(k)}$ may be defined by (see [4, 7, 8])

$$(t\csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi. \tag{1.6}$$

By (1.1), (1.6), and note that $\csc t = 2i/(e^{it} - e^{-it})$ (where $i^2 = -1$), we can get

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)}\left(\frac{k}{2}\right). \tag{1.7}$$

Taking $k = 1, 2$ in (1.7), and note that $B_{2n}^{(1)}(1/2) = (2^{1-2n} - 1)B_{2n}$, $B_{2n}^{(2)}(1) = (1-2n)B_{2n}$ (see [4, page 22, page 145]), we have

$$D_{2n}^{(1)} = \left(2 - 2^{2n}\right)B_{2n}, \qquad D_{2n}^{(2)} = 4^n(1-2n)B_{2n}. \tag{1.8}$$

The D numbers $D_{2n}^{(k)}$ satisfy the recurrence relation (see [7])

$$D_{2n}^{(k)} = \frac{(2n-k+2)(2n-k+1)}{(k-2)(k-1)} D_{2n}^{(k-2)} - \frac{2n(2n-1)(k-2)}{k-1} D_{2n-2}^{(k-2)}. \tag{1.9}$$

By (1.9), we may immediately deduce the following (see [4, page 147]):

$$D_{2n}^{(2n+1)} = \frac{(-1)^n(2n)!}{4^n}\binom{2n}{n}, \qquad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1}(n!)^2, \tag{1.10}$$

$$D_{2n}^{(2n+3)} = \frac{(-1)^n(2n)!}{2\cdot 4^{2n}}\binom{2n+2}{n+1}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2}\right). \tag{1.11}$$

The numbers $D_{2n}^{(2n)}$ are called the D-Nörlund numbers that satisfy the recurrence relation (see [7])

$$\sum_{j=0}^{n} \frac{(-1)^j}{4^j(2j+1)}\binom{2j}{j}\frac{D_{2n-2j}^{(2n-2j)}}{(2n-2j)!} = \frac{(-1)^n}{4^n}\binom{2n}{n}, \tag{1.12}$$

so we find $D_0^{(0)} = 1, D_2^{(2)} = -2/3, D_4^{(4)} = 88/15, D_6^{(6)} = -3056/21, D_8^{(8)} = 319616/45, D_{10}^{(10)} = -18940160/33, \ldots$.

A generating function for the D-Nörlund numbers $D_{2n}^{(2n)}$ is (see [7])

$$\frac{t}{\sqrt{1+t^2}\log\left(t+\sqrt{1+t^2}\right)}=\sum_{n=0}^{\infty}D_{2n}^{(2n)}\frac{t^{2n}}{(2n)!},\quad |t|<1. \tag{1.13}$$

These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [4, page 246])

$$\int_0^{\pi/2}\frac{\sin t}{t}dt=\sum_{n=0}^{\infty}\frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!},\qquad \int_0^{\pi/2}\frac{\sin t}{t}dt=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)^2},$$
$$\frac{2}{\pi}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \tag{1.14}$$

The main purpose of this paper is to prove a recurrence formula for D numbers $D_{2n}^{(2n-1)}$ and to obtain a generating function for D numbers $D_{2n}^{(2n-1)}$. That is, we will prove the following main conclusion.

Theorem 1.1. *Let $n\in\mathbb{N}$. Then*

$$\sum_{j=1}^{n}\binom{2n}{2j}(-1)^{j-1}4^{j-1}((j-1)!)^2D_{2n-2j}^{(2n-1-2j)}=\frac{(-1)^{n-1}2(2n)!}{4^n}\binom{2n-2}{n-1}, \tag{1.15}$$

so one finds $D_2^{(1)}=-1/3$, $D_4^{(3)}=17/5$, $D_6^{(5)}=-1835/21$, $D_8^{(7)}=195013/45$, $D_{10}^{(9)}=-3887409/11,\ldots$.

Theorem 1.2. *Let t be a complex number with $|t|<1$. Then*

$$\sum_{n=0}^{\infty}D_{2n}^{(2n-1)}\frac{t^{2n}}{(2n)!}=\frac{1}{\sqrt{1+t^2}}\left(\frac{t}{\log(t+\sqrt{1+t^2})}\right)^2. \tag{1.16}$$

2. Proof of the Theorems

Proof of Theorem 1.1. Note the identity (see [4, page 203])

$$B_{2n}^{(k)}\left(x+\frac{k}{2}\right)=\sum_{j=0}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}x^2\left(x^2-1^2\right)\left(x^2-2^2\right)\cdots\left(x^2-(j-1)^2\right), \tag{2.1}$$

we have

$$\frac{B_{2n}^{(k)}(x+k/2)-B_{2n}^{(k)}(k/2)}{x^2}=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}\left(x^2-1^2\right)\left(x^2-2^2\right)\cdots\left(x^2-(j-1)^2\right). \tag{2.2}$$

Therefore,

$$\lim_{x\to 0}\frac{B_{2n}^{(k)}(x+k/2)-B_{2n}^{(k)}(k/2)}{x^2}=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(-1)^{j-1}\left((j-1)!\right)^2. \tag{2.3}$$

By (2.3) and (1.2), we have

$$\lim_{x\to 0}\frac{2n(2n-1)B_{2n-2}^{(k)}(x+k/2)}{2}=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(-1)^{j-1}\left((j-1)!\right)^2. \tag{2.4}$$

That is,

$$n(2n-1)B_{2n-2}^{(k)}\left(\frac{k}{2}\right)=\sum_{j=1}^{n}\binom{2n}{2j}\frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}}(-1)^{j-1}\left((j-1)!\right)^2. \tag{2.5}$$

By (2.5) and (1.7), we have

$$D_{2n-2}^{(k)}=\frac{1}{n(2n-1)}\sum_{j=1}^{n}\binom{2n}{2j}(-1)^{j-1}4^{j-1}\left((j-1)!\right)^2D_{2n-2j}^{(k-2j)}. \tag{2.6}$$

Setting $k=2n-1$ in (2.6), and note (1.10), we immediately obtain Theorem 1.1. This completes the proof of Theorem 1.1. □

Remark 2.1. Setting $k=2n$ in (2.6), and note (1.10), we may immediately deduce the following recurrence formula for D-Nörlund numbers $D_{2n}^{(2n)}$:

$$\sum_{j=1}^{n}\binom{2n}{2j}(-1)^{j}4^{j}\left((j-1)!\right)^2D_{2n-2j}^{(2n-2j)}=(-1)^n n4^n\left((n-1)!\right)^2 \quad (n\in\mathbb{N}). \tag{2.7}$$

Proof of Theorem 1.2. Note the identity (see [9])

$$\sum_{n=0}^{\infty}(-1)^n4^n(n!)^2\frac{t^{2n}}{(2n)!}=\frac{1}{1+t^2}\left(1-\frac{t}{\sqrt{1+t^2}}\log\left(t+\sqrt{1+t^2}\right)\right), \tag{2.8}$$

where $|t|<1$. We have

$$\sum_{n=0}^{\infty}(-1)^n4^n(n!)^2\frac{t^{2n+2}}{(2n+2)!}=\frac{1}{2}\left(\log\left(t+\sqrt{1+t^2}\right)\right)^2, \tag{2.9}$$

That is,

$$\sum_{n=1}^{\infty}(-1)^{n-1}4^{n-1}((n-1)!)^2\frac{t^{2n}}{(2n)!}=\frac{1}{2}\left(\log\left(t+\sqrt{1+t^2}\right)\right)^2. \tag{2.10}$$

On the other hand,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2(2n)!}{4^n}\binom{2n-2}{n-1}\frac{t^{2n}}{(2n)!}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}t^{2n+2}=\frac{t^2}{2\sqrt{1+t^2}}. \tag{2.11}$$

Thus, by (2.10), (2.11), and Theorem 1.1, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}4^{n-1}((n-1)!)^2\frac{t^{2n}}{(2n)!}\sum_{n=1}^{\infty}D_{2n}^{(2n-1)}\frac{t^{2n}}{(2n)!}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2(2n)!}{4^n}\binom{2n-2}{n-1}\frac{t^{2n}}{(2n)!}. \tag{2.12}$$

That is,

$$\frac{1}{2}\left(\log\left(t+\sqrt{1+t^2}\right)\right)^2\sum_{n=1}^{\infty}D_{2n}^{(2n-1)}\frac{t^{2n}}{(2n)!}=\frac{t^2}{2\sqrt{1+t^2}}. \tag{2.13}$$

By (2.13), and note that

$$\lim_{t\to 0}\frac{t}{\log\left(t+\sqrt{1+t^2}\right)}=1, \qquad D_0^{(-1)}=1, \tag{2.14}$$

we immediately obtain Theorem 1.2. This completes the proof of Theorem 1.2. □

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