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Research Article

Existence of Solutions for a Nonlinear Algebraic System

Guang Zhang[1] and Liang Bai[2]

[1] *School of Science, Tianjin University of Commerce, Tianjin 300134, China*
[2] *School of Science, Qingdao Technological University, Fushun Road 11, Qingdao 266033, China*

Correspondence should be addressed to Guang Zhang, qd_gzhang@126.com

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As well known, the existence and nonexistence of solutions for nonlinear algebraic systems are very important since they can provide the necessary information on limiting behaviors of many dynamic systems, such as the discrete reaction-diffusion equations, the coupled map lattices, the compartmental systems, the strongly damped lattice systems, the complex dynamical networks, the discrete-time recurrent neural networks, and the discrete Turing models. In this paper, both the existence of nonzero solution pairs and the nonexistence of nontrivial or nonzero solutions for a nonlinear algebraic system will be considered by using the critical point theory and Lusternik-Schnirelmann category theory. The process of proofs on the obtained results is simple, the conditions of theorems are also easy to be verified, however, some of them improve the known ones even if the system is reduced to the precial cases, in particular, others of them are still new.

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1. Introduction

In this paper, the nonlinear algebraic system,

$$Ax = \lambda f(x), \tag{1.1}$$

will be considered, where $\lambda > 0$ is a parameter,

$$x = (x_1, x_2, \ldots, x_n)^T, \qquad f(x) = \left(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\right)^T \tag{1.2}$$

are column vectors with f_k is a continuous function defined on $\mathbf{R}$ and $f_k(-u) = -f_k(u)$ for $u \in \mathbf{R}$ and $k \in \{1, 2, \ldots, n\} = [1, n]$, and n is a positive integer. Also $A = (a_{ij})_{n\times n}$ is an $n \times n$ square matrix that there exists a positive $n \times n$ diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ such that DA is nonnegative definite. The letter T will denote transposition.

For a given $\lambda > 0$, a column vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$ is said to be a solution of (1.1) corresponding to it if substitution of λ and x into (1.1) renders it an identity. The vector x is said to be positive if $x_k > 0$ for $k \in [1, n]$, negative if $x_k < 0$ for $k \in [1, n]$, and nonzero if $x_k \neq 0$ for $k \in [1, n]$. Positive, negative, and (strongly) nonzero vector x are denoted by $x > 0$, $x < 0$ and $x \nparallel 0$ respectively. If there exists $k_0 \in [1, n]$ such that $x_{k_0} \neq 0$, it will be called nontrivial solution of (1.1). In this case, it is denoted by $x \neq 0$.

First note that $x = 0$ is always a trivial solution of (1.1). Also note that if x is a solution of (1.1), then $-x$ is a solution of (1.1) as well. Therefore, we always consider solution pairs, $\pm x$.

Nonlinear systems of the form (1.1) arise in many applications such as the discrete models of steady-state equations of reaction–diffusion equations (see [1–6]), the discrete analogue of the periodic boundary value problems (see [7–11]), the steady-state equations of coupled map lattices (see [12–25]), the discrete periodic boundary value problems (see [26–29]), the steady-state equations of compartmental system (see [30–34]), the steady-state models on complex dynamical network ([35–39]), the steady-state systems of discrete Turing instability models (see [5, 33, 40–68]). Thus, the existence and the nonexistence of solutions on (1.1) are very important.

In fact, the special cases of (1.1) have been extensively studied by a number of authors, see [26–29, 36–38] and the listed references therein. However, our results improve and extend the known ones even if (1.1) is reduced to their cases, in particular, some of them are new.

In this paper, the existence of solution pairs for the nonlinear algebraic system (1.1) will be considered by using the critical point theory and Lusternik-Schnirelmann category theory [69] or [70]. The nonexistence of nontrivial and nonzero solutions of (1.1) will also be established. The present paper is organized as follows. In Section 2, problems in various areas are transformed into system (1.1). In Section 3, we discuss turing instability. Then the nonexistence of solutions for (1.1) will be studied in Section 4, here, all results are new. Furthermore, in Section 5, the existence of solution pairs for (1.1) will be considered, some known results will be extended and improved, in particular, the method of proofs is different from previous ones. Some applications will be presented in Section 6.

2. Problems Expressed by (1.1)

A lot of problems in various areas can be expressed by (1.1). In this section, we will pick some typical examples.

2.1. Periodic Boundary Value Problems

As well known, steady-state equations of many important models in application, such as the nonlinear reaction—diffusion equations [6], the generalized reaction Duffing model [4], and the Fisher equation [1], can be expressed by the following equation:

$$-\left(r(t)x'(t)\right)' + q(t)x = f(t, x). \tag{2.1}$$

Thus, existence of solutions for second-order differential equation with periodic boundary value condition has been extensively studied by a number of authors (see [8–11]).

By using finite differences [7], discrete analogue of the periodic boundary value problem

$$
\begin{gathered}
-\left(r(t)x'(t)\right)' + q(t)x = f(t,x), \quad t \in (0,1), \\
x(0) = x(1), \qquad x'(0) = x'(1),
\end{gathered}
\tag{2.2}
$$

can be written by

$$
\begin{gathered}
-a_k x_{k-1} + b_k x_k - c_k x_{k+1} = \lambda f_k(x_k) \quad \text{for } k \in [1,n], \\
x_0 = x_n, \qquad x_{n+1} = x_1,
\end{gathered}
\tag{2.3}
$$

where $r \in C^1[0,1]$ with $r(t) \geq r_{\min} > 0$, $q(t) \in C[0,1]$ with $q(t) \geq 0$, $f(t,-x) = -f(t,x)$ for all $(t,x) \in [0,1] \times \mathbf{R}$, $x_k = kh$, $h = 1/n$ and

$$
\begin{gathered}
a_k = \frac{2r_{k-1/2}}{h}, \qquad c_k = \frac{2r_{k+1/2}}{h}, \\
b_k = \frac{2(r_{k-1/2} + r_{k+1/2})}{h} + 2hq_k, \\
f_k(x_k) = f(k, x_k), \\
\lambda = 2h.
\end{gathered}
\tag{2.4}
$$

In view of (2.3), we can obtain (1.1) with the $n \times n$ square matrix A having the form

$$
A = \begin{pmatrix}
b_1 & -c_1 & 0 & \cdots & -a_1 \\
-a_2 & b_2 & -c_2 & \cdots & 0 \\
 & \cdots & & \cdots & \\
0 & \cdots & -a_{n-1} & b_{n-1} & -c_{n-1} \\
-c_n & \cdots & 0 & -a_n & b_n
\end{pmatrix}.
\tag{2.5}
$$

The conditions $r(t) \geq r_{\min} > 0$ and $q(t) \geq 0$ lead to $a_k, b_k, c_k > 0$, $b_k \geq a_k + c_k$ for $k \in [1,n]$. At the same time, the matrix A has a zero eigenvalue and all other eigenvalues are positive, see [7].

Let

$$
d_2 = d > 0, \qquad d_k = \prod_{i=2}^{k-1} \frac{c_i}{a_{i+1}} \quad \text{for } k \in [3, n-1],
\tag{2.6}
$$

the conditions

$$
c_1 = da_2, \qquad a_1 = c_n, \qquad a_n = c_{n-1} \prod_{i=2}^{n-2} \frac{c_i}{a_{i+1}}
\tag{2.7}
$$

imply that the matrix DA is symmetric. As an simiple example, we consider

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -0.5 & 1 & -0.5 \\ -1 & -1 & 2 \end{pmatrix}, \tag{2.8}$$

then there exists the positive diagonal matrix $D = \mathrm{diag}(1,2,1)$ such that

$$DA = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{2.9}$$

which is nonnegative definite.

Pattern dynamics in coupled map lattices (CMLs) have been extensively studied (see [12–19, 22]). It has been found that CMLs exhibit a variety of space-time patterns such as kink-antikinks, traveling waves, space-time periodic structures, space-time intermittence and spatiotemporal chaos. It is believed that CMLs possess the potential to explain phenomena associated with turbulence and other spatiotemporal systems.

Consider the following coupled map lattice:

$$u_n^{t+1} = u_n^t + \alpha\left(u_{n-1}^t - 2u_n^t + u_{n+1}^t\right) + \beta f\left(u_n^t\right), \tag{2.10}$$

where $t \in \mathbf{N}$ denotes the time and $n \in \mathbf{Z}$ denotes the spatial coordinate, β/α is positive and treated as a parameter. This is a discrete analogue of the well-known Nagumo equation of the form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbf{R},\ t \in \mathbf{R}^+, \tag{2.11}$$

where D is a positive constant. The continuous Nagumo equation (2.11) is used as a model for the spread of genetic traits [2] and for the propagation of nerve pulses in a nerve axon, neglecting recovery [3, 5].

Solution $\{u_n^t\}$ of (2.10) is said to be stationary wave solution if $u_n^{t+1} = u_n^t$ for all $n \in \mathbf{Z}$ and $t \in \mathbf{N}$. In view of (2.10), we have

$$0 = \alpha\left(\varphi(n-1) - 2\varphi(n) + \varphi(n+1)\right) + \beta f\left(\varphi(n)\right). \tag{2.12}$$

Now, we consider the existence of ω-periodic solution of (2.12). Clearly, this is equal to the existence of solutions for (1.1), $\omega \times \omega$ nonnegative definite matrix,

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & 2 & -1 \\ -1 & \cdots & 0 & -1 & 2 \end{pmatrix}_{\omega\times\omega}. \tag{2.13}$$

Recently, Zhou et al. [29] consider the discrete time second order dynamical systems

$$X_{k+1} - 2X_k + X_{k-1} + g(k, X_k) = 0, \quad k \in Z, \tag{2.14}$$

where $g(g_1, g_2, \ldots, g_l)^T \in C(Z \times R^l, R^l)$ and $g(k+\omega, U) = g(k, U)$ for any $(k, U) \in Z \times R^l$. Our results are also valid for the problem (2.14). In this case, the corresponding results also improve and extend the main theorem in [29].

Cai et al. [27] considered existence and multiplicity of periodic solution for the fourth-order difference equation

$$\Delta^4 x_{k-2} - \lambda f(k, x_k) = 0, \quad k \in \mathbf{Z}, \tag{2.15}$$

by using linking theorem, where the function $f(k, u)$ is defined on $\mathbf{Z} \times \mathbf{R}$ with $f(k+\omega, u) = f(k, u)$ for a given positive integer ω and $f(k, -u) = -f(k, u)$ for all $(k, u) \in \mathbf{Z} \times \mathbf{R}$.

However, (2.15) is equal to (1.1), where

$$A = \begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 1 & -4 \\ -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 1 \\ 1 & -4 & 6 & -4 & & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \cdots & & & & \\ 0 & 0 & & & \cdots & & & & 0 \\ \cdots & & \cdots & & & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 & 1 \\ 1 & 0 & 0 & 0 & & 1 & -4 & 6 & -4 \\ -4 & 1 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}_{\omega\times\omega} \tag{2.16}$$

which is nonnegative definite.

Clearly, x_k and f of (2.15) can also be replaced by X_k and g of (2.14), respectively. In this case, our all results are new.

2.2. Compartmental System

Dynamic models of many processes in the biological and physical sciences give systems of ordinary differential equations called compartmental systems (see [30–34] and references therein). For example, Jacquez and Simon [32] considered the following system:

$$\dot{q}_i(t) = \sum_{j=1}^{n} f_{ij} q_j(t) + I_i, \quad i \in [1, n], \tag{2.17}$$

where q_i represents the mass of compartment i. $f_{ii} = -(f_{0i} + \sum_{j \neq i} f_{ji})$, f_{ij} is the transfer or rate coefficient from compartment j to compartment i, f_{0i} is the transfer coefficient from compartment i to environment, I_i represents the flows into the compartment i from outside the system, or inflow. The entries of the matrix $F = (f_{ij})_{n \times n}$ have three properties:

$$\begin{gathered} f_{ii} \leq 0 \quad \forall i \geq 0, \\ f_{ij} \geq 0 \quad \forall i \neq j, \\ \sum_{i=1}^{n} f_{ij} = \sum_{i \neq j} f_{ij} + f_{jj} \leq 0 \quad \forall j \geq 0. \end{gathered} \tag{2.18}$$

A system of the form of (2.17) for which $F = (f_{ij})_{n \times n}$ satisfies (2.18) is called a compartmental system.

Considering the compartmental system that the transfer coefficient from compartment j to compartment i is equal to that from compartment i to compartment j for all the compartments in the system, and the inflow I_i is determined by the mass of compartment i, say, $I_i = g_i(q_i)$. Then steady-state equation of this kind compartmental system is

$$-\sum_{j=1}^{n} f_{ij} q_j = g_i(q_i), \quad i \in [1, n]. \tag{2.19}$$

When g_i is an odd function, we can obtain (1.1) with the $n \times n$ square matrix $-F$ having the form

$$\begin{pmatrix} -f_{11} & -f_{12} & \cdots & -f_{1n} \\ -f_{21} & -f_{22} & \cdots & -f_{2n} \\ \cdots & & & \\ -f_{n1} & -f_{n2} & \cdots & -f_{nn} \end{pmatrix}. \tag{2.20}$$

By using (2.18), we can get that the matrix $-F$ is nonnegative definite. To our knowledge, few results on the system (2.19) are found in literature.

2.3. Strongly Damped Lattice System

Recently, Li and Zhou [21] considered the following second-order lattice dynamic system:

$$\ddot{u}_i + k(B\dot{u})_i + (Au)_i + h(\dot{u}_i) + f(u_i) = g_i, \tag{2.21}$$

where $i = (i_1, i_2, \ldots, i_n) \in \mathbf{Z}_m^n = \mathbf{Z}^n \cap \{i_1, i_2, \ldots, i_n \in [1, m]\}$, $k \geq 0$, $h, f \in C^1(\mathbf{R}, \mathbf{R})$, $g_i \in \mathbf{R}$ $(i \in \mathbf{Z}_m^n)$ are given, $u = (u_i)_{i \in \mathbf{Z}_m^n}$ is a vector with the components u_i and can be ordered as the following form of 1-dimensional vector in $\mathbf{R}^{m^n}$:

$$\begin{aligned} u &= \left(u_{(1,1,\ldots,1)}, u_{(2,1,\ldots,1)}, \ldots, u_{(m,1,\ldots,1)}, \ldots, u_{(1,m,\ldots,m)}, \ldots, u_{(m,m,\ldots,m)}\right)^T \\ &= (u_1, u_2, \ldots, u_\nu, \ldots, u_{m^n})^T \in \mathbf{R}^{m^n}, \end{aligned} \tag{2.22}$$

where $\nu = i_1 + m(i_2 - 1) + \cdots + m^{n-1}(i_n - 1)$, $i_1, i_2, \ldots, i_n \in [1, m]$, $\dot{u} = (\dot{u}_i)_{i \in \mathbf{Z}_m^n}$. A is a nonnegative definite matrix on $\mathbf{R}^{m^n}$ with eigenvalues $\lambda_s \geq 0$ $(0 \leq s \leq m^n - 1)$, and 0 is the simple and minimal eigenvalue with corresponding eigenvector $e = (1, \ldots, 1)^T \in \mathbf{R}^{m^n}$. Also $(A\dot{u})_i$, $(Au)_i$ denote the ith component of $A\dot{u}$, Au, respectively. An example of A is $A = -\Delta$, at this time, (2.21) can be regarded as the discrete analogue of the initial-boundary value problem of the following continuous strongly damped wave equation:

$$u_{tt} - k\Delta u_t - \Delta u + h(u_t) + f(u) = g(x), \tag{2.23}$$

which arises in wave phenomena in various areas in mathematical physics (see [20, 21, 23–25] and references therein).

Steady-state equation of (2.21) is

$$(Au)_k = F_k(u_k), \quad k \in [1, m^n], \tag{2.24}$$

where

$$F_k(u_k) = -f(u_k) - h(0) + g_k. \tag{2.25}$$

When F_k is an odd function, (2.24) can be expressed by (1.1). Thus, the existence of (2.24) is important, however, to our knowledge few results are seen in literature.

2.4. Complex Dynamical Network

Recently, complex dynamical network have been considered by Li et al. in [35]. Suppose that a complex network consists of N identical linearly and diffusively coupled nodes, with each node being an m-dimensional dynamical system. The state equations of this dynamical network are given by

$$x_i' = f(x_i) + \sum_{j=1, j \neq i}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \quad i \in [1, N], \tag{2.26}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{im})^T \in \mathbf{R}^m$ are the state variables of node i, the constant $c_{ij} > 0$ represents the coupling strength between node i and node j, $\Gamma = (\tau_{ij}) \in \mathbf{R}^{m\times m}$ is a matrix linking coupled variables, and if some pairs (i, j), $1 \le i$, $j \le m$, with $\tau_{ij} \neq 0$, then it means two coupled nodes are linked through their ith and jth state variables, respectively. The coupling matrix $A = (a_{ij}) \in \mathbf{R}^{N\times N}$ represents the coupling configuration of the network, which is assumed as a random network described by the E-R model or a scale-free network described by the B-A model. If there is a connection between node i and node j $(i \neq j)$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$ $(i \neq j)$. If the degree k_i of node i is defined to be the number of its outreaching connections, then

$$\sum_{j=1, j\neq i}^{N} a_{ij} = \sum_{j=1, j\neq i}^{N} a_{ji} = k_i, \quad i \in [1, N]. \tag{2.27}$$

Let the diagonal elements be $a_{ii} = -k_i$, $i \in [1, N]$.

In [35] the authors assumed there exists a generous stationary state for network (2.26) which is defined as

$$x_1 = x_2 = \cdots = x_n = \overline{x}, \quad f(\overline{x}) = 0. \tag{2.28}$$

They suppose that Γ is positive semidefinite and apply the pinning control strategy on a small fraction of the nodes to achieves the stabilization control of the goal (2.28). The network (2.26) can be rewritten by the system

$$X' = -DX + F(X), \tag{2.29}$$

where $X = (x_1, x_2, \ldots, x_{mN})^T$ is the state vector, and the $F(X)$ denotes the mN-dimensional functional value vector of X, and

$$D = \left(d_{ij}\right)_{mN\times mN} \tag{2.30}$$

is nonnegative definite in [35].

When $F(-X) = -F(X)$, steady-state equation of (2.29) can be expressed by (1.1). We think that the assumption (2.28) is not fact for a complex dynamical network. Thus, it is necessary to consider the existence of the other solutions. On the other hand, for any $i \in [1, N]$, we also obtain a nonlinear algebraic system from (2.26), which is the special case of (1.1).

2.5. Discrete Neural Networks

Recently, Zhou et al. [39] considered the following discrete-time recurrent neural networks, which is thought to describe the dynamical characteristics of transiently chaotic neural network:

$$v_i(t+1) = kv_i(t) + \sum_{j=1}^{n} w_{ij}u_j(t) + a_i - w_{ii}a_{0i}, \quad i \in [1, n], \tag{2.31}$$

where v_i is the internal state of neuron i, u_i is the output of neuron i, a_i is the input bias of neuron i, a_{0i} is the self-recurrent bias of neuron i, k represents the damping factor of nerve membrane, and w_{ij} is the connection weight from neuron j to neuron i which is written as

$$W = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix}. \tag{2.32}$$

In order to obtain the results on asymptotically stability, authors in [39], assume that the input-output function is $u_i(t) = s(v_i(t))$, inverse function $s^{-1}(y)$ of $s(x)$ exists, $0 < s'(x) \leq M$, there exists a matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ with $d_i > 0$ for $i \in [1, n]$ such that $(DW)^T = DW$, and $((1 + k)/M)D + DW$ $(k \geq 0)$ is positive definite, this implies that the matrix DW is nonnegative definite.

The steady-state equation of (2.31) is

$$\sum_{j=1}^{n} w_{ij} u_j = f_i(u_i), \quad i \in [1, n], \tag{2.33}$$

where

$$f_i(u_i) = (1 - k)s^{-1}(u_i) - a_i + w_{ii} a_{0i}. \tag{2.34}$$

When f_i is an odd function, (2.33) can be expressed by (1.1).

On the other hand, Wang and Cheng [36–38] considered the existence of steady-state solutions for the discrete neural networks

$$\begin{gathered} x_i^{t+1} - x_i^t = x_{i-1}^t + x_{i+1}^t - f\left(i, x_i^t\right), \\ x_i^{t+1} = x_{i-1}^t + x_{i+1}^t + g_i\left(x_i^t\right), \end{gathered} \tag{2.35}$$

with the periodic boundary value conditions:

$$x_0^t = x_\omega^t, \qquad x_1^t = x_{\omega+1}^t. \tag{2.36}$$

Our results are also valid for their problems and improve their theorems. In particular, for the more general system of the form

$$X_{k+1} + X_{k-1} - g(k, X_k) = 0, \tag{2.37}$$

our results are also valid, where X_k and g are similar with (2.14).

3. Turing Instability

In 1952, Turing [68] suggested that, under certain conditions, chemicals can react and diffusion in such a way as to produce steady-state heterogeneous spatial patterns of chemical of morphogenic concentration. His idea is a simple but profound one. In view of Turing's theory framework, many Turing's patterns have been obtained by the observations, the numerical simulations, the animal coat patterns, the wavelength of the electrochemical system, the vegetation in many semiarid regions, the skeletal pattern formation of chick limb, and so forth, see [33, 40–43, 45–47, 49, 53–57, 62–67].

However, the mathematical theory of Turing's patterns is not clear, see the recent papers [44, 50–52, 59–61]. In fact, when the diffusion term is added, the steady-state solutions are different with the primary model. Some new solutions will be increased. On the other hand, all numerical simulations will use the discrete analogue of the corresponding reaction diffusion equations or systems. Thus, it is necessary to consider the existence of solutions for the discrete steady-state equations. In general, such equation can be expressed by a partial difference equation of the form

$$\Delta_1^2 x_{i-1,j} + \Delta_2^2 x_{i,j-1} + \lambda f_{ij}\left(x_{ij}\right) = 0 \tag{3.1}$$

with the periodic boundary value conditions

$$\begin{aligned} &x_{0,j} = x_{n,j}, \quad x_{1,j} = x_{n+1,j}, \quad j \in [1, m], \\ &x_{i,0} = x_{i,m}, \quad x_{i,1} = x_{i,m+1}, \quad i \in [1, n]. \end{aligned} \tag{3.2}$$

However, we will give the other explanation in the later.

Because the existence of solutions of (3.1)-(3.2) is equal to (1.1), see Zhang et al. [71] or Zhang and Feng [72]. Clearly, x_k and f of (3.1) can also be replaced by X_k and g of (2.14), respectively.

4. Nonexistence

Usually, the existence of solutions is important. In fact, the nonexistence is also important because it can give some useful information for the existence of solutions. Thus, in this section, we will firstly give the nonexistence results of nontrivial solutions and nonzero solutions on the system (1.1). The obtained results are new.

When the matrix DA is nonnegative definite, we know that its eigenvalues are nonnegative and denote

$$0 = \gamma_1 = \gamma_2 = \cdots = \gamma_{m_0} < \gamma_{m_0+1} \leq \cdots \leq \gamma_n, \tag{4.1}$$

and the corresponding orthonormal eigenvectors are $v_1, v_2, \ldots, v_n$.

First of all, we let x be a nontrivial solution of (1.1). Multiplying (1.1) by $x^T D$ on the left we get

$$x^T B x - \lambda \sum_{k=1}^{n} d_k x_k f_k(x_k) = 0, \tag{4.2}$$

which implies that

$$\frac{x^T B x}{x^T x} = \lambda \frac{\sum_{k=1}^{n} d_k x_k f_k(x_k)}{x^T x}. \tag{4.3}$$

In view of the reference [73], we know that

$$\max_{x \neq 0} \frac{x^T B x}{x^T x} = \gamma_n, \qquad \min_{x \neq 0} \frac{x^T B x}{x^T x} = \gamma_1 = 0. \tag{4.4}$$

Thus, we have

$$0 = \gamma_1 = \gamma_2 = \cdots = \gamma_{m_0} \leq \lambda \frac{\sum_{k=1}^{n} d_k x_k f_k(x_k)}{x^T x} \leq \gamma_n, \tag{4.5}$$

which implies that the following nonexistence result is fact.

Theorem 4.1. *If there exists $\lambda > 0$ such that*

$$\gamma_n < \lambda \inf_{x \neq 0} \frac{\sum_{k=1}^{n} d_k x_k f_k(x_k)}{\|x\|^2} \tag{4.6}$$

or

$$\sup_{x \neq 0} \frac{\sum_{k=1}^{n} d_k x_k f_k(x_k)}{\|x\|^2} < 0, \tag{4.7}$$

then the system (1.1) *has no nontrivial solutions.*

Now, we assume that $f_1 = f_2 = \cdots = f_n = f$ and denote

$$\lim_{|u| \to 0} \frac{f(u)}{u} = l, \qquad \lim_{|u| \to \infty} \frac{f(u)}{u} = L. \tag{4.8}$$

Clearly, the condition $l = L = +\infty$ implies that the infimum

$$\inf_{u \neq 0} \frac{f(u)}{u} \tag{4.9}$$

exists and

$$\sum_{k=1}^{n} d_k x_k f_k(x_k) \geq \min_{k\in[1,n]}\{d_k\}\|x\|^2 \inf_{u\neq 0}\frac{f(u)}{u}. \tag{4.10}$$

The condition $l = L = -\infty$ implies that the supremum

$$\sup_{u\neq 0}\frac{f(u)}{u} \tag{4.11}$$

exists and

$$\sum_{k=1}^{n} d_k x_k f_k(x_k) \leq \max_{k\in[1,n]}\{d_k\}\|x\|^2 \sup_{u\neq 0}\frac{f(u)}{u}. \tag{4.12}$$

Thus, Theorem 4.1 implies that the following results hold.

Corollary 4.2. *Assume that $f_1 = f_2 = \cdots = f_n = f$. Then the condition $l = L = +\infty$ implies that the system* (1.1) *has no nontrivial solutions when there exists $\lambda > 0$ such that*

$$\gamma_n < \lambda \inf_{u\neq 0}\frac{f(u)}{u}\min_{i\in[1,n]}\{d_i\} \tag{4.13}$$

holds. The condition $l = L = -\infty$ implies that for any $\lambda > 0$, the system (1.1) *has no nontrivial solutions when*

$$\sup_{u\neq 0}\frac{f(u)}{u} < 0 \tag{4.14}$$

holds.

When l and L satisfy $0 < l,\ L < \infty$, we easily obtain the following results.

Corollary 4.3. *Assume that $f_1 = f_2 = \cdots = f_n = f$ and $0 < l,\ L < \infty$, then condition* (4.13) *or* (4.14) *implies that the system* (1.1) *has no nontrivial solutions.*

Similarly, when l and L satisfy the conditions: (H1) $l = +\infty$ and $0 < L < \infty$, (H2) $L = +\infty$ and $0 < l < \infty$, (H3) $l = -\infty$ and $0 < L < \infty$, (H4) $L = -\infty$ and $0 < l < \infty$, we have the following result.

Corollary 4.4. *Assume that $f_1 = f_2 = \cdots = f_n = f$. If condition (H1) or (H2) holds, then* (4.13) *implies that the system* (1.1) *has no nontrivial solutions. If the condition (H3) or (H4) holds, then* (4.14) *implies that the system* (1.1) *has no nontrivial solutions.*

Now, we again assume that the system (1.1) has a nonzero solution x, then we have

$$DAx = \lambda \operatorname{diag}\left(\frac{d_1 f_1(x_1)}{x_1}, \frac{d_2 f_2(x_2)}{x_2}, \dots, \frac{d_n f_n(x_n)}{x_n}\right)x,$$

$$v_i^T DAx = \lambda v_i^T \operatorname{diag}\left(\frac{d_1 f_1(x_1)}{x_1}, \frac{d_2 f_2(x_2)}{x_2}, \dots, \frac{d_n f_n(x_n)}{x_n}\right)x, \tag{4.15}$$

$$(DAv_i)^T x = \lambda \left[\operatorname{diag}\left(\frac{d_1 f_1(x_1)}{x_1}, \frac{d_2 f_2(x_2)}{x_2}, \dots, \frac{d_n f_n(x_n)}{x_n}\right)v_i\right]^T x,$$

$$\gamma_i {v_i}^T x = \left[\operatorname{diag}\left(\frac{\lambda d_1 f_1(x_1)}{x_1}, \frac{\lambda d_2 f_2(x_2)}{x_2}, \dots, \frac{\lambda d_n f_n(x_n)}{x_n}\right)v_i\right]^T x. \tag{4.16}$$

Thus, we have the following result.

Theorem 4.5. *Assume that there exist $k_0 \in [m_0, n-1]$ and $\lambda > 0$ such that $\gamma_{k_0} < \gamma_{k_0+1}$ and*

$$\gamma_{k_0} < \frac{\lambda d_k f_k(u)}{u} < \gamma_{k_0+1} \quad \text{for } u \in (0,\infty),\ k \in [1,n]. \tag{4.17}$$

Then the system (1.1) *has no nonzero solutions.*

In view of Theorem 4.5, we can also obtain some corollaries, here, we only give a clear result.

Corollary 4.6. *The conditions $uf_k(u) > 0$ or $uf_k(u) < 0$ for $k \in [1,n]$ and $u \neq 0$ imply that the system* (1.1) *has no positive-negative solutions.*

Corollary 4.6 can be immediately obtained by (4.16).

5. Existence

In view of Section 4, we find that the existence of solutions may become fact when the function f_k crosses the eigenvectors spaces. This motivates us to use Lusternik-Schnirelmann category theory and leads to new methods compared with previous ones, see [26–29, 36–38].

For a given symmetric matrix B, the index of the corresponding quadratic form on $\mathbf{R}^n$, $q(x) = x^T Bx$, is the largest dimension of a subspace $S \subset \mathbf{R}^n$ such that $q(x) < 0$ for all $x \in S$, $x \neq 0$. The following result is specialized to our cases, see the references [69] or [70].

Lemma 5.1. *If $H(x)$ with $H(0) = 0$ is a C^1 even function on $\mathbf{R}^n$ of the form $H(x) = q(x) + v(x)$, where $q(x)$ is a quadratic form of index m, and such that $H(x) \geq 0$ for large $\|x\|$ (where $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$) and that $v(x) = o(\|x\|^2)$ as $\|x\| \to 0$, then $H(x)$ has at least m pairs, $\pm x$, nonzero critical points.*

For using Lemma 5.1, we reformulate our problem as a critical point problem. For any $k \in [1,n]$ and $u \in \mathbf{R}$, we let

$$F_k(u) = \int_0^u f_k(s)ds. \tag{5.1}$$

At this time, we can define the functions $H : \mathbf{R}^n \to \mathbf{R}$ by

$$H(x) = -\frac{1}{2}x^T Bx + \lambda \sum_{k=1}^{n} d_k F_k(x_k), \tag{5.2}$$

where $B = DA$. Since

$$\frac{\partial H(x)}{\partial x_k} = -(Bx)_k + \lambda d_k f_k(x_k) \quad \text{for } k \in [1, n], \tag{5.3}$$

we see that a column vector $w = (w_1, w_2, \ldots, w_n)^T$ is a critical point of the functional H corresponding to λ if and only if w is a solution of (1.1) corresponding to λ.

Let

$$f_{i0} = \lim_{|u| \to 0} \frac{f_i(u)}{u}, \qquad \xi_i \in C(\mathbf{R}, \mathbf{R}) \quad \text{for } i \in [1, n], \tag{5.4}$$

such that

$$f_i(u) = f_{i0}u + \xi_i(u) \quad \text{for } i \in [1, n], \tag{5.5}$$

where

$$\lim_{|u| \to 0} \frac{\xi_i(u)}{u} = 0 \quad \text{for } i \in [1, n]. \tag{5.6}$$

In this case, we have

$$\begin{aligned} H(x) &= -\frac{1}{2}x^T Bx + \lambda \sum_{k=1}^{n} d_k \int_0^{x_k} f_k(s)ds \\ &= \frac{1}{2}x^T(\lambda F_0 - B)x + \lambda \sum_{k=1}^{n} d_k \int_0^{x_k} \xi_k(s)ds, \\ -H(x) &= \frac{1}{2}x^T Bx - \lambda \sum_{k=1}^{n} d_k \int_0^{x_k} f_k(s)d \\ &= \frac{1}{2}x^T(B - \lambda F_0)x - \lambda \sum_{k=1}^{n} d_k \int_0^{x_k} \xi_k(s)ds, \end{aligned} \tag{5.7}$$

where $F_0 = \operatorname{diag}(d_1 f_{10}, d_2 f_{20}, \ldots, d_n f_{n0})$.

By using the condition (5.6), we easily get that

$$\pm\lambda \sum_{k=1}^{n} d_k \int_0^{x_k} \xi_k(s)ds = o\left(\|x\|^2\right) \quad \text{as } \|x\| \longrightarrow 0. \tag{5.8}$$

For the functional $-H(x)$, we ask that $-H(x) \geq 0$ for large $\|x\|$, which implies that $F_k(x_k) \leq 0$ for $k \in [1,n]$ and large $|x_k|$. In this case, if there exists $m \in [m_0, n-1]$ and $\lambda > 0$ such that

$$\gamma_m < \lambda d_k f_{k0} < \gamma_{m+1}, \tag{5.9}$$

then the matrix $B - \lambda F_0$ has exactly m negative eigenvalues. Lemma 5.1 implies that the following result holds.

Theorem 5.2. *If there exist $m \in [m_0, n-1]$ and $\lambda > 0$ such that $\gamma_m < \gamma_{m+1}$ and that*

$$\gamma_m < \lambda d_k f_{k0} < \gamma_{m+1} \quad \text{for } k \in [1,n], \tag{5.10}$$

further suppose that there is $R_\lambda > 0$ such that $F_k(x_k) \leq 0$ for $|x_k| > R_\lambda$ and $k \in [1,n]$. Then the system (1.1) *has at least m nonzero solution pairs.*

By using Theorem 5.2, we clearly obtain the following results.

Corollary 5.3. *For any $k \in [1,n]$, if there exists $\lambda > 0$ such that $\lambda d_k f_{k0} > \gamma_n$ and there is $R_\lambda > 0$ such that $F_k(x_k) \leq 0$ for $|x_k| > R_\lambda$, then the system* (1.1) *has at least n nonzero solution pairs.*

Corollary 5.4. *If there exists $\lambda > 0$ such that*

$$0 < \lambda d_k f_{k0} < \gamma_{m_0+1} \quad \text{for } k \in [1,n], \tag{5.11}$$

further suppose that there is $R_\lambda > 0$ such that $F_k(x_k) \leq 0$ for $|x_k| > R_\lambda$ and $k \in [1,n]$. Then the system (1.1) *has at least m_0 nonzero solution pairs, particularly, the system* (1.1) *has at least one positive-negative solution pair.*

Corollary 5.5. *For any $k \in [1,n]$ and $\lambda > 0$, the conditions $f_{k0} = +\infty$ and $f_{k\infty} = -\infty$ imply that the system* (1.1) *has at least n nonzero solution pairs.*

Now, we consider the functional $H(x)$. Similarly, we have the following result by using Lemma 5.1.

Theorem 5.6. *If there exist $m \in [m_0+1, n]$ and $\lambda > 0$ such that $\gamma_{m-1} < \gamma_m$ and that*

$$\gamma_{m-1} < \lambda d_k f_{k0} < \gamma_m \quad \text{for } k \in [1,n], \tag{5.12}$$

further suppose that there is $R_\lambda > 0$ such that

$$\lambda d_k F_k(x_k) \geq \frac{1}{2}\gamma_k x_k^2 \quad \text{for } |x_k| > R_\lambda,\ k \in [1,n]. \tag{5.13}$$

Then the system (1.1) *has at least $n-m+1$ nonzero solution pairs.*

Corollary 5.7. *Assume that the condition $f_{k0} < 0$ holds for any $k \in [1,n]$, and there exist $\lambda > 0$ and $R_\lambda > 0$ such that the condition* (5.13) *is valid, then the system* (1.1) *has at least n nonzero solution pairs.*

Corollary 5.8. *For any* $k \in [1, n]$ *and* $\lambda > 0$, *the conditions* $f_{k0} = -\infty$ *and* $f_{k\infty} = +\infty$ *imply that the system* (1.1) *has at least* n *nonzero solution pairs.*

Consider the algebraic equations

$$\begin{aligned} x - y &= \lambda x^3, \\ -x + y &= \lambda y^3, \end{aligned} \tag{5.14}$$

which is the special case of (1.1). Obviously, for any $\lambda > 0$ all conditions of Theorem 5.6 hold. Thus, (5.14) has at least one nonzero solution pair. In fact, it has the exact nonzero solution pair $\pm(\sqrt{2/\lambda}, -\sqrt{2/\lambda})$. Thus, the conditions of Theorem 5.6 are sharp for (5.14).

6. Applications

Clearly, the theorems and corollaries established earlier are useful to solve the problems listed in Section 2. Some simple illustrative examples and remarks will be listed in this section.

6.1. Periodic Solutions

Guo and Yu [28] considered the existence of pm-periodic solution for

$$\Delta^2 x_{k-1} + f(k, x_k) = 0 \quad \text{for } k \in \mathbf{Z}. \tag{6.1}$$

The main result they derived is described as follows.

Assume $f(t, z)$ satisfies the following condition:

(i) $f \in C(\mathbf{Z} \times \mathbf{R}, \mathbf{R})$, and there exists positive integer m such that for any $(t, z) \in \mathbf{Z} \times \mathbf{R}$, $f(t + m, z) = f(t, z)$;

(ii) for any $z \in \mathbf{R}$, $\int_0^z f(t, s)ds \geq 0$, and $f(t, z) = o(z), (z \to 0)$;

(iii) there exists $R > 0$, $\beta > 2$, such that for any $|z| \geq R$,

$$z f(t, z) \geq \beta \int_0^z f(t, s)ds > 0, \tag{6.2}$$

then for any positive integer p, (6.1) has at least three pm-periodic solutions.

Now, we consider the existence of pT-periodic solution of the following nonlinear second-order difference equation:

$$\Delta^2 x_{k-1} + \lambda f_k(x_k) = 0, \quad k \in \mathbf{Z}, \tag{6.3}$$

where p is a given positive integer, where $\lambda > 0$ is a parameter, $f_k \in C(\mathbf{R}, \mathbf{R})$ and $f_k(-u) = -f_k(u)$ for $u \in \mathbf{R}$, and there exists a positive integer T such that for any $k \in \mathbf{Z}$, $u \in \mathbf{R}$, $f_{k+T}(u) = f_k(u)$. A solution of (6.3) is called to be nonzero if its every term is not zero.

The above problem is equal to the system (1.1) where the matrix is defined by (2.13) with $\omega = pT$ and has the eigenvalues

$$\delta_k = 4\sin^2\frac{(k-1)\pi}{pT}, \quad k \in [1, pT]. \tag{6.4}$$

Note that

$$\sin^2\frac{k\pi}{pT} = \sin^2\frac{(pT-k)\pi}{pT}, \tag{6.5}$$

then we have

$$0 = \delta_1 < \delta_2 = \delta_{pT} < \delta_3 = \delta_{pT-1} < \cdots < \delta_{(pT+2)/2} \tag{6.6}$$

when pT is even and

$$0 = \delta_1 < \delta_2 = \delta_{pT} < \delta_3 = \delta_{pT-1} < \cdots < \delta_{(pT+1)/2} = \delta_{(pT+3)/2} \tag{6.7}$$

when pT is odd.

Again let

$$\gamma_1 = \delta_1 < \gamma_2 = \gamma_3 = \delta_2, \ldots. \tag{6.8}$$

By using Theorems 5.2 and 5.6, we can obtain the following facts.

Theorem 6.1. *For any $k \in [1, pT]$, if there exist*

$$m \in \begin{cases} \left[1, \dfrac{pT}{2}\right], & \text{when } pT \text{ is even}, \\ \left[1, \dfrac{(pT-1)}{2}\right], & \text{when } pT \text{ is odd}, \end{cases} \tag{6.9}$$

and $\lambda > 0$ such that

$$4\sin^2\frac{(m-1)\pi}{pT} < \lambda f_{k0} < 4\sin^2\frac{m\pi}{pT} \quad \text{for } k \in [1, pT], \tag{6.10}$$

further suppose that there is $R_\lambda > 0$ such that $F_k(x_k) \leq 0$ for $|x_k| > R_\lambda$ and $k \in [1, pT]$. Then the problem (6.3) *has at least $4m-2$ nonzero pT-periodic solutions.*

Theorem 6.2. *When pT is a even positive integer and there exist $m \in [2, pT/2+1]$ and $\lambda > 0$ such that* (6.10) *holds, further suppose that there is $R_\lambda > 0$ such that*

$$\lambda F_k(x_k) \geq 2\sin^2\frac{(k-1)\pi}{pT}x_k^2 \quad \text{for } |x_k| > R_\lambda,\ k \in [1, pT], \tag{6.11}$$

then the system (6.3) *has at least* $2(pT-2m)$ *nonzero pT-periodic solutions; If pT is an odd positive integer and there exist $m \in [2, (pT+1)/2]$ and $\lambda > 0$ such that* (6.10) *and* (6.11) *hold, then the system* (6.3) *has at least* $2(pT-2m)+2$ *nonzero pT-periodic solutions.*

In view of Theorem 6.1, we can immediately obtain the following result.

Corollary 6.3. *For any $k \in [1, pT]$,*

$$f_{k0} = 0, \qquad f_{k\infty} = \infty, \tag{6.12}$$

or

$$f_{k0} = \infty, \qquad f_{k\infty} = 0 \tag{6.13}$$

hold, then the problem (6.3) *has at least* $2(pT-1)$ *nonzero pT-periodic solutions.*

Remark 6.4. Corollary 6.3 improves the main result in [28].

When $f_k(u) = u^3$ for any $k \in \mathbf{Z}$, we have

$$\lim_{|u|\to 0}\frac{u^3}{u} = 0, \qquad \lim_{|u|\to\infty}\frac{u^3}{u} = \infty. \tag{6.14}$$

Corollary 6.3 implies that for any positive integer p and positive number λ, the equation

$$\Delta^2 x_{k-1} + \lambda x_k^3 = 0 \tag{6.15}$$

has at least $2(p-1)$ nonzero p-periodic solutions. For example, let $p = 3$ and $\lambda = 1$, we can consider the existence of solutions for nonlinear system:

$$\begin{aligned} 2x_1 - x_2 - x_3 &= x_1^3, \\ -x_1 + 2x_2 - x_3 &= x_2^3, \\ -x_1 - x_2 + 2x_3 &= x_3^3, \end{aligned} \tag{6.16}$$

which has at least 4 nonzero solution pairs in view of Corollary 6.3. However, Theorem in [28] can only obtain two nonzero solutions because they conclude a zero solution. In fact, we can solve its the nonzero numerical solutions:

$$\begin{gathered}
x_3 = -1.5033, \qquad x_1 = 1.894, \qquad x_2 = -1.5033, \\
x_3 = 1.5033, \qquad x_1 = -1.894, \qquad x_2 = 1.5033, \\
x_3 = -1.894, \qquad x_1 = 1.5033, \qquad x_2 = 1.5033, \\
x_3 = 1.894, \qquad x_1 = -1.5033, \qquad x_2 = -1.5033, \\
x_3 = -1.5033, \qquad x_1 = -1.5033, \qquad x_2 = 1.894\ , \\
x_3 = 1.5033, \qquad x_1 = 1.5033, \qquad x_2 = -1.894.
\end{gathered} \tag{6.17}$$

However, we find that the number of nonzero solutions of (6.16) is more than 4. In fact, (6.16) has also nontrivial solutions

$$\begin{gathered}
x_3 = -1.7321, \qquad x_1 = 1.7321, \qquad x_2 = 0.0, \\
x_3 = -1.7321, \qquad x_1 = 0.0, \qquad x_2 = 1.7321, \\
x_3 = 1.7321, \qquad x_1 = -1.7321, \qquad x_2 = 0.0, \\
x_3 = 1.7321, \qquad x_1 = 0.0, \qquad x_2 = -1.7321, \\
x_3 = 0.0, \qquad x_1 = -1.7321, \qquad x_2 = 1.7321, \\
x_3 = 0.0, \qquad x_1 = 1.7321, \qquad x_2 = -1.7321.
\end{gathered} \tag{6.18}$$

Thus, we have the following open problem.

Open Problem 1

Obtain better existence results for the system (1.1) when $n \geq 3$.

When $f_k(u) = u(1-u^2)$ for any $k \in \mathbf{Z}$, we have

$$\lim_{|u|\to 0} \frac{f(u)}{u} = 1, \qquad \lim_{|u|\to \infty} \frac{f(u)}{u} = -\infty. \tag{6.19}$$

For any

$$\lambda > \begin{cases} 4, & \text{when } n \text{ is even}, \\ 4\cos^2 \dfrac{\pi}{2n}, & \text{when } n \text{ is odd}, \end{cases} \tag{6.20}$$

Theorem 6.1 implies that the periodic boundary value problem of the form

$$\Delta^2 x_{k-1} + \lambda x_k \left(1 - x_k^2\right) = 0, \quad k \in [1, n]$$
$$x_0 = x_n, \qquad x_1 = x_{n+1} \tag{6.21}$$

has at least $2n$ nonzero solutions.

On the other hand, Zhou et al. [29] consider the discrete time second-order dynamical systems:

$$X_{k+1} - 2X_k + X_{k-1} + g(k, X_k) = 0, \quad k \in Z, \tag{6.22}$$

where $g(g_1, g_2, \ldots, g_l)^T \in C(Z \times R^l, R^l)$ and $g(k + \omega, U) = g(k, U)$ for any $(k, U) \in Z \times R^l$. Our results are also valid for the problem (6.22). In this case, the corresponding results also improve the main theorem in [29]. In fact, we let I be an $\omega \times \omega$ unit matrix, then the problem (6.22) is equal to the system (1.1) where

$$A = \begin{pmatrix} 2I & -I & 0 & \cdots & -I \\ -I & 2I & -I & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -I & 2I & -I \\ -I & \cdots & 0 & -I & 2I \end{pmatrix}, \tag{6.23}$$

which has the eigenvalues

$$\delta_{k_i} = 4 \sin^2 \frac{(k_i - 1)\pi}{\omega}, \quad k_i = i \in [1, \omega]. \tag{6.24}$$

6.2. Steady-State Solutions on Discrete Neural Networks

In [36], Wang and Cheng considered the existence of steady-state solutions for the discrete neural network

$$x_i^{t+1} - x_i^t = x_{i-1}^t + x_{i+1}^t - f\left(i, x_i^t\right) \tag{6.25}$$

with the periodic boundary value conditions

$$x_0^t = x_\omega^t, \qquad x_1^t = x_{\omega+1}^t. \tag{6.26}$$

In fact, the steady-state equation can be written by

$$x_{i-1} + x_{i+1} = f(i, x_i), \quad i \in [1, \omega]$$
$$x_0 = x_\omega, \qquad x_1 = x_{\omega+1} \tag{6.27}$$

which can be rewritten by the system (1.1), where $\lambda = 1$, the coefficient matrix is defined by(2.13)

$$F_k(u) = u^2 - \int_0^u f(k,s)ds,$$
$$f_{k0} = \lim_{u\to 0}\frac{f(k,u)}{u}, \tag{6.28}$$
$$F(x) = \left(2x_1 - f(1,x_1),\ldots,2x_\omega - f(\omega,x_\omega)\right)^T.$$

In this case, in Theorems 6.1 and 6.2 pT and f_{k0} are, respectively, replaced by ω and $2 - f_{k0}$, then, they are valid for the problem (6.27). At this time, Theorem 6.1 improves the corresponding result in [36], but Theorem 6.2 is new.

On the other hand, for the more general system of the form

$$X_{k+1} + X_{k-1} - g(k, X_k) = 0, \tag{6.29}$$

our results are also valid, where X_k and g are similar with (2.14) and the method is similar with Section 6.2. Thus, the main results in [36, 38] are also extended and improved.

We also find that Wang and Cheng [37] considered the existence of the steady-state solutions for the discrete neural network:

$$x_i^{t+1} = x_{i-1}^t + x_{i+1}^t + f_i\left(x_i^t\right) \tag{6.30}$$

with the periodic boundary value condition (6.26). Similarly, the problem (6.30)–(6.26) be rewritten by the system (1.1), where $\lambda = 1$, A is defined by (2.13), and

$$F_k(u) = \frac{1}{2}u^2 + \int_0^u f_k(s)ds. \tag{6.31}$$

By using similar method, we can obtain the improved and extended results. Clearly, we can also consider the existence of steady-state solutions for the general system of the form

$$X_i^{t+1} = X_{i-1}^t + X_{i+1}^t + f_i\left(X_i^t\right), \tag{6.32}$$

where X_i^t is a k-vector for each $i \in Z$.

6.3. Periodic Solutions for Fourth-Order Difference Equation

When $\omega = 2$, the problem (2.15) exists 2-periodic solutions if and only if the nonlinear system

$$8\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} f(1,x_1) \\ f(2,x_2) \end{pmatrix} \tag{6.33}$$

has nontrivial solutions. The matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{6.34}$$

has the eigenvalues $\gamma_1 = 0$ and $\gamma_2 = 2$. When $\omega = 3$, the problem (2.15) exists 3-periodic solutions if and only if the nonlinear system

$$3\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda\begin{pmatrix} f(1, x_1) \\ f(2, x_2) \\ f(3, x_3) \end{pmatrix} \tag{6.35}$$

has nontrivial solutions. The matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{6.36}$$

has the eigenvalues $\gamma_1 = 0$, $\gamma_{2,3} = 3$.

When $\omega = 4$, the problem (2.15) exists 4-periodic solutions if and only if the nonlinear system

$$2\begin{pmatrix} 3 & -2 & 1 & -2 \\ -2 & 3 & -2 & 1 \\ 1 & -2 & 3 & -2 \\ -2 & 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda\begin{pmatrix} f(1, x_1) \\ f(2, x_2) \\ f(3, x_3) \\ f(4, x_4) \end{pmatrix} \tag{6.37}$$

has nontrivial solutions. The matrix

$$\begin{pmatrix} 3 & -2 & 1 & -2 \\ -2 & 3 & -2 & 1 \\ 1 & -2 & 3 & -2 \\ -2 & 1 & -2 & 3 \end{pmatrix} \tag{6.38}$$

has the eigenvalues $\gamma_1 = 0$, $\gamma_{2,3} = 2$, and $\gamma_4 = 8$.

When $\omega \geq 5$, (2.15) exists ω-periodic solutions if and only if the nonlinear system (1.1) has nontrivial solutions, where the matrix is defined in Subsection 2.1.

We can obtain eigenvalues of A from the fourth-order linear difference equation

$$x_{k-2} - 4x_{k-1} + 6x_k - 4x_{k+1} + x_{k+2} = \lambda x_k \quad \text{for } k \in [1, \omega] \tag{6.39}$$

with the periodic boundary value conditions:

$$x_{-1} = x_{\omega-1}, \qquad x_0 = x_\omega, \qquad x_{\omega+1} = x_1, \qquad x_{\omega+2} = x_2. \tag{6.40}$$

Let

$$x_k = t^k. \tag{6.41}$$

In view of (6.39), we have

$$t^{k-2} - 4t^{k-1} + 6t^k - 4t^{k+1} + t^{k+2} = \lambda t^k, \tag{6.42}$$

which implies that

$$\lambda = \frac{(1-t)^4}{t^2}. \tag{6.43}$$

From (6.40), we see that

$$t^\omega = 1, \tag{6.44}$$

thus, we have

$$\begin{gathered} t_p = \exp\left(\frac{2(p-1)\pi}{\omega} i\right), \quad p \in [1,\omega], \\ 1 - t_p = 2\sin\left(\frac{(p-1)\pi}{\omega}\right)\exp\left(\left(\frac{\pi}{2} + \frac{(p-1)\pi}{\omega} + j\pi\right)i\right) \end{gathered} \tag{6.45}$$

for $p \in [1,\omega]$ and $j \in Z$. In view of (6.43), we have obtained

$$\lambda_p = \frac{(1-t_p)^4}{(t_p)^2} = 2^4 \sin^4\left(\frac{(p-1)\pi}{\omega}\right) > 0, \quad p \in [1,\omega]. \tag{6.46}$$

Note that

$$\sin^4 \frac{p\pi}{\omega} = \sin^4 \frac{(\omega - p)\pi}{\omega}, \tag{6.47}$$

then

$$\begin{gathered} 0 = \delta_1 < \delta_2 = \delta_\omega < \delta_3 = \delta_{\omega-1} < \cdots < \delta_{\omega/2+1} \quad (\omega \text{ is even}) \\ 0 = \delta_1 < \delta_2 = \delta_\omega < \delta_3 = \delta_{\omega-1} < \cdots < \delta_{[\omega/2]+1} = \delta_{[\omega/2]+2} \quad (\omega \text{ is odd}). \end{gathered} \tag{6.48}$$

By using Theorems 5.2 and 5.6, we can clearly obtain similar results as Theorems 6.1 and 6.2 and improve and extend the results in [27]. They will be omitted.

6.4. On Partial Difference Equation

In Turing pattern analysis, the positive steady-state solutions are usually needed. The authors in [41] think that persistent puzzle in the field of biological electron transfer is the conserved iron-sulfur cluster motif in both high potential iron-sulfur protein (HiPIP) and ferredoxin (Fd) active sites. However, the voltage in cell can be negative and there exists the negative threshold, see [45]. Thus, the negative steady-state should be also considered for Turing pattern analysis. Our results will likely find important implications in other real evolutionary processes.

By the discussion in Section 2, we have known that the models of many applied problems can be expressed by the partial difference equation of the form

$$\Delta_1^2 x_{i-1,j} + \Delta_2^2 x_{i,j-1} + \lambda f_{ij}\left(x_{ij}\right) = 0 \tag{6.49}$$

with periodic boundary value conditions

$$\begin{aligned} &x_{0,j} = x_{n,j}, \quad x_{1,j} = x_{n+1,j}, \quad j \in [1,m], \\ &x_{i,0} = x_{i,m}, \quad x_{i,1} = x_{i,m+1}, \quad i \in [1,n]. \end{aligned} \tag{6.50}$$

In fact, we can also give the other explanation.

Indeed, let us consider $n \times m$ neuron units placed on a torus. Let x_{ij}^t denote the state value of the ijth neuron unit during the time periodic $t \in \{0,1,2,\dots\}$. Assume that each neuron unit is random activated by its four neighbors so that the change of state values between two consecutive time periods is given by

$$x_{ij}^{t+1} - x_{ij}^t = \alpha\left(\Delta_1^2 x_{i-1,j}^t + \Delta_2^2 x_{i,j-1}^t\right) + f_{ij}\left(x_{ij}^t\right) \tag{6.51}$$

with the periodic boundary value conditions

$$\begin{aligned} &x_{0,j}^t = x_{n,j}^t, \quad x_{1,j}^t = x_{n+1,j}^t, \quad j \in [1,m], \\ &x_{i,0}^t = x_{i,m}^t, \quad x_{i,1}^t = x_{i,m+1}^t, \quad i \in [1,n], \end{aligned} \tag{6.52}$$

where α is the connection weight, f_{ij} stands for the bias mechanism inherent in the ijth neuron unit.

In order to utilize the neural network modeled by the aforementioned evolutionary system, it is of interest to predict the existence of steady-state solution $\{x_{ij}^t, (i,j) \in [1,n] \times [1,m]\}_{t=0}^{\infty}$ such that $x_{ij}^t = x_{ij}$ for $(i,j) \in [1,n] \times [1,m]$ and $t \geq 0$. This then leads us to finding solutions of the steady system of (6.49)-(6.50) or more generally (1.1).

We can similarly obtain the eigenvalues of corresponding (6.49)–(6.50):

$$\mu_{ij} = 8\left(\sin^2\frac{(i-1)\pi}{n} + \sin^2\frac{(j-1)\pi}{m}\right) \tag{6.53}$$

for $(i,j) \in [1,n] \times [1,m]$, thus, the existence and nonexistence of solutions for (6.49)–(6.50) can also be established. They will omitted.

In the present paper, we ask that the coefficient matrix and the nonlinear term of the system (1.1), respectively, satisfy the symmetry and the odd symmetry. Clearly, a number of application problems are not valid. They will be considered in the further paper.

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