Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2009, Article ID 827087, 12 pages
doi:10.1155/2009/827087

Research Article

On the Cauchy Problem of a Quasilinear Degenerate Parabolic Equation

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Received 21 April 2009; Accepted 25 August 2009

Recommended by Guang Zhang

By Oleinik's line method, we study the existence and the uniqueness of the classical solution of the Cauchy problem for the following equation in $[0,T]\times R^2$: $\partial_{xx}u+u\partial_y u-\partial_t u=f(\cdot,u)$, provided that T is suitable small. Results of numerical experiments are reported to demonstrate that the strong solutions of the above equation may blow up in finite time.

1. Introduction

We consider the following Cauchy problem:

$$\partial_{xx}u+u\partial_y u-\partial_t u=f(\cdot,u),\quad (t,x,y)\in(0,T]\times R^2, \tag{1.1}$$

$$u(0,\cdot)=u_0(x,y),\quad (x,y)\in R^2. \tag{1.2}$$

This problem arises in financial mathematics recently; more and more mathematicians have been interested in it. In [1], Antonelli et al. introduced a new model for agents' decision under risk, in which the utility function is the solution to (1.1)-(1.2); they also proved, by means of probability methods, the existence of a continuous viscosity solution of (1.1)-(1.2), which satisfies

$$\left|u(x,y,t)-u(\xi,\eta,\tau)\right|\le C_T\left(|x-\xi|+\left|y-\eta\right|\right) \tag{1.3}$$

for every $(x,y),(\xi,\eta)\in R^2$, $t\in[0,T)$, under the assumption that f is uniformly Lipschitz continuous function. In [2], Citti et al. studied the interior regularity properties of this problem; they proved that the viscosity solutions are indeed classical solutions. On the other hand, Antonelli and Pascucci [3] showed that the solution u found in [1] can be also considered as a distributional solution.

However, all the above results are obtained when T is suitably small; say, the solution is local. The global weak solutions of the Cauchy problem for a more general class of equations, that contains (1.1), are obtained in [4–7], and so forth. This kind of solutions, however, is few regular and does not satisfy condition (1.3) in general.

In this paper, we will solve the Cauchy problem (1.1)-(1.2) in another simpler way and get the result as [2] again. Moreover, some examples are provided by numerical computation. The results of computation show that the strong solutions of the above equation may blow-up in finite time, though there exist the global weak solutions.

2. Line Method

In order to describe our method, we have to quote the well-known Prandtl system for a nonstationary boundary layer arising in an axially symmetric incompressible flow past a solid body, it has the form

$$\begin{gathered} \partial_t u + u\partial_x u + v\partial_y u = \partial_t U + U\partial_x U + \partial_y^2 u, \\ \partial_x(ru) + \partial_y(rv) = 0 \end{gathered} \tag{2.1}$$

in a domain $D = \{0 < t < T,\ 0 < x < X,\ 0 < y < \infty\}$, where $U(t,x)$ and $r(x)$ are given functions. If we introduce the Crocco variables:

$$\tau = t, \qquad \xi = x, \qquad \eta = \frac{u(t,x)}{U(t,x)}, \tag{2.2}$$

we obtain the following equation for $w(\tau,\xi,\eta) = \partial_y u/U$:

$$w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0. \tag{2.3}$$

Oleinik and Samokhin [8] had done excellent work in the boundary theory by the line method. Comparing this equation with (1.1), it is natural to conjecture that we are able to solve problem (1.1)-(1.2) by Qleinik's method.

Consider the following initial boundary problem:

$$w_{\eta\eta} - w_\tau + w w_\xi = f(\tau,\xi,\eta,w), \tag{2.4}$$

$$w(0,\xi,\eta) = u_0(\xi,\eta), \tag{2.5}$$

where $u_0 \in C^2(R^2)$; its first-order derivatives and $u_{0\eta\eta}$ are all bounded.

Definition 2.1. A function $w(\tau,\xi,\eta)$ is said to be a solution of problem (2.4)-(2.5) if w has first-order derivatives in (2.4) which is continuous in $(0,T] \times R^2$, and its derivative $w_{\eta\eta}$ is continuous; w satisfies (2.4) in $(0,T] \times R^2$, together with condition (2.5).

The solution of problem (2.4)-(2.5) will be constructed as the limit of a sequence w^n, $n \to \infty$, which consists of solutions of the equations

$$L_n(w^n) = w^n_{\eta\eta} - w^n_\tau + w^{n-1}w^n_\xi - f(\cdot, w^n) = 0, \tag{2.6}$$

$$w^n(0,\xi,\eta) = w_0(\xi,\eta). \tag{2.7}$$

As $w^0(\tau,\xi,\eta)$ we take a function which is smooth in $[0,T] \times R^2$.

Suppose that for some nonnegative number p

$$|f(\cdot, v)| \le c(1 + |v|^p), \tag{2.8}$$

and when $v_1 - v_2 \ge 0$,

$$\begin{gathered} c_1(v_1 - v_2) \ge f(\cdot, v_1) - f(\cdot, v_2) \ge c_2(v_1 - v_2), \\ \max\left\{\left|\frac{\partial f}{\partial \tau}\right|, \left|\frac{\partial f}{\partial \xi}\right|, \left|\frac{\partial f}{\partial \eta}\right| \left|\frac{\partial^2 f}{\partial v^2}\right|\right\} \le c. \end{gathered} \tag{2.9}$$

Lemma 2.2. *Let V be a smooth function such that $L_n(V) \ge 0$ in $(0,T) \times R^2$, $V \le w^n$ for $\tau = 0$. Then $V \le w^n$ everywhere $(0,T) \times R^2$. Let V_1 be a smooth function such that $L_n(V_1) \le 0$ in $(0,T) \times R^2$, $V \ge w^n$ for $\tau = 0$. Then $V_1 \ge w^n$ everywhere in $(0,T) \times R^2$.*

Proof. Let us prove the first statement of Lemma 2.2. The difference $z^n = w^n - V$ satisfies the inequality

$$0 \ge L_n(z^n) = L_n(w^n) - L_n(V) = z^n_{\eta\eta} - z^n_\tau + w^{n-1}z^n_\xi - \left(f(\cdot, w^n) - f(\cdot, V)\right). \tag{2.10}$$

Let $z^n_1 = e^{-\alpha\tau}z^n$. Then

$$\begin{aligned} 0 &\ge z^n_{1\eta\eta} - z^n_{1\tau} + \alpha z^n_1 + w^{n-1}z^n_{1\xi} - e^{-\alpha\tau}\left(f(\cdot, w^n) - f(\cdot, V)\right) \\ &\ge z^n_{1\eta\eta} - z^n_{1\tau} - \alpha z^n_1 + w^{n-1}z^n_{1\xi} - c_1 z^n_1. \end{aligned} \tag{2.11}$$

If we choose α large enough, by the maximal principle, we know $V \le w^n$ everywhere in $(0,T) \times R^2$.

Let us construct functions satisfying the conditions of Lemma 2.2. To this end, we define a twice continuously differentiable even function such that $V_1 = (1 - e^{-\beta|\eta|})e^{\beta\tau}$ for $|\eta| > 1$, $V_1 = \varphi(\eta)e^{\beta\tau}$ for $|\eta| \le 1$, where $\varphi(\eta)$ is a C^2 function, $|\varphi_{\eta\eta}| \le c$.

When $|\eta| > 1$,

$$\begin{aligned} L_n(V_1) &= V^n_{1\eta\eta} - V_{1\tau} - w^{n-1}V^n_{1\xi} - f(\cdot, V_1) \\ &= -\beta_1^2 e^{-\beta_1|\eta|}e^{\beta\tau} - \beta\left(1 - e^{-\beta_1|\eta|}\right)e^{\beta\tau} - f(\cdot, V_1) \\ &\le -\beta_1^2 e^{-\beta_1|\eta|}e^{\beta\tau} - \beta\left(1 - e^{-\beta_1|\eta|}\right)e^{\beta\tau} + c\left(1 - e^{-\beta_1|\eta|}\right)^p e^{p\beta\tau} + c < 0 \end{aligned} \tag{2.12}$$

if we chose β large enough and $\beta\tau \le T_0$ small enough.

When $|\eta| \le 1$,

$$\begin{aligned} L_n(V_1) &= \varphi_{\eta\eta} e^{\beta\tau} - \beta\varphi e^{\beta\tau} - f(\cdot, V_1) \\ &\le \varphi_{\eta\eta} e^{\beta\tau} - \beta\varphi e^{\beta\tau} + c\left(1 + \varphi^p e^{\beta\tau p}\right) < 0 \end{aligned} \tag{2.13}$$

by the same reason.

Let $V = \psi(\eta) e^{-\alpha\tau}$, $\alpha_1 > \psi(\eta) \ge \alpha_0 > 0$, $|\psi_{\eta\eta}| \le c$. Then

$$\begin{aligned} L_n(V) &= \psi_{\eta\eta} e^{-\alpha\tau} + \alpha\psi e^{\alpha\tau} - f(\cdot, V) \\ &\ge \psi_{\eta\eta} e^{-\alpha\tau} + \alpha\psi e^{\alpha\tau} - c\left(1 + \psi^p e^{\alpha\tau p}\right) \ge 0 \end{aligned} \tag{2.14}$$

if we chose α large enough and $\alpha\tau \le T_0$ small enough.

Similarly, we are able to prove the second statement of Lemma 2.2. □

Thus we have the following.

Lemma 2.3. *Suppose that f satisfies (2.9) and $V(0, \xi, \eta) \le w_0 \le V_1(0, \xi, \eta)$, then*

$$V \le w^n \le V_1. \tag{2.15}$$

The smooth functions V, V_1 can be constructed as in [8], and we omit details here.

Let

$$\Phi_n = \Phi = (u_\tau^n)^2 + \left(u_\xi^n\right)^2 + \left(u_\eta^n\right)^2, \tag{2.16}$$

where $u^n = w^n$. We will show that there exist positive constants M and T such that the conditions $\Phi_\mu \le M$ for $\tau \le T$, $\mu \le n-1$, imply that $\Phi_n \le M$ for $\tau \le T$.

First, we rewrite (2.6) as

$$u_{\eta\eta}^n - u_\tau + u^{n-1} u_\xi^n - f(\cdot, u^n) = 0, \quad (\tau, \xi, \eta) \in (0, T] \times R^2. \tag{2.17}$$

Applying the operator $2u_\tau^n(\partial/\partial\tau) + 2u_\xi^n(\partial/\partial\xi) + 2u_\eta^n(\partial/\partial\eta)$ to (2.17),

$$\begin{aligned} &2u_\tau^n u_{\tau\eta\eta}^n + 2u_\tau^n\left(u_\tau^{n-1} u_\xi^n + u^{n-1} u_{\xi\tau}^n\right) - 2u_\tau^n u_{\tau\tau}^n - 2\frac{\partial f}{\partial u}(u_\tau^n)^2 - 2u_\tau^n \frac{\partial f}{\partial \tau}, \\ &2u_\xi^n u_{\xi\eta\eta}^n + 2u_\xi^n\left(u_\xi^{n-1} u_\xi^n + u^{n-1} u_{\xi\xi}^n\right) - 2u_\xi^n u_{\tau\xi}^n - 2\frac{\partial f}{\partial u}\left(u_\xi^n\right)^2 - 2u_\xi^n \frac{\partial f}{\partial \xi}, \\ &2u_\eta^n u_{\eta\eta\eta}^n + 2u_\eta^n\left(w_\eta^{n-1} u_\xi^n + w^{n-1} u_{\xi\eta}^n\right) - 2u_\eta^n u_{\tau\eta}^n - 2\frac{\partial f}{\partial u}\left(u_\eta^n\right)^2 - 2u_\eta^n \frac{\partial f}{\partial \xi} = 0, \end{aligned} \tag{2.18}$$

then

$$u^{n-1}\Phi_\xi = \left(2u_\tau^n u_{\tau\xi}^n + 2u_\xi^n u_{\xi\xi}^n + 2u_\eta^n u_{\eta\xi}^n\right)u^{n-1},$$

$$-\Phi_\tau = -2u_\tau^n u_{\tau\tau}^n - 2u_\xi^n u_{\xi\tau}^n - 2u_\eta^n u_{\eta\tau}^n,$$

$$\Phi_{\eta\eta} = 2\left(u_{\tau\eta}\right)^2 + 2u_\tau^n u_{\tau\eta\eta}^n + 2\left(u_{\xi\eta}\right)^2 + 2u_\xi^n u_{\xi\eta\eta}^n + 2\left(u_{\eta\eta}\right)^2 + 2u_\eta^n u_{\eta\eta\eta}^n,$$

$$\Phi_{\eta\eta}+u^{n-1}\Phi_\xi - \Phi_\tau - 2\frac{\partial f}{\partial u}\Phi+2u_\tau^n u_\xi^n u_\xi^{n-1}+2\left(u_\xi^n\right)^2 u_\xi^{n-1}+2u_\eta^n u_\xi^n u_\eta^{n-1} - 2u_\tau^n\frac{\partial f}{\partial \tau} - 2u_\xi^n\frac{\partial f}{\partial \xi} - 2u_\eta^n\frac{\partial f}{\partial \xi} = 0. \tag{2.19}$$

By (2.9), (2.15), and Cauchy inequality, we are able to get

$$\Phi_{\eta\eta} + u^{n-1}\Phi_\xi - \Phi_\tau + R^n\Phi \geq 0, \tag{2.20}$$

where R^n depends on u^{n-1} and its derivatives are up to the second. Let $\Phi_1 = \Phi e^{-\gamma\tau}$ with a positive constant γ to be chosen later. Then

$$\Phi_{1\eta\eta} + u^{n-1}\Phi_{1\xi} - \Phi_{1\tau} + \left(R^n - \gamma\right)\Phi \geq 0 \tag{2.21}$$

if we choose γ according to M such that $R^n - \gamma \leq -1$. If Φ_1 attains its positive maximum at $\tau = 0$, then

$$\Phi_1|_{\tau=0} = \Phi e^{-\gamma\tau}|_{\tau=0} = \Phi|_{\tau=0} \leq c, \tag{2.22}$$

where the constant c does not depend on n. At the same time, the positive maximum of Φ_1 in $(0,T] \times R^2$ cannot be attained by maximal principle. Thus we have

$$\Phi_1 \leq c. \tag{2.23}$$

So, if we let $T_1 \leq T$ small enough such that $e^{\gamma T_1} = 2$ and set $M = 2c$, then

$$\Phi \leq c e^{\gamma T_1} = M. \tag{2.24}$$

In order to estimate the second derivatives of u^n in $[0,T_1] \times R^2$, consider the function

$$F = \left(u_{\tau\tau}^n\right)^2 + \left(u_{\xi\xi}^n\right)^2 + \left(u_{\eta\eta}^n\right)^2 + \left(u_{\tau\xi}^n\right)^2 + \left(u_{\xi\eta}^n\right)^2 + \left(u_{\tau\eta}^n\right)^2. \tag{2.25}$$

Applying the operator

$$P = 2u_{\tau\tau}^n\frac{\partial^2}{\partial\tau^2} + 2u_{\xi\xi}^n\frac{\partial^2}{\partial\xi^2} + 2u_{\eta\eta}^n\frac{\partial^2}{\partial\eta^2} + 2u_{\tau\xi}^n\frac{\partial^2}{\partial\tau\partial\xi} + 2u_{\tau\eta}^n\frac{\partial^2}{\partial\tau\partial\eta} + 2u_{\xi\eta}^n\frac{\partial^2}{\partial\xi\partial\eta} \tag{2.26}$$

to both sides of (2.17), we find that

$$
\begin{aligned}
0 = {} & 2u^n_{\tau\tau}u^n_{\eta\eta\tau\tau} + 2u^n_{\tau\tau}\left(u^{n-1}_{\tau\tau}u^n_{\xi} + 2u^{n-1}_{\tau}u^n_{\xi\tau} + u^{n-1}u^n_{\xi\tau\tau}\right) \\
& - 2u^n_{\tau\tau}u^n_{\tau\tau\tau} + -2u^n_{\tau\tau}\left(\frac{\partial^2 f}{\partial u^2}(u^n_{\tau})^2 + \frac{\partial f}{\partial u}u^n_{\tau\tau}\right) \\
& + 2u^n_{\xi\xi}u^n_{\eta\eta\xi\xi} + 2u^n_{\xi\xi}\left(u^{n-1}_{\xi\xi}u^n_{\xi} + 2u^{n-1}_{\xi}u^n_{\xi\xi} + u^{n-1}u^n_{\xi\xi\xi}\right) \\
& - 2u^n_{\xi\xi}u^n_{\tau\xi\xi} - 2u^n_{\xi\xi}\left(\frac{\partial^2 f}{\partial u^2}\left(u^n_{\xi}\right)^2 + \frac{\partial f}{\partial u}u^n_{\xi\xi}\right) \\
& + 2u^n_{\eta\eta}u^n_{\eta\eta\eta\eta} + 2u^n_{\eta\eta}\left(u^{n-1}_{\eta\eta}u^n_{\xi} + 2u^{n-1}_{\eta}u^n_{\xi\eta} + u^{n-1}u^n_{\xi\eta\eta}\right) \\
& - 2u^n_{\eta\eta}u^n_{\tau\xi\xi} - 2u^n_{\eta\eta}\left(\frac{\partial^2 f}{\partial u^2}\left(u^n_{\eta}\right)^2 - \frac{\partial f}{\partial u}u^n_{\eta\eta}\right) \\
& + 2u^n_{\tau\xi}u^n_{\eta\eta\tau\xi} + 2u^n_{\tau\xi}\left(u^{n-1}_{\tau\xi}u^n_{\xi} + u^{n-1}_{\tau}u^n_{\xi\xi} + u^{n-1}_{\xi}u^n_{\xi\tau} + u^{n-1}u^n_{\xi\xi\tau}\right) \\
& - 2u^n_{\tau\xi}u^n_{\tau\tau\xi} - 2u^n_{\tau\xi}\left(\frac{\partial^2 f}{\partial u^2}u^n_{\tau}u^n_{\xi} + \frac{\partial f}{\partial u}u^n_{\tau\xi}\right) \\
& + 2u^n_{\xi\eta}u^n_{\eta\eta\xi\eta} + 2u^n_{\xi\eta}\left(u^{n-1}_{\xi\eta}u^n_{\xi} + u^{n-1}_{\xi}u^n_{\xi\eta} + u^{n-1}_{\eta}u^n_{\xi\xi} + u^{n-1}u^n_{\xi\xi\eta}\right) \\
& - 2u^n_{\xi\eta}u^n_{\tau\tau\xi} - 2u^n_{\xi\eta}\left(\frac{\partial^2 f}{\partial u^2}u^n_{\eta}u^n_{\xi} + \frac{\partial f}{\partial u}u^n_{\xi\eta}\right) \\
& + 2u^n_{\tau\eta}u^n_{\eta\eta\tau\eta} + 2u^n_{\tau\eta}\left(u^{n-1}_{\tau\eta}u^n_{\xi} + u^{n-1}_{\tau}u^n_{\xi\eta} + u^{n-1}_{\eta}u^n_{\xi\tau} + u^{n-1}u^n_{\xi\tau\eta}\right) - 2u^n_{\tau\eta}u^n_{\tau\tau\eta} \\
& - 2u^n_{\tau\eta}\left(\frac{\partial^2 f}{\partial u^2}u^n_{\eta}u^n_{\tau} + \frac{\partial f}{\partial u}u^n_{\tau\eta}\right).
\end{aligned}
\tag{2.27}
$$

At the same time, we can calculate that

$$
\begin{aligned}
F_{\eta} &= 2u^n_{\tau\tau}u^n_{\tau\tau\eta} + 2u^n_{\xi\xi}u^n_{\xi\xi\eta} + 2u^n_{\eta\eta}u^n_{\eta\eta\eta} + 2u^n_{\tau\xi}u^n_{\tau\xi\eta} + 2u^n_{\xi\eta}u^n_{\xi\eta\eta} + 2u^n_{\tau\eta}u^n_{\tau\eta\eta}, \\
F_{\eta\eta} &= 2\left(u^n_{\tau\tau\eta}\right)^2 + 2u^n_{\tau\tau}u^n_{\tau\tau\eta\eta} + 2\left(u^n_{\xi\xi\eta}\right)^2 + 2u^n_{\xi\xi}u^n_{\eta\eta\xi\xi} + 2\left(u^n_{\eta\eta\eta}\right)^2 + 2u^n_{\eta\eta}u^n_{\eta\eta\eta\eta} \\
&\quad + 2\left(u^n_{\tau\xi\eta}\right)^2 + 2u^n_{\tau\xi}u^n_{\tau\xi\eta\eta} + 2\left(u^n_{\eta\xi\xi}\right)^2 + 2u^n_{\xi\eta}u^n_{\xi\eta\eta\eta} + 2\left(u^n_{\tau\eta\eta}\right)^2 + 2u^n_{\tau\eta}u^n_{\tau\eta\eta\eta}, \\
u^{n-1}F_{\xi} &= u^{n-1}\left(2u^n_{\tau\tau}u^n_{\tau\tau\xi} + 2u^n_{\xi\xi}u^n_{\xi\xi\xi} + 2u^n_{\eta\eta}u^n_{\eta\eta\xi} + 2u^n_{\tau\xi}u^n_{\tau\xi\xi} + 2u^n_{\xi\eta}u^n_{\xi\eta\xi} + 2u^n_{\tau\eta}u^n_{\tau\eta\xi}\right), \\
-F_{\tau} &= -\left(2u^n_{\tau\tau}u^n_{\tau\tau\tau} + 2u^n_{\xi\xi}u^n_{\xi\xi\tau} + 2u^n_{\eta\eta}u^n_{\eta\eta\tau} + 2u^n_{\tau\xi}u^n_{\tau\xi\tau} + 2u^n_{\xi\eta}u^n_{\xi\eta\tau} + 2u^n_{\tau\eta}u^n_{\tau\eta\tau}\right),
\end{aligned}
\tag{2.28}
$$

and so we have

$$
\begin{aligned}
&F_{\eta\eta}+u^{n-1}F_{\xi}-F_{\tau}-2\frac{\partial f}{\partial u}F-2u^{n}_{\tau\tau}\left(w^{n-1}_{\tau\tau}u^{n}_{\xi}+2w^{n-1}_{\tau}u^{n}_{\xi\tau}\right)\\
&\quad-2u^{n}_{\xi\xi}\left(u^{n-1}_{\xi\xi}u^{n}_{\xi}+2u^{n-1}_{\xi}u^{n}_{\xi\xi}\right)-2u^{n}_{\eta\eta}\left(u^{n-1}_{\eta\eta}u^{n}_{\xi}+2w^{n-1}_{\eta}u^{n}_{\xi\eta}\right)\\
&\quad-2u^{n}_{\tau\xi}\left(u^{n-1}_{\tau\xi}u^{n}_{\xi}+u^{n-1}_{\tau}u^{n}_{\xi\xi}+u^{n-1}_{\xi}u^{n}_{\xi\tau}\right)-2u^{n}_{\xi\eta}\left(u^{n-1}_{\xi\eta}u^{n}_{\xi}+u^{n-1}_{\xi}u^{n}_{\xi\eta}+u^{n-1}_{\eta}u^{n}_{\xi\xi}\right)\\
&\quad-2u^{n}_{\tau\eta}\left(u^{n-1}_{\tau\eta}u^{n}_{\xi}+u^{n-1}_{\tau}u^{n}_{\xi\eta}+u^{n-1}_{\eta}u^{n}_{\xi\tau}\right)-2u^{n}_{\tau\tau}\frac{\partial^2 f}{\partial u^2}(u^{n}_{\tau})^2-2u^{n}_{\xi\xi}\frac{\partial^2 f}{\partial u^2}\left(u^{n}_{\xi}\right)^2-2u^{n}_{\eta\eta}\frac{\partial^2 f}{\partial u^2}\left(u^{n}_{\eta}\right)^2\\
&\quad-2u^{n}_{\tau\xi}\frac{\partial^2 f}{\partial u^2}u^{n}_{\tau}u^{n}_{\xi}-2u^{n}_{\xi\eta}\frac{\partial^2 f}{\partial u^2}u^{n}_{\eta}u^{n}_{\xi}-2u^{n}_{\tau\eta}\frac{\partial^2 f}{\partial u^2}u^{n}_{\eta}u^{n}_{\tau}\\
&\quad-2\frac{\partial f}{\partial u}\left[(u^{n}_{\tau})^2+\left(u^{n}_{\xi}\right)^2+\left(u^{n}_{\eta}\right)^2\right]=0.
\end{aligned}
\tag{2.29}
$$

By the introduced assumption that the first-order and second-order derivatives of u^{n-1}, $\partial f/\partial u$, and $\partial^2 f/\partial u^2$ are all bounded and using Cauchy inequality, we can get from (2.29) that

$$
F_{\eta\eta}-2\alpha F_{\eta}-u^{n-1}F_{\xi}-F_{\tau}+R^{n}_{1}F\geq 0. \tag{2.30}
$$

By the transformation $F_1 = Fe^{-\gamma\tau}$, if we chose γ large enough, we are able to show that there exist positive constants M and T such that the conditions $F_{\mu}\leq M$ for $\tau\leq T$, $\mu\leq n-1$, imply that $F_n\leq M$ for $\tau\leq T$. Thus we have the following.

Theorem 2.4. *Let w^n be the solutions of problems* (2.4)-(2.5)*, then the derivatives of w^n up to the second-order are uniformly bounded with respect to n in the domain $(0,T]\times R^2$ with a small positive number T.*

Now let us establish uniform convergence of $w^n = u^n$ in $[0,T]\times R^2$. For $v^n = w^n - w^{n-1}$ we obtain the following equation from (2.6):

$$
\begin{gathered}
v^{n}_{\eta\eta}-v^{n}_{\tau}+w^{n-1}v^{n}_{\xi}-v^{n-1}w^{n-1}_{\xi}-\left(f(\cdot,w^{n})-f\left(\cdot,w^{n-1}\right)\right)=0,\\
v^{n}(0,\xi,\eta)=0.
\end{gathered}
\tag{2.31}
$$

Let $v^n = e^{\alpha\tau}v^{\eta}_{1}$. Then

$$
\begin{aligned}
&v^{n}_{1\eta\eta}-v^{n}_{1\tau}+w^{n-1}v^{n}_{1\xi}-v^{n-1}_{1}w^{n-1}_{\xi}-\alpha v^{n}_{1}-e^{-\alpha\tau}\left(f(\cdot,w^{n})-f\left(\cdot,w^{n-1}\right)\right)=0,\\
&v^{n}_{1\eta\eta}-v^{n}_{1\tau}+w^{n-1}v^{n}_{1\xi}-v^{n-1}_{1}w^{n-1}_{\xi}\\
&\quad=\alpha v^{n}_{1}+e^{-\alpha\tau}\left(f(\cdot,w^{n})-f\left(\cdot,w^{n-1}\right)\right)=\alpha v^{n}_{1}+e^{-\alpha\tau}\frac{\partial f}{\partial w}v^{n}_{1}\geq(\alpha-c)v^{n}_{1},
\end{aligned}
\tag{2.32}
$$

where we have chosen $\tau\leq T$ small enough such that $e^{-\alpha\tau}=2$, and $2(\partial f/\partial w)\geq -c$.

If v_1 attains its positive maximal value in $(0,T] \times R^2$, we can choose α large enough such that

$$\left|\frac{w_{\xi}^{n-1}}{\alpha - c}\right| < 1, \tag{2.33}$$

and then at the maximal point we have

$$(\alpha - c)v_1^n \le -v_1^{n-1} w_{\xi}^{n-1}. \tag{2.34}$$

If v_1^n attains its negative minimal value in $(0,T] \times R^2$, we have

$$(\alpha - c)\left(-v_1^n\right) \le -v_1^{n-1} w_{\xi}^{n-1}. \tag{2.35}$$

Notice that at $\tau = 0, v_1^n = v^n = 0$. By (2.34) and (2.35),

$$\max\left|v_1^n\right| \le q \max\left|v_1^{n-1}\right|, \quad q < 1, \tag{2.36}$$

which means that the series $v_1^1 + v_1^2 + \cdots + v_1^n + \cdots$, whose sum has the form $w^n e^{-\alpha\tau}$, is majorized by a geometrical progression and, therefore, is uniformly convergent. The fact that w^n and its derivatives up to the second-order are bounded implies that the first derivatives of w^n are uniformly convergent as $n \to \infty$.

It follows from (2.6) that $w_{\eta\eta}^n$ are also uniformly convergent as $n \to \infty$.

Now, we can take $w^{-1} = w^0 = w_0$; then by the above discussion, we have the following theorem.

Theorem 2.5. *Suppose that* $V(0,\xi,\eta) \le w_0 \le V_1(0,\xi,\eta)$ *and* f *satisfies* (2.9) *and is suitable smooth, then there exists a small positive number* T *such that the Cauchy problem* (2.4) *has a classical solution.*

By the way, it is easy to prove the uniqueness of the solution for the Cauchy problem (2.4), and we omit the details here.

3. Computational Examples

In this section, a numerical simulate is made for the equations by differential method. Numerical computation of these examples shows that the strong solutions for the corresponding Cauchy problem of (1.1)-(1.2) will blow-up in finite time.

Let $\Omega = [0, L_x] \times [0, L_y]$ and $u(x,y,0) = u_0(x,y)$, $(x,y) \in \Omega$, but $u(x,y,0) = 0$, $(x,y) \in R^2/\Omega$. Then instead of studying the Cauchy Problem (1.1)-(1.2), we can study the following

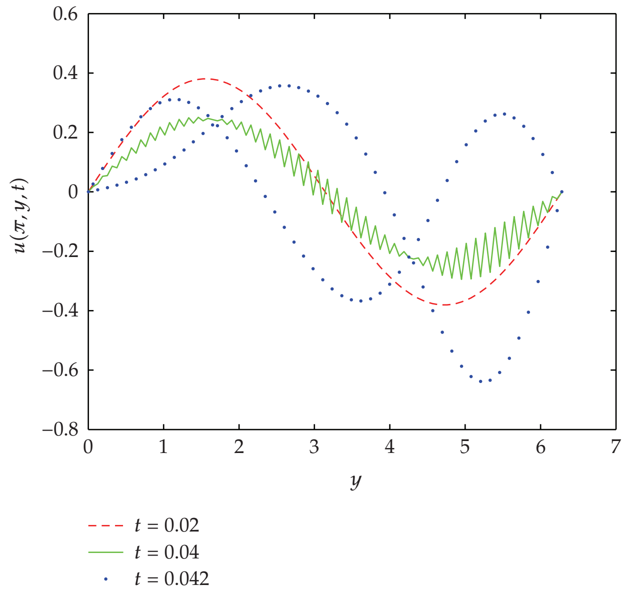


Figure 1: $f(\cdot, u) = u$.

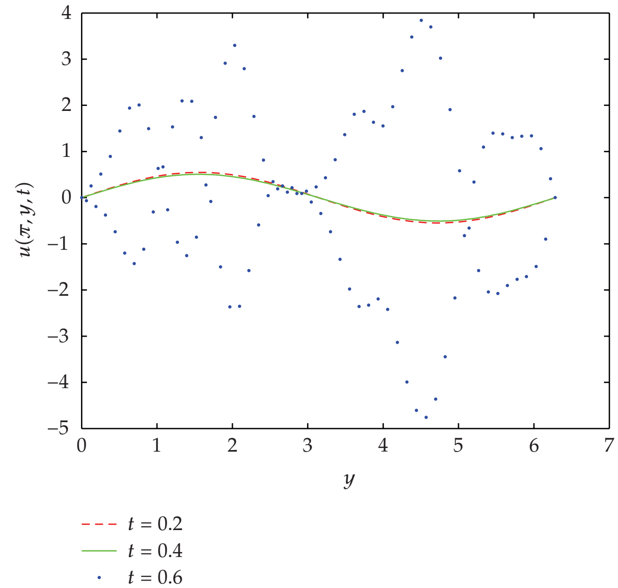
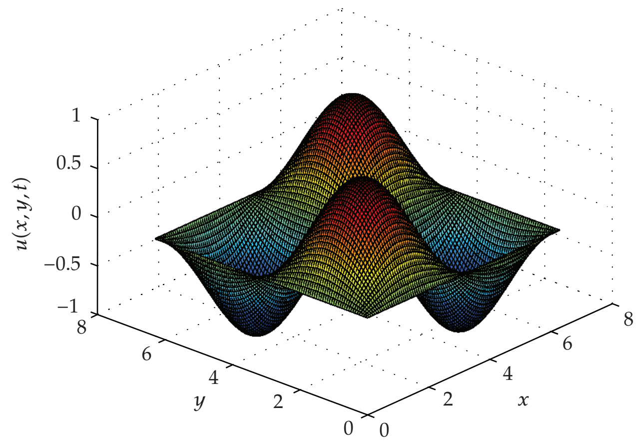


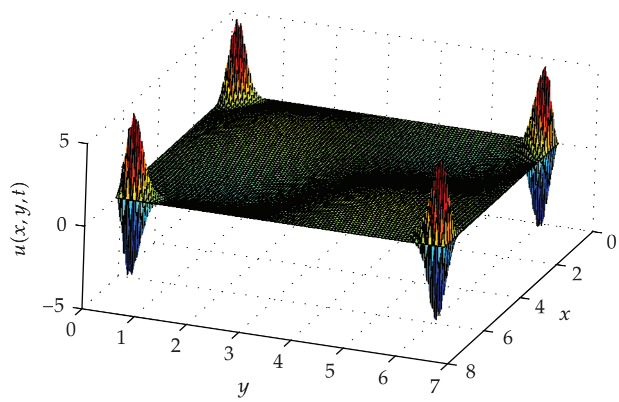
Figure 2: $f(\cdot, u) = \sin u$.

initial boundary problem:

$$\begin{gathered}
\partial_{xx} u + u \partial_y u - \partial_t u = f(\cdot, u), \quad (x, y, t) \in \Omega \times (0, T], \\
u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \\
u|_{\partial\Omega} = 0.
\end{gathered} \tag{3.1}$$



(a)



(b)

Figure 3: The numerical results in (a) at $t = 0$ and in (b) at $t = 0.046$ when $f(\cdot, u) = u^2$.

If $f(\cdot, 0) = 0$, it is clear that if $u(x, y, t)$ is a classical solution of (3.1), then $u(x, y, t)$ is a strong solution of the Cauchy problem (1.1)-(1.2).

To dissect domain Ω, suppose that $L_x = L_y = 2\pi$ and $h_x = 2\pi/N$, $h_y = 2\pi/M$ stands for the space step-length in the axis x and axis y, and $k = T/J$ stands for the time step-length. Let $\Omega_h = \{(ih_x, jh_y) \mid 0 \le i \le N;\ 0 \le j \le M\}$ and define $u_{ij}^n = u(ih_x, jh_y, nk)$. The differential scheme of the original equation is (to ensure numerical stability, here we apply arithmetic averages in order to avoid "oscillation" and "shifting" of the numerical solution)

$$\begin{gathered}
\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h_x^2} + \frac{u_{i+1,j}^n + u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j}^n + u_{i,j-1}^n}{6} \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h_y} \\
- \frac{u_{i,j}^{n+1} - (1/4)\left(u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n\right)}{k} \\
= f\left(ih_x, jh_y, nk, \frac{u_{i+1,j}^n + u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j}^n + u_{i,j-1}^n}{6}\right), \\
u^n|_{\partial\Omega_h} = 0, \quad (n = 1, 2, \ldots), \qquad u_{i,j}^0 = u_0(ih_x, jh_y).
\end{gathered} \tag{3.2}$$

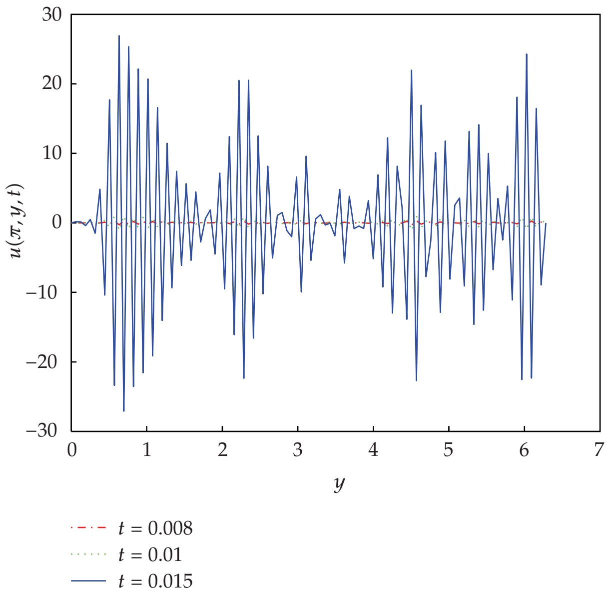


Figure 4: $f(\cdot, u) = u$.

So we get the following explicit formula:

$$\begin{aligned} u_{i,j}^{n+1} = {} & \frac{1}{4}\left(u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n}\right) + \frac{k}{h_x^2}\left(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) \\ & + \frac{k}{12h_y}\left(u_{i+1,j}^{n} + 2u_{i,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n}\right)\left(u_{i,j+1}^{n} - u_{i,j-1}^{n}\right) \\ & - kf\left(ih_x, jh_y, nk, \frac{u_{i+1,j}^{n} + 2u_{i,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n}}{6}\right). \end{aligned} \tag{3.3}$$

Experiment 1. Suppose $\Omega = [0, 2\pi] \times [0, 2\pi]$, $h_x = h_y = 2\pi/40$, $k = 0.001$, $u_0(x, y) = \sin x \sin y$ which itself does not satisfy (1.1); we get the graphs (see Figures 1–3) where $u(x, y, t)$ changes according to the changes of t when different functions are given to $f(\cdot, u)$.

Figure 1 shows that when $f(\cdot, u) = u$, at $t = 0.04$, the numerical solutions become oscillatory; at $t = 0.042$, the bifurcation of solutions occurs; when $t > 0.042$, the solutions will blow-up. Similarly Figure 2 shows that when $f(\cdot, u) = \sin u$, at $t = 0.6$, the bifurcation of solutions occurs; when $t > 0.6$, the solutions will blow-up. Figure 3 is the spatiotemporal graphs of solutions when $f(\cdot, u) = u^2$ at $t = 0$ (initial value) and $t = 0.0046$. When $t > 0.0046$, the solutions will blow-up.

Experiment 2. The initial value is unknown in the general situation; so we use random numbers ($[-0.01, 0.01]$) as the initial value and draw graphs (see Figures 4 and 5) where $u(x, y, t)$ changes as t changes when different functions are given to $f(\cdot, u)$.

Figures 4 and 5 show that even though the initial value is sufficiently small, the blow-up will appear in finite time for the different functions.

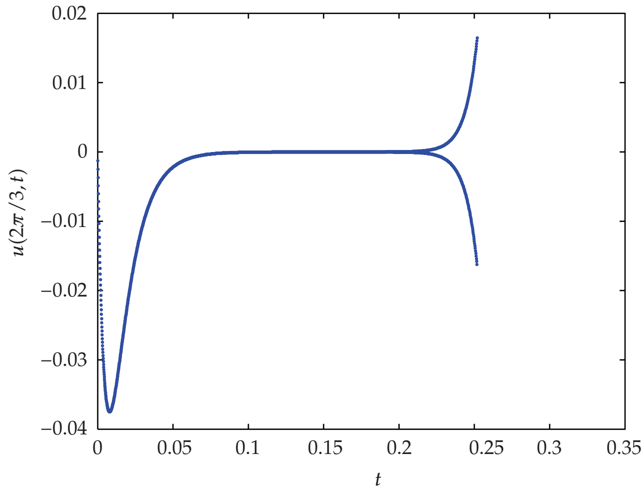


Figure 5: $f(\cdot, u) = 1 - \sin u$.

The numerical result shows that there is a locality solution of the equation. When t becomes larger, the bifurcation of solutions occurs in finite time and blow-up appears. For this problem, it is essential to have a further research.

Acknowledgments

The research was supported by the Fujian National Science Foundation of China Grant 2008J0198, 2009J1009. The authors would like to thank Professor Zhao Junning for insightful discussions and Professor Xu Chuanju for helpful suggestions and comments.

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