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Research Article

The Numerical Convergence of the Landau-Lifshitz Equations and Its Simulation

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A difference scheme of Landau-Lifshitz (LL for short) equations is studied. Their convergence and stability are proved. Furthermore, a new solution of LL equation is given for testing our scheme. At the end, three subcases of this LL equation are concerned about, and some properties about these equations are shown by a numeric simulation way.

1. Introduction and the Number Scheme

The LL equation [1] has aroused considerable interest among physicists and mathematicians. For the one-dimensional case, there have been many contributions to the study of the soliton solution, the interaction of solitary waves, and other properties of the solitary waves [2–4]. However, we point out here that it will be a more challenging task study the high-dimensional dynamics [5, 6] about LL equation. In the classical study of ferromagnetic chain, we often consider the following system (vector form):

$$\frac{\partial u}{\partial t} = u \times \Delta u + f(x,t,u), \tag{1.1}$$

where "×" denotes a 3-dimensional cross product. The isotropic Heisenberg chain [2] is the special case of (1.1). Moreover, another special case of (1.1) is the LL equation with an easy plane

$$\frac{\partial u}{\partial t} = u \times \Delta u + u \times (Pu), \quad P = \operatorname{diag}\{P_1, P_2, P_3\}, \ P_1 \le P_2 \le P_3. \tag{1.2}$$

Equation (1.2) has been studied by the inverse scattering method in [7]. Its integrability theories [8] are established. As far as we know, no explicit solutions of (1.2) are given by various direct methods such as Jacobi elliptical function method [9]. Therefore, the numerical method to study this equation and its more generalized form, which gives some of the visual characteristics of equations, is necessary. However, we do not intend to construct efficient (or optimal control) and high-precision algorithms [10–15] here. In this paper, we are concerned about the convergence of the discrete scheme. Furthermore, we hope that by means of numerical simulation, more intuitive understanding of the nature of the equation will be obtained.

For universality, we consider the following system with the Gilbert damping term which covers the above situations. The periodic condition is about the space variables x

$$\begin{gathered}\frac{\partial u}{\partial t} = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 (u \times \Delta u) + f(x,t,u) \quad (x,t) \in R \times I, \\ u(x+2\pi,t) = u(x,t) \quad (x,t) \in R \times I, \\ u(x,0) = u_0(x) \quad x \in R,\end{gathered} \tag{1.3}$$

where the spin vector $u(x,t) = (u^1,u^2,u^3)^T$ is a 3-dimensional vector-valued unknown function with respect to space variables $x = (x_1,x_2,\ldots,x_n)$ and time t, and period about $u(x,t)$ is 2π. $f(x,t,u)$ is continuous with respect to x, t, and u. $u_0(x)$ is the initial value function which is also a 3-dimensional function. $I = [0,T]$, $(T > 0)$, R^n is the n-dimensional real value dominion. $\alpha_1 \geq 0$ is damping coefficient. $\alpha_2 > 0$, Laplace operator $\Delta u = (\partial^2 u^1/\partial x^2,\ \partial^2 u^2/\partial x^2,\ \partial^2 u^3/\partial x^2)^T$.

Although the existence of the global attractor of (1.3) has been proved in [16], no exact solutions have been proposed as far as we know. In this paper, we give an exact solution of it, which can test the property of numerical scheme of (1.3). From a physical point of view, the value of $u(x,t)$ is finite, just like the situation which is mentioned in [5, 16]. In this section, we just suppose that $|u(x,t)| = 1$ for convenience. So the first equation of problem (1.3) is equal to the following form:

$$\frac{\partial u}{\partial t} = -\alpha_1 \Delta u + \alpha_1 |\nabla u|^2 u + \alpha_2 A(u) \Delta u + f(x,t,u), \tag{1.4}$$

where $\nabla u = (u_x^1, u_x^2, u_x^3)^T$,

$$A(u) = \begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix} \in R^{3\times 3}. \tag{1.5}$$

Furthermore, we consider a more general type of Landau-Lifshitz equation (as the variety of (1.4)):

$$\begin{gathered}\frac{\partial u}{\partial t} = -\alpha_1 \Delta u + \alpha_1 g(\nabla u) u + \alpha_2 A(u) \Delta u + f(x,t,u) \quad (x,t) \in R^n \times I, \\ u(x+2\pi,t) = u(x,t) \quad (x,t) \in R^n \times I, \\ u(x,0) = u_0(x) \quad x \in R^n,\end{gathered} \tag{1.6}$$

where $g(v) : R^3 \to R$ is a continuous differential function. When $g(s) = |s|^2$, (1.6) takes the form of (1.4).

For convenien, we discuss our scheme in one-dimensional case which the n-dimensional case can be discussed similarly. Firstly, we give some notion and symbol. Set $\Omega = [0, 2\pi]$, J is a positive integer. We divide the region $\Omega \times I$ as a discrete mesh which $h = 2\pi/J$ in its space step and k in its time step. $x_j = jh$, $t_n = nk$ $(j = 0, 1, \ldots, J; n = 0, 1, \ldots, [T/k])$. Here $(u^m)_j^n$ denotes the approximate value of u^m $(m = 1, 2, 3)$ on (x_j, t_n). Let u^n denote the layer mesh function, that is, $u^n = (u_1^n, u_2^n, \ldots, u_J^n)$ where $u_j^n = ((u^1)_j^n, (u^2)_j^n, (u^3)_j^n)^T$.

Define the discrete inner product and norm as follows:

$$(u^n, v^n) = h\sum_{j=1}^{J} u_j^n \cdot v_j^n = h\sum_{j=1}^{J}\sum_{m=1}^{3} (u^m)_j^n \cdot (v^m)_j^n; \qquad \|u^n\| = (u^n, u^n)^{1/2}. \tag{1.7}$$

We define the following finite difference approximations of derivatives along the space direction:

$$u_{jx}^n = \frac{u_{j+1}^n - u_j^n}{h}, \qquad u_{j\overline{x}}^n = \frac{u_j^n - u_{j-1}^n}{h}, \qquad u_{j\widehat{x}}^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h}. \tag{1.8}$$

Similarly, we can define the derivatives along the time direction and the Laplace operator: $u_{jt}^n, u_{j\overline{t}}^n, u_{j\widehat{t}}^n$, and $u_{jx\overline{x}}^n$.

Under the definition and the symbol setting as above, we now define a finite difference approximation of (1.6) by

$$\begin{gathered} u_{jt}^n = \alpha_1 u_{jx\overline{x}}^n + \alpha_1 g\left(u_{j\widehat{x}}^n\right) u_j^n + \alpha_2 A\left(u_j^n\right) u_{jx\overline{x}}^n + f_j^n\left(u_j^n\right) \quad 0 \le j \le J,\ 0 \le n \le \left[\frac{T}{k}\right] - 1, \\ u_{j+rJ}^n = u_j^n, \quad 0 \le j \le J,\ r \in Z, \\ u_j^0 = u_0(x_j) \quad 0 \le j \le J. \end{gathered} \tag{1.9}$$

Obviously, according to difference approximation (1.9), we can start from the zero layer to any layer about u^n $(n = 1, 2, \ldots, [T/k])$.

2. Existence of the Solution and the Stability of the Scheme

For studying the convergence and the stability, we first introduce several lemmas.

Lemma 2.1. *Considering mesh-function u^n, v^n defined on mesh-points, one has the following relationship:*

(1) $(u_{x\overline{x}}^n, v^n) = -(u_x^n, v_x^n)$,

(2) $(u_t^n, u^n) \ge (1/2)\|u^n\|_t^2 - (k/2)\|u_t^n\|^2$,

(3) $\|u_{\widehat{x}}^n\| \le \|u_x^n\|$,

(4) $\|u_x\| \le (2/h)\|u\|$.

Lemma 2.2. *Let $A(u)$ stand for an antisymmetric matrix, v is the three-dimension vector, then the following relationships hold:*

(1) $A(u)u = 0$,

(2) $(w, A(u)v) = -(A(u)w, v)$,

(3) $A(u)v = -A(v)u$.

Let $U(x,t)$ denote the solution of (1.6) (here $U(x,t)$ will be regarded as a function with smoothness in some degree). U_j^n denotes the value of $U(x,t)$ on (x_j, t_n). $U^n = (U_1^n, U_2^n, \ldots, U_j^n)^T$ denotes the value of n-layer mesh about $U(x,t)$.

Definition 2.3. One has

$$C_b^0\left(R^3, R^{3\times 3}\right) = \left\{B(u) \in R^{3\times 3} \mid \|B(u)\| < \infty,\ \forall u \in R^3\right\},$$
$$C_b^1\left(R^3\right) = \left\{f(u) \in R \mid \|f(u)\|_\infty + \|f'(u)\|_\infty < \infty,\ \forall u \in R^3\right\}. \tag{2.1}$$

Lemma 2.4. *Assume that $A(u) \in C_b^0(R^3; R^{3\times 3})$ and $U \in C^0(I; (C^3(0; 2\pi))^3)$ are the solution of* (1.6), *then for any $\varepsilon > 0$*

$$\alpha_2\left(A(U^n)U_{x\bar{x}}^n - A(u^n)u_{x\bar{x}}^n, e^n\right) \le \varepsilon \|e_x^n\|^2 + \frac{\alpha_2^2 M_1^2}{4\varepsilon}\|e^n\|^2, \tag{2.2}$$

where $M_1 = \sup_{u\in R^3}\|A(u)\|$, and $e^n = U^n - u^n$.

Proof. We have, by (1) of Lemma 2.2, $-\alpha_2(U_{x\bar{x}}^n, A(e^n)e^n) = 0$. According to the antiproperty of antisymmetric matrix, we have $\alpha_2(e_x^n, A(u^n)e_x^n) = 0$. Here, we set $f \in C_b^1(R, I)$, hence

$$\begin{aligned}
&\left(f(x,t_n)A(U^n)U_{x\bar{x}}^n - f(x,t_n)A(u^n)u_{x\bar{x}}^n, e^n\right)\\
&\quad = \left(A(e^n)U_{x\bar{x}}^n + A(u^n)e_{x\bar{x}}^n, f(x,t_n)e^n\right)\\
&\quad = -\left(U_{x\bar{x}}^n, f(x,t_n)A(e^n)e^n\right) - \left(e_{x\bar{x}}^n, f(x,t_n)A(u^n)e^n\right)\\
&\quad = \left(e_x^n, \left(f(x,t_n)A(u^n)e^n\right)_x\right)\\
&\quad = h\sum_{j=0}^{J-1} e_{jx}^n \frac{\left[f(x_{j+1},t_n)A\left(u_{j+1}^n\right)e_{j+1}^n + f(x_j,t_n)A\left(u_j^n\right)e_j^n\right]}{h}\\
&\quad = h\sum_{j=0}^{J-1} e_{jx}^n f(x_{j+1},t_n)A\left(u_{j+1}^n\right)e_{jx}^n + h\sum_{j=0}^{J-1} e_{jx}^n \frac{\partial f(x_j,t_n)}{\partial x}A\left(u_j^n\right)e_{j+1}^n.
\end{aligned} \tag{2.3}$$

Let $f(x_j, t_n) = 1$ $(j = 0, 1, \ldots, J;\ n = 0, 1, \ldots, [T/k])$. By ε-inequality, we have

$$\begin{aligned}\alpha_2\big(A(U^n)U^n_{x\overline{x}} - A(u^n)u^n_{x\overline{x}}, e^n\big) &\le |\alpha_2|\|e^n_x\|\|A(u^n_x)\|\|e^n\| \\ &\le \varepsilon\|e^n_x\|^2 + \frac{\alpha_2^2 M_1^2}{4\varepsilon}\|e^n\|^2.\end{aligned} \tag{2.4}$$

□

Lemma 2.5. *Under the same condition of Lemma 2.4, one has*

$$\big\|A(U^n)U^n_{x\overline{x}} - A(u^n)u^n_{x\overline{x}}\big\| \le M_1\left(\|e^n\| + \frac{2}{h}\|e^n_x\|\right). \tag{2.5}$$

Proof. We first note that, by (3) of Lemma 2.2, we have, $A(e^n)U^n_{x\overline{x}} = -A(U^n_{x\overline{x}})e^n$; recalling (4) of Lemma 2.1, we have $\|e^n_{x\overline{x}}\| \le (2/h)\|e^n_x\|$, then we obtain

$$\begin{aligned}\big\|A(U^n)U^n_{x\overline{x}} - A(u^n)u^n_{x\overline{x}}\big\| &= \big\|A(e^n)U^n_{x\overline{x}} - A(u^n)e^n_{x\overline{x}}\big\| \\ &\le \big\|A(e^n)U^n_{x\overline{x}}\big\| + \big\|A(u^n)e^n_{x\overline{x}}\big\| \\ &\le \big\|-A(U^n_{x\overline{x}})e^n\big\| + \big\|A(u^n)e^n_{x\overline{x}}\big\| \\ &\le M_1\left(\|e^n\| + \frac{2}{h}\|e^n_x\|\right).\end{aligned} \tag{2.6}$$

□

Lemma 2.6. *If* $g(u) \in C^1_b(R^3)$ *and* $U \in C^0(I; (C^3(0, 2\pi)))^3$ *are the solution of* (1.6)*, then for any* $\varepsilon > 0$

$$\begin{aligned}&\alpha_1\big(g(U^n_{\hat{x}})U^n - g(u^n_{\hat{x}})u^n, e^n\big) + \big(f(U^n) - f(u^n), e^n\big) \\ &\quad\le \varepsilon\|e^n_x\|^2 + \left(\frac{\alpha_1^2 M_2^2 M_3^2}{4\varepsilon} + \alpha_1 M_4 + \frac{M_0}{\alpha_1}\right)\|e^n\|^2,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}&M_0 = \sup_{1\le n\le[T/K]}\left\|\frac{\partial f^n}{\partial u}\right\|, \qquad M_2 = \sup_{z\in R^3}\big\|\nabla g(z)\big\|, \qquad M_3 = \sup_{1\le n\le[T/K]}\|U^n\|_\infty, \\ &M_4 = \sup_{z\in R^3}\big\|g(z)\big\|, \qquad e^n = U^n - u^n.\end{aligned} \tag{2.8}$$

Proof. By (3) of Lemma 2.1, we have

$$\begin{aligned}
&\alpha_1\left(g(U_{\hat{x}}^n)U^n - g(u_{\hat{x}}^n)u^n, e^n\right) + \left(f(U^n) - f(u^n), e^n\right) \\
&\quad \le \alpha_1\left[\left(\left[g(U_{\hat{x}}^n) - g(u_{\hat{x}}^n)\right]U^n, e^n\right) + \left(g(u_{\hat{x}}^n)e^n, e^n\right)\right] + M_0\|e^n\|^2 \\
&\quad \le \alpha_1\left[\left\|\nabla g(U_{\hat{x}}^n + \theta e_{\hat{x}}^n)\right\|\|U^n\|_\infty\left\|e_{\hat{x}}^n\right\|\|e^n\| + \left\|g(u_{\hat{x}}^n)\right\|\|e^n\|^2\right] + M_0\|e^n\|^2 \\
&\quad \le \varepsilon\|e_x^n\|^2 + \left(\frac{\alpha_1^2 M_2^2 M_3^2}{4\varepsilon} + \alpha_1 M_4 + \frac{M_0}{\alpha_1}\right)\|e^n\|^2.
\end{aligned} \tag{2.9}$$

□

Lemma 2.7. *One has*

$$\alpha_1\left\|g(U_{\hat{x}}^n)U^n - g(u_{\hat{x}}^n)u^n\right\| + \left\|f(U^n) - f(u^n)\right\| \le \alpha_1\left(\frac{2M_2M_3}{h} + M_4 + \frac{M_0}{\alpha_1}\right)\|e^n\|. \tag{2.10}$$

Proof. According to (1) and (2) of Lemma 2.1, we have

$$\begin{aligned}
&\alpha_1\left\|g(U_{\hat{x}}^n)U^n - g(u_{\hat{x}}^n)u^n\right\| + \left\|f(U^n) - f(u^n)\right\| \\
&\quad \le \alpha_1\left(\left\|\nabla g(U_{\hat{x}}^n + \theta e_{\hat{x}}^n)\right\|\|U^n\|_\infty\left\|e_{\hat{x}}^n\right\|\|e^n\| + \left\|g(u_{\hat{x}}^n)\right\|\|e^n\|^2\right) + M_0\|e^n\|.
\end{aligned} \tag{2.11}$$

□

Definition 2.8. $S_\varepsilon(U) = \{v \in H_p^1(\Omega) \mid \|U - v\|_{H_p^1(\Omega)} \le \varepsilon\}$.

According to the lemmas mentioned above, we now come to discuss the convergence of difference equation (1.9).

Theorem 2.9. *Let $U \in C^2(I; (H_p^4(0, 2\pi))^3)$ be the solution of (1.6), u^n is the solution of (1.9), $g(u) \in C^1(R^3)$, $f(u) \in C(R^3)$. For any positive integer σ, if $(k/h^2) \le (\alpha_1 - \sigma)/6(\alpha_1 + |\alpha_2|M_1)^2$, then there exists a positive constant C_i $(i = 1, 2, 3, 4)$ which independent of h and k. If $h \le C_1$, $k \le C_2$, one has*

$$\sup_{0 \le n \le [T/k]} \|U^n - u^n\| + |||U_x^n - u_x^n||| \le C_3\left(k + h^2\right), \tag{2.12}$$

furthermore,

$$\|U^n - u^n\|_\infty \le C_4 h, \tag{2.13}$$

where $M_1 = m \sup_{u \in S_\varepsilon(U)}\|A(u)\| (m > 1)$, $|||\cdot||| = (k\sum_{n=1}^{[T/k]}\|\cdot\|^2)^{1/2}$, $\|v\|_\infty = \max_{1 \le j \le J}|v_j|$.

Proof. According to the condition, the solution of (1.6) $U \in C^2(I; (H_p^4(0, 2\pi))^3)$, substituting the according part of (1.9) with $U(x, t)$, we have

$$U_t^n = \alpha_1 U_{x\overline{x}}^n + \alpha_2 A(U^n)U_{x\overline{x}}^n + \alpha_1 g(U_{\hat{x}}^n)U^n + r^n, \tag{2.14}$$

where r^n is the error estimate of (1.9). According to (2.14) and (1.9), denote $e^n = U^n - u^n$, we have

$$e_t^n = \alpha_1 e_{x\bar{x}}^n + \alpha_2\left(A(U^n)U_{x\bar{x}}^n - A(u^n)u_{x\bar{x}}^n\right) + \alpha_1\left(g(U_{\hat{x}}^n)U^n - g(u_{\hat{x}}^n)u^n\right) + f(U^n) - f(u^n) + r^n. \quad (2.15)$$

Take the inner product of (2.15) and e^n, combine them with (1) and (2) of Lemma 2.1, we have

$$\frac{1}{2}\|e^n\|_t^2 - \frac{k}{2}\left\|e_t^n\right\|^2 + \alpha_1\|e_x^n\|^2 = \alpha_2\left(A(U^n)U_{x\bar{x}}^n - A\left(u^n u_{x\bar{x}}^n\right), e^n\right) + \alpha_1\left(g(U_{\hat{x}}^n)U^n - g(u_{\hat{x}}^n)u^n, e^n\right) + \left(f(U^n) - f(u^n), e^n\right) + (r^n, e^n). \quad (2.16)$$

Just like the method mentioned in [17], we can assume that $A^*(u) \in C_0^b(R^3, R^{3\times3})$ and $g^*(u) \in C_1^b(R^3)$. Conveniently, we can denote $A^*(u)$ as $A(u)$ and $g^*(u)$ as $g(u)$. By Lemmas 2.4 and 2.6, (1.9) can change into the following form:

$$\begin{aligned}\frac{1}{2}\|e^n\|_t^2 - \frac{k}{2}\left\|e_t^n\right\|^2 + (\alpha_1 - 2\varepsilon)\|e_x^n\|^2 &\le M_5\|e^n\|^2 + \frac{1}{2}\|r^n\|^2 + M_0\|e^n\|^2 \\ &\le M_5\|e^n\|^2 + C^2\left(k + h^2\right)^2 + M_0\|e^n\|^2,\end{aligned} \quad (2.17)$$

where

$$M_0 = \max_{1\le n\le[T/k]}\left\|\frac{\partial f^n}{\partial u}\right\|, \qquad M_5 = \frac{\alpha_1^2 M_2^2 M_3^2}{4\varepsilon} + \alpha_1 M_4 + \frac{\alpha_2^2 M_1^2}{4\varepsilon} + \frac{1}{2}. \quad (2.18)$$

In the following part, we propose the estimation about $\|e_t^n\|$ of (2.17).

By Lemmas 2.5 and 2.7 and (2.15), we have

$$\begin{aligned}k\left\|e_t^n\right\|^2 &\le |\alpha_2|\left\|A(U^n)U_{x\hat{x}}^n - A(u^n)u_{x\hat{x}}^n\right\| + \alpha_1\left\|g(U_{\hat{x}}^n)U^n - g(u_{\hat{x}}^n)u^n\right\| \\ &\quad + \left\|f(U^n) - f(u^n)\right\| + \|r^n\| \\ &\le \frac{2\alpha_1}{h}\|e_x^n\| + \alpha_1\left(\frac{2M_2M_3}{h} + M_4\right)\|e^n\| + |\alpha_2|M_1\left(\|e^n\| + \frac{2}{h}\|e_x^n\|\right) + M_0\|e^n\| + r^n \\ &\le \frac{M_6}{h}\|e_x^n\| + \left(\frac{M_7}{h} + M_8 + M_0\right)\|e^n\| + C\left(k + h^2\right),\end{aligned} \quad (2.19)$$

where $M_6 = 2(\alpha_1 + |\alpha_2 M_1|)$, $M_7 = 2\alpha_1 M_2 M_3$, and $M_8 = \alpha_1 M_4 + |\alpha_2|M_1$.

So according to the inequality given above, we have

$$k\left\|e_t^n\right\|^2 \le 3M_6^2\frac{k}{h^2}\|e_x^n\|^2 + \left(6M_7^2\frac{k}{h^2} + 6M_8^2 k + 3kM_0^2\right)\|e^n\|^2 + 3kC^2\left(k + h^2\right)^2. \quad (2.20)$$

Substitute (2.16) into (2.15), let $\varepsilon = \alpha_1/4$, according to the condition given in Theorem 2.9, we have

$$\frac{1}{2}\|e^n\|_t^2 + \left(\frac{\alpha_1}{2} - 3M_6^2\frac{k}{k^2}\right)\|e_x^n\|^2 \le M_5\|e^n\|^2 + C\left(k + h^2\right)^2, \tag{2.21}$$

where $M_9 = M_0 + M_5 + 6M_8^2T + M_7^2((\alpha_1 - \sigma)/(\alpha_1 + |\alpha_2|M_1)^2)$.

According to (2.17) and the condition given in Theorem 2.9,

$$\left\|e_t^n\right\|^2 + \sigma\|e_x^n\|^2 \le 2M_9\|e^n\|^2 + C\left(k + h^2\right)^2 \tag{2.22}$$

By $\|e^0\| = 0$ and Gronwall inequality, we have

$$\|e^n\| \le C\left(k + h^2\right)^2 \exp(M_9T) \le C_3\left(k + h^2\right)^2. \tag{2.23}$$

By (2.22),

$$\sigma k\sum_{n=1}^{m}\|e_x^n\|^2 \le CT\left(k + h^2\right)^2 + 2M_9k\sum_{n=1}^{m}\|e^n\|^2. \tag{2.24}$$

So we have

$$k\sum_{n=1}^{[T/k]}\|e_x^n\|^2 \le M_{10}^2\left(k + h^2\right)^2, \tag{2.25}$$

where $M_{10}^2 = (CT + 2M_9TC_3^2)/\sigma$ by (2.25), hence

$$\left(k\sum_{n=1}^{m}\|e_x^n\|^2\right)^{1/2} \le M_{10}\left(k + h^2\right)^2. \tag{2.26}$$

So by (2.23) and (2.26), when $A(u) \in C_b(R^3, R^{3\times 3})$, $g \in C_b^1(R^3)$, and $f \in C_b(R^3)$, Theorem 2.9 is right. Similarly to [17], these presuppositions can be omitted according to the finite extensive method of nonlinear function. In fact, by (2.23) and $\|e_x^n\| \le (2/h)\|e^n\| \le Ch$, when $k, h \to 0$, $u^n \in S_\varepsilon(U)$, according to the definition about $A^*(u)$ and $g^*(v)$ mentioned in [17] ($f^*(u)$ can be defined accordingly), we have $A^*(u) = A(u)$ and $g^*(u_{\hat{x}}^n) = g(u_{\hat{x}}^n)$ and $f^*(u^n) = f(u^n)$. So this theorem and also hold when $A(u) \in C^0(R^3, R^{3\times 3})$, $g(u) \in C^1(R^3)$, $f(u) \in C(R^3)$. Here, we finish the proof of the first inequality of Theorem 2.9. □

By discrete Sobolev's inequality [18] and the first inequality of Theorem 2.9, the second inequality in this theorem can be proved.

Similarly, we have the stability theorem about the difference equation.

Theorem 2.10. *Let u^n be the solution of* (1.9)*, v^n is the solution of* (1.9) *under the disturbance $u_0(x) + \delta(x)$. $g(u) \in C^1(R^3)$. For any positive constant δ, if $(k/h^2) \le (\alpha_1 - \sigma)/6(\alpha_1 + |\alpha_2|M_1)^2$, then there exist constant C_i $(i = 5, 6, 7)$ which is independent on h and k. When $h \le C_5$, $k \le C_6$, one has*

$$\sup_{0 \le n \le [T/k]} \left(\|u^n - v^n\| + |||u_x^n - v_x^n|||\right) \le C_7 \|\delta_0\|, \tag{2.27}$$

where $M_1 = m \sup_{u \in S_\varepsilon(U)} \|A(u)\|$ $(m > 1)$.

3. Numerical Experiment and Its Error Analysis

In this section, we propose the numerical examples and the error analysis of the solutions. Three subcases of (1.3) mentioned in Section 1 will be performed in our simulation respectively.

Conveniently for computation, first we rewrite (1.9) as the following form:

$$\begin{gathered} u_j^{n+1} = u_j^n + k\left[\left(\alpha_1 + \alpha_2 A\left(u_j^n\right)\right) u_{jx\hat{x}}^n + \alpha_1 \left|u_{j\hat{x}}^n\right|^2 u_j^n + f\left(u_j^n\right)\right], \quad j = 1, \ldots, J;\ n = 0, \ldots, \left[\frac{T}{k}\right] - 1, \\ u_j^0 = u_0(x_j), \quad j = 1, \ldots, J, \\ u_0^n = u_J^n; \quad u_{J+1}^n = u_1^n, \quad n = 1, \ldots, \left[\frac{T}{k}\right]. \end{gathered} \tag{3.1}$$

According to (3.1), in the first step, we can get the value of u^0 on u_j^0. Second, according to second equation of (3.1), we can also get the solution u^{n+1} step by step. In each step of computation, the third equation of (3.1) will be used repeatedly.

(i) Setting $\alpha_1 = \alpha_2 = 1$ and $f(x, t, u) = u \times (0, 0, 1)^T$, we consider the following spin wave of $u = u(x - ct)$:

$$u = \left(\sqrt{1 - s_0^2} \cos \xi, \sqrt{1 - s_0^2} \sin \xi, s_0\right)^T, \tag{3.2}$$

where $\xi = x - t, s_0 = 0$. In fact, we found that (3.2) is the solution of (1.3), where $\xi = ax - bt + c, s_0 = 0$. As far as we know, exact solutions of this case were still not constructed. For the simplicity, we have omitted the details of our constructing.

In accordance with (3.2), the initial and boundary conditions of this equation can be proposed. Let us first consider a domain $x \in [0, 6\pi]$ with the Dirichlet boundary condition on the spin vector. We have implemented (3.1) where $k = 0.0001, h = 1/5$ in numerical resolution. Figures 1(a) and 1(b) show the numerical solutions in time $t = 0.1$, and $t = 0.5$ respectively.

From Figure 1, we observe that the numerical solutions exhibit an irregular changing at the beginning of the space steps. This will happen in the range of probably 1–10 space steps in Figure 1(a) as well as mainly in 1–20 steps in Figure 1(b). This can be seen more clearly in Figure 2 which exhibit the error about the solution in Figure 1 accordingly.

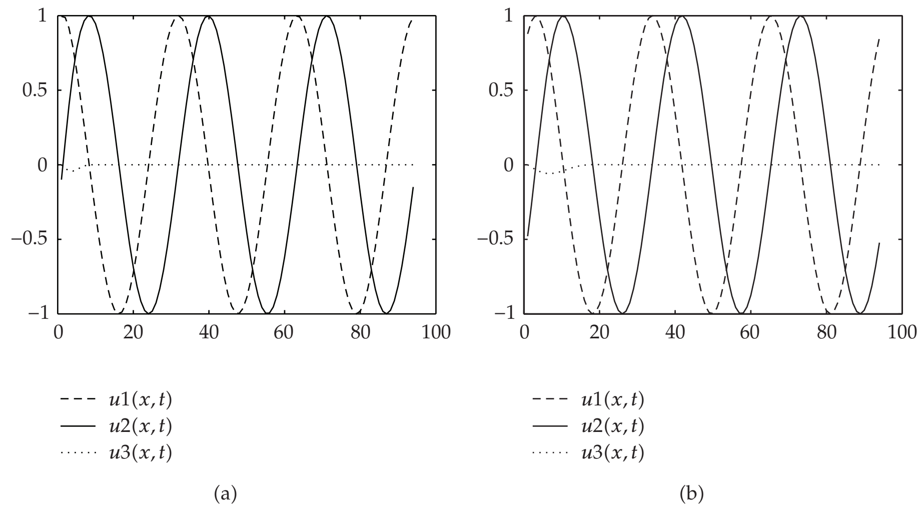

Figure 1: The solution u at (a) $t = 0.1$, and (b) $t = 0.5$; $k = 0.0001, h = 1/5$.

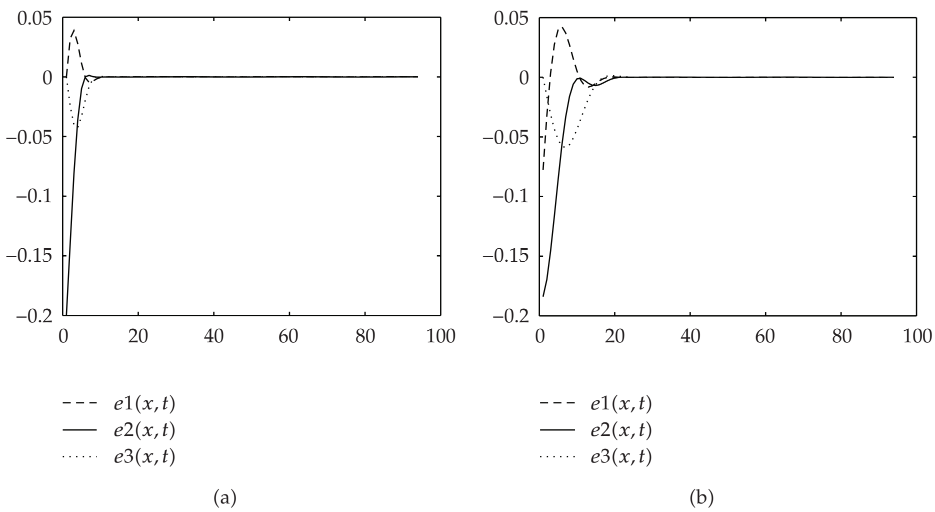

Figure 2: The error of u at (a) $t = 0.1$ and (b) $t = 0.5$; $k = 0.0001, h = 1/5$.

Observed from Figures 1 and 2, probably from the 10 or 20 space steps, the numerical solution is credible. But from then on, the creditable region about the solution is gradually shrinking when the time increase. At space step 50, by amplifying solution error image Figure 2, we found that the magnitude of error is approximately 10^{-4} which is consist with our Theorem 2.9. These details can be seen in Figure 3.

(ii) Setting $\alpha_1 = \alpha_2 = 1$, and $f(x,t,u) = 0$ in (1.3), the solitary solution which proposed in [19] can be written as (3.3) (here we can set $\delta = 2$). For this case, according to (3.3), the initial boundary conditions can be given in our numerical scheme (3.1). Furthermore, we mention here at (3.3) is not a periodic solution, but we can extend the problem into a periodic one. In fact, we take the truncated domain as $\Omega = [0, 4\pi]$. $u^j(4\pi, t) \approx 0$ and the smoothness of the solution ensure the extend can be done. Thus the finite difference scheme

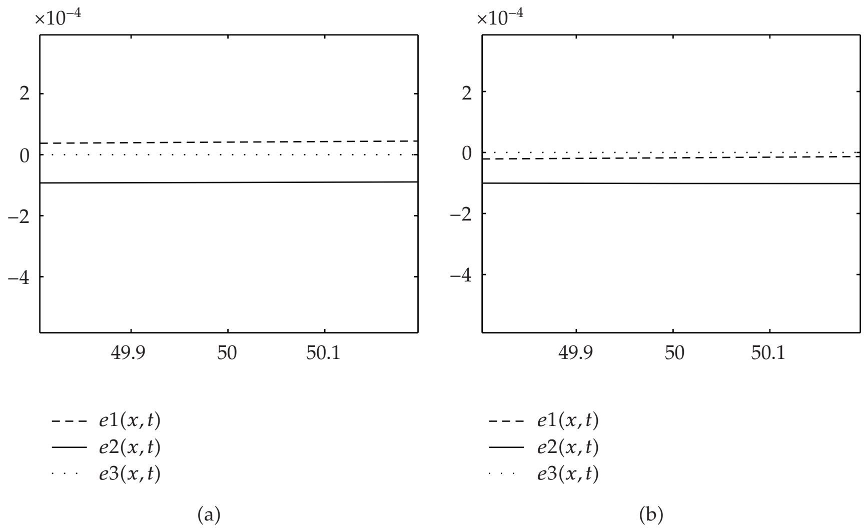

Figure 3: The error of u at (a) $t = 0.1$ and (b) $t = 0.5$ at space step 50; $k = 0.0001, h = 1/5$.

(3.1) still can be used for computation. Figure 4 shows the numerical solution in the $t = 0.3$ and the corresponding error image. Seen in Figure 4, we find that solution error is quite large. Therefore, we suspect that (3.3) which is proposed in [19] is not correct. Moreover, a simple symbol computation by Maple confirms our speculation:

$$\begin{aligned}
u^1 &= \operatorname{sech}\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\left\{\tanh\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\sin\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\right.\\
&\qquad\left.-\cos\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\right\},\\
u^2 &= -\operatorname{sech}\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\left\{\tanh\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\cos\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\right.\\
&\qquad\left.+\sin\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right]\right\},\\
u^3 &= \tanh^2\left[\left(\frac{\sqrt{3}x}{4}\right)(t+\delta)^{-1/2}\right].
\end{aligned} \tag{3.3}$$

(iii) Let $\alpha_1 = 0, \alpha_2 = 1$, and $f(x,t,u) = (u^1,u^2,u^3)^T \times (\operatorname{diag}(1,2,3)\cdot(u^1,u^2,u^3)^T)$ in (1.3). This is special case of LL equation (1.2) with an easy plane. Under situation (3.2), we can easily offer the initial-boundary condition about (3.1).

When $\alpha_1 = 0$, Theorem 2.9 obviously can not give estimate about k/h^2. But when we follow (3.2) to set initial boundary value, in addition to numerical viscosity began in the outside of space step, solution is very irregular. Therefore, we believe that the numerical

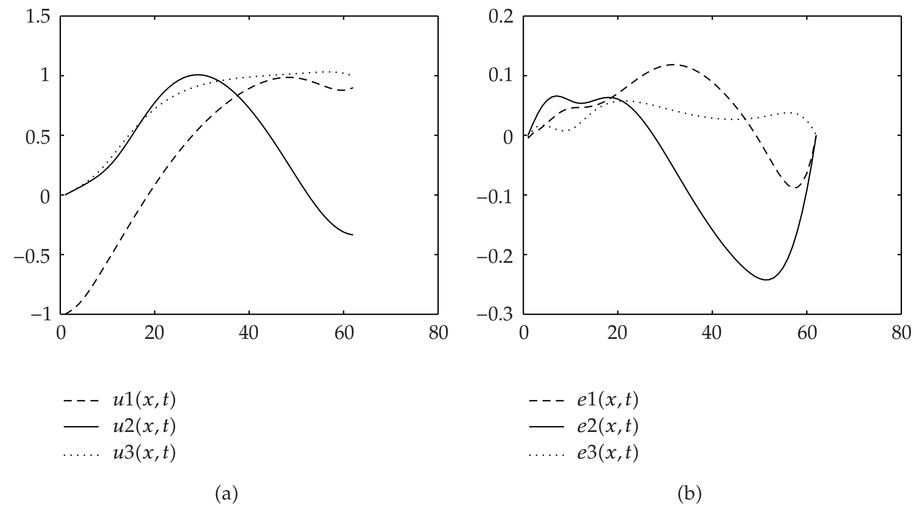

Figure 4: (a) Solution of u at $t = 0.3$, (b) error of u at $t = 0.3$; $k = 0.0001, h = 1/5$.

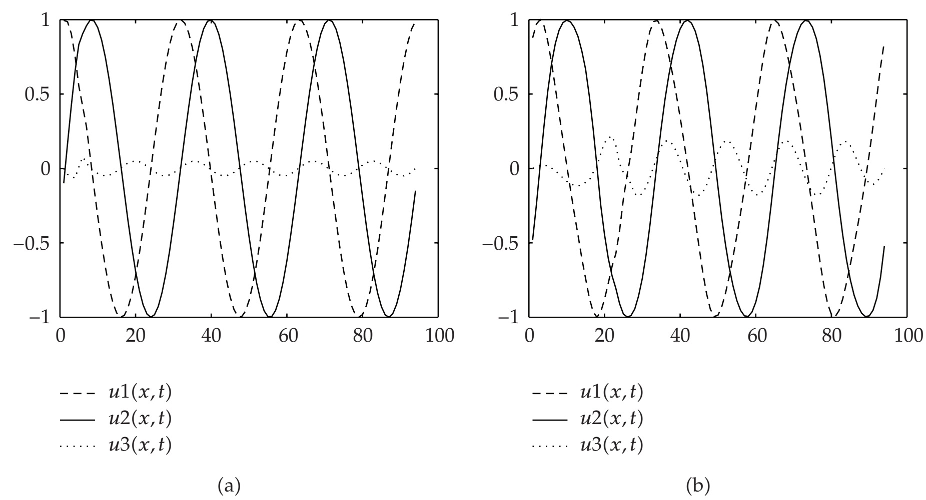

Figure 5: The solution u at (a) $t = 0.1$ and (b) $t = 0.5$; $k = 0.0001, h = 1/5$.

solution is convergent. Nevertheless, for a better estimation, we should try to use other methods to estimate convergence rate of the discrete form solution. From Figure 5, we found that the evolution of u^1 and u^2 is similar to subcase (i) mentioned above; compared to subcase (i), the evolution of u^3 exhibits a larger undulate behavior.

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