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Research Article

Sign-Changing Solutions for Discrete Second-Order Three-Point Boundary Value Problems

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We consider the second-order three-point discrete boundary value problem. By using the topological degree theory and the fixed point index theory, we provide sufficient conditions for the existence of sign-changing solutions, positive solutions, and negative solutions. As an application, an example is given to demonstrate our main results.

1. Introduction

In this paper, we consider the following second-order three-point discrete boundary value problem (BVP):

$$\begin{gathered} \Delta^2 u(t-1) + f(t, u(t)) = 0, \quad t \in [1, n], \\ u(0) = 0, \qquad u(n+1) = \alpha u(m), \end{gathered} \tag{1.1}$$

where $n \in \{2, 3, \ldots\}$, $[1, n]$ is the discrete interval $\{1, 2, \ldots, n\}$, $m \in [1, n]$, $0 \le \alpha \le 1$, $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, and $f : [1, n] \times \mathbf{R} \to \mathbf{R}$ is a continuous function.

Boundary value problems for difference equations arise in different areas of applied mathematics and physics. Existence and multiplicity of positive solutions or nontrivial solutions for boundary value problems of difference equations have been extensively studied in the literature; see [1–9] and the references therein.

On the other hand, in the existing literature, there are some papers studying the sign-changing solutions for boundary value problems of differential equations; for example, see [10–12]. But the problems of the existence of sign-changing solutions to discrete multipoint boundary value problems have received very little attention in the literature to the best knowledge of the authors. In this paper, motivated by [12, 13], we aim to study the existence of multiple sign-changing solutions to the second-order three-point discrete boundary value problem (1.1). Under some suitable conditions, we prove that the three-point discrete boundary value problem (1.1) has at least two sign-changing solutions, two positive solutions, and two negative solutions. The main approach is the topological degree theory and the fixed point index theory.

The organization of this paper is as follows. In Section 2, we present some preliminary knowledge about the topological degree theory and the fixed point index theory and use the knowledge to obtain some lemmas which are very crucial in our main results. In Section 3, by computing the topological degree and the fixed point index, we discuss the existence of multiple sign-changing solutions to BVP (1.1), and a simple example is given.

2. Preliminaries

As we have mentioned, we will use the theory of the Leray-Schauder degree and the fixed point index in a cone to prove our main existence results. Let us collect some results that will be used below. One can refer to [13–16] for more details.

Lemma 2.1 (see [13, 14]). *Let E be a Banach space and, $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If there exists $x_0 \in X \setminus \{\theta\}$ such that*

$$x - Ax \neq \mu x_0, \quad \forall x \in X \cap \partial\Omega, \ \mu \geq 0, \tag{2.1}$$

then the fixed point index $i(A, X \cap \Omega, X) = 0$.

Lemma 2.2 (see [13, 14]). *Let E be a Banach space and let $X \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E, $\theta \in \Omega$. Suppose that $A : X \cap \overline{\Omega} \to X$ is a completely continuous operator. If*

$$Ax \neq \mu x, \quad \forall x \in X \cap \partial\Omega, \ \mu \geq 1. \tag{2.2}$$

then the fixed point index $i(A, X \cap \Omega, X) = 1$.

Lemma 2.3 (see [15]). *Let E be a Banach space, let Ω be a bounded open subset of E, $\theta \in \Omega$, and $A : \overline{\Omega} \to E$ be completely continuous. Suppose that*

$$\|Ax\| \leq \|x\|, \quad Ax \neq x, \quad \forall x \in \partial\Omega, \tag{2.3}$$

then $\deg(I - A, \Omega, \theta) = 1$.

Lemma 2.4 (see [16]). *Let A be a completely continuous operator which is defined on a Banach space E. Let $x_0 \in E$ be a fixed point of A and assume that A is defined in a neighborhood of x_0 and Fréchet differentiable at x_0. If 1 is not an eigenvalue of the linear operator $A'(x_0)$, then x_0 is an isolated singular point of the completely continuous vector field $I - A$ and for small enough $r > 0$,*

$$\deg(I - A, B(x_0, r), \theta) = (-1)^k, \tag{2.4}$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(x_0)$ in $(1, +\infty)$.

Lemma 2.5 (see [16]). *Let A be a completely continuous operator which is defined on a Banach space E. Assume that 1 is not an eigenvalue of the asymptotic derivative. Then the completely continuous vector field $I - A$ is nonsingular on spheres $S_\rho = \{x \in E : \|x\| = \rho\}$ of sufficiently large radius ρ and*

$$\deg(I - A, B(\theta, \rho), \theta) = (-1)^k, \tag{2.5}$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(\infty)$ in $(1, +\infty)$.

From [12, Lemma 2.4], we have the following lemma.

Lemma 2.6. *Let X be a solid cone of a Banach space E (X° is nonempty), let Ω be a relatively bounded open subset of X, and let $A : X \to X$ be a completely continuous operator. If any fixed point of A in Ω is an interior point of X, there exists an open subset O of E ($O \subset \Omega$) such that*

$$\deg(I - A, O, \theta) = i(A, \Omega, X). \tag{2.6}$$

Now we shall consider the space

$$E = \{u : [0, n+1] \longrightarrow \mathbf{R} \mid u(0) = 0,\ \ u(n+1) = \alpha u(m)\} \tag{2.7}$$

equipped with the norm $\|u\| = \max_{t\in[0,n+1]}|u(t)|$. Clearly E is a $n-$dimensional Banach space. Choose the cone $P \subset E$ defined by

$$P = \{u \in E \mid u(t) \geq 0, t \in [1, n]\}. \tag{2.8}$$

Obviously, the interior of P is $P^\circ = \{u \in E \mid u(t) > 0, t \in [1, n]\}$. For each $u, v \in E$, we write $u \geq v$ if $u(t) \geq v(t)$ for $t \in [1, n]$. A solution u of BVP (1.1) is said to be a positive solution (a negative solution, resp.) if $u \in P \setminus \{\theta\}$ ($u \in (-P) \setminus \{\theta\}$, resp.). A solution u of BVP (1.1) is said to be a sign-changing solution if $u \notin P \cup (-P)$.

Lemma 2.7. *Let $v : [1, n] \to R$ be fixed. Then the problem*

$$\Delta^2 u(t-1) + v(t) = 0, \quad t \in [1, n], \qquad u(0) = 0, \quad u(n+1) = \alpha u(m) \tag{2.9}$$

has a unique solution

$$u(t) = \sum_{k=1}^{n} G(t,k) v(k), \tag{2.10}$$

where $G(t,k)$ is given by

$$G(t,k) = \begin{cases} \dfrac{k(n+1-\alpha m - t + \alpha t)}{n+1-\alpha m}, & k \in [1, t-1] \cap [1, m-1]; \\ \dfrac{t(n+1-\alpha m - k + \alpha k)}{n+1-\alpha m}, & k \in [t, m-1]; \\ \dfrac{k(n+1-\alpha m - t) + \alpha m t}{n+1-\alpha m}, & k \in [m, t-1]; \\ \dfrac{t(n+1-k)}{n+1-\alpha m}, & k \in [t, n] \cap [m, n]. \end{cases} \tag{2.11}$$

Proof. We use a similar approach to that in [17, Lemmas 3.1, 3.3]. From $\Delta^2 u(t-1) + v(t) = 0$, we have

$$\Delta u(1) - \Delta u(0) + v(1) = 0, \quad \Delta u(2) - \Delta u(1) + v(2) = 0, \ldots, \Delta u(t) - \Delta u(t-1) + v(t) = 0. \tag{2.12}$$

Suming the above equations, one gets

$$\Delta u(t) = \Delta u(0) - \sum_{i=1}^{t} v(i), \tag{2.13}$$

where, and in what follows, we denote $\sum_{k=s}^{l} x(k) = 0$ when $l < s$. Again summing (2.13) from 0 to $t-1$, it follows that

$$u(t) = u(0) + t\Delta u(0) - \sum_{k=1}^{t-1} (t-k) v(k) \tag{2.14}$$

where $t \in [0, n+1]$. Since $u(0) = 0$ and $u(n+1) = \alpha u(m)$, one gets

$$\Delta u(0) = \frac{1}{n+1-\alpha m} \sum_{k=1}^{n} (n+1-k) v(k) - \frac{\alpha}{n+1-\alpha m} \sum_{k=1}^{m-1} (m-k) v(k). \tag{2.15}$$

By (2.14) and (2.15), we have

$$u(t) = \frac{t}{n+1-\alpha m}\sum_{k=1}^{n}(n+1-k)v(k) - \frac{\alpha t}{n+1-\alpha m}\sum_{k=1}^{m-1}(m-k)v(k) - \sum_{k=1}^{t-1}(t-k)v(k), \quad t \in [0, n+1]. \tag{2.16}$$

When $t > m$, it follows from (2.16) that

$$\begin{aligned} u(t) &= \frac{t}{n+1-\alpha m}\left(\sum_{k=1}^{m-1}(n+1-k)v(k) + \sum_{k=m}^{t-1}(n+1-k)v(k) + \sum_{k=t}^{n}(n+1-k)v(k)\right) \\ &\quad - \frac{\alpha t}{n+1-\alpha m}\sum_{k=1}^{m-1}(m-k)v(k) - \sum_{k=1}^{m-1}(t-k)v(k) - \sum_{k=m}^{t-1}(t-k)v(k) = \sum_{k=1}^{n}G(t,k)v(k). \end{aligned} \tag{2.17}$$

When $t \le m$, it follows from (2.16) that

$$\begin{aligned} u(t) &= \frac{t}{n+1-\alpha m}\left(\sum_{k=1}^{t-1}(n+1-k)v(k) + \sum_{k=t}^{m-1}(n+1-k)v(k) + \sum_{k=m}^{n}(n+1-k)v(k)\right) \\ &\quad - \frac{\alpha t}{n+1-\alpha m}\left(\sum_{k=1}^{t-1}(m-k)v(k) + \sum_{k=t}^{m-1}(m-k)v(k)\right) - \sum_{k=1}^{t-1}(t-k)v(k) \\ &= \sum_{k=1}^{n}G(t,k)v(k). \end{aligned} \tag{2.18}$$

Then, the unique solution of (2.9) can be written as $u(t) = \sum_{k=1}^{n} G(t,k)v(k)$. □

Remark 2.8. Green's function $G(t,k)$ defined by Lemma 2.7 is positive on $[1,n] \times [1,n]$.

Define operators $K, \mathrm{f}, A : E \to E$, respectively, by

$$\begin{aligned} (Ku)(t) &= \sum_{k=1}^{n}G(t,k)u(k), \quad u \in E, t \in [1,n]; \\ (\mathrm{f}u)(t) &= f(t,u(t)), \quad u \in E, t \in [1,n]; \end{aligned} \tag{2.19}$$

$$A = K\mathrm{f}. \tag{2.20}$$

Now from Lemma 2.7, it is easy to see that BVP (1.1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator A. It follows from the continuity of f that $A : E \to E$ is completely continuous.

We shall use the following assumptions.

(H_1) We have $0 \le \alpha < 1$, or

$$\alpha = 1, \qquad \Lambda = \phi, \tag{2.21}$$

where $\Lambda = A \cap B$, and

$$A = \left\{ \frac{(2k-1)\pi}{n+1+m} : k = 1,2,\ldots,\left[\frac{n+1+m}{2}\right] \right\},$$
$$B = \begin{cases} \left\{ \dfrac{2t\pi}{n+1-m} : t = 1,2,\ldots,\left[\dfrac{n-m}{2}\right] \right\}, & n-m \ge 2, \\ \phi, & n-m < 2. \end{cases} \tag{2.22}$$

$[x]$ denotes the integer part of the real number x.

(H_2) For any $t \in [1,n]$, $f(t,0) = 0$; for any $t \in [1,n]$ and $x \in \mathbf{R}$, $xf(t,x) \ge 0$.

(H_3) There exists an even number $k_0 \in [1,n]$ such that

$$\frac{1}{\lambda_{k_0}} < \beta_0 < \frac{1}{\lambda_{k_0+1}}, \tag{2.23}$$

where $\lim_{x\to 0}(f(t,x)/x) = \beta_0$ uniformly for $t \in [1,n]$, and $\lambda_k^{-1} = 4\sin^2(\xi_k/2), k \in [1,n]$, $\xi_1,\ldots,\xi_n$ are given in Lemma 2.9, $\lambda_{n+1}^{-1} \triangleq +\infty$.

(H_4) There exists an even number $k_1 \in [1,n]$ such that

$$\frac{1}{\lambda_{k_1}} < \beta_\infty < \frac{1}{\lambda_{k_1+1}}, \tag{2.24}$$

where $\lim_{|x|\to\infty}(f(t,x)/x) = \beta_\infty$ uniformly for $t \in [1,n]$, and $\lambda_1^{-1},\ldots,\lambda_n^{-1},\lambda_{n+1}^{-1}$ are given in condition (H_3).

(H_5) There exists a constant $T > 0$ such that for any $(t,x) \in [1,n] \times [-T,T]$,

$$|f(t,x)| < \rho^{-1}T, \tag{2.25}$$

where $\rho = [(n+1)^2 - \alpha m^2]^2/8(n+1-\alpha m)^2$.

Lemma 2.9. *Suppose that* (H_1) *holds; then there exist* $\xi_1,\xi_2,\ldots,\xi_n$ *with* $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \pi$ *such that* $\sin(n+1)\xi_i = \alpha \sin m\xi_i$, $i = 1,2,\ldots,n$.

Proof. First, suppose that $0 \le \alpha < 1$. Let $h(x) = \sin(n+1)x - \alpha \sin mx$, then we have

$$h\left(\frac{\pi}{2(n+1)}\right) > 0, \qquad h\left(\frac{\pi}{n+1}\right) \le 0, \qquad h\left(\frac{3\pi}{2(n+1)}\right) < 0,$$
$$h\left(\frac{5\pi}{2(n+1)}\right) > 0, \qquad h\left(\frac{7\pi}{2(n+1)}\right) < 0, \ldots, (-1)^n h\left(\frac{(2n+1)\pi}{2(n+1)}\right) > 0. \tag{2.26}$$

It follows from the intermediate value theorem that there exist $\xi_1 \in (\pi/2(n+1), \pi/n+1]$ and $\xi_i \in ((2i-1)\pi/2(n+1), (2i+1)\pi/2(n+1)), i = 2,\ldots,n$, such that $\sin(n+1)\xi_i = \alpha \sin m\xi_i$, $i = 1,2,\ldots,n$.

Now suppose that $\alpha = 1$ and $\Lambda = \phi$. Let $\Lambda_1 = A \cup B$. It is easy to see that there exist $\zeta_i \in \Lambda_1, i = 1,2,\ldots,n$ with $0 < \zeta_1 = (\pi/n+1+m) < \zeta_2 < \cdots < \zeta_n < \pi$ such that $\sin(n+1)\zeta_i = \sin m\zeta_i$. Then Lemma 2.9 is proved. □

Remark 2.10. Condition $\Lambda = \phi$ is reasonable. For example, let $n = 5$, $m = 3$, then $\Lambda = \phi$. Let $n = 18$, $m = 3$, then $\xi = \pi/2 \in \Lambda \ne \phi$.

Lemma 2.11. *Suppose that* (H_1) *holds; then the set of eigenvalues of the linear operator* K *consists of the strictly decreasing finite sequence of* $\lambda_k, k = 1,2,\ldots,n$, *with corresponding eigenfunctions* $\varphi_k(t) = \sin(t\xi_k)$, *where* $\lambda_k = (4\sin^2(\xi_k/2))^{-1}$, $k = 1,2,\ldots,n$, *and* $\xi_1,\ldots,\xi_n$ *are given in Lemma 2.9. In addition, the algebraic multiplicity of each eigenvalue* λ_k *of the linear operator* K *is equal to* 1.

Proof. It is easy to see that

$$Ku(t) = \lambda u(t), \quad t \in [1,n], \ u \in E \tag{2.27}$$

is equivalent to the following equation:

$$\lambda \Delta^2 u(t-1) + u(t) = 0, \quad t \in [1,n],$$
$$u(0) = 0, \qquad u(n+1) = \alpha u(m). \tag{2.28}$$

By Lemma 2.9, we suppose that $\varphi_k(t) = \sin(t\xi_k)$ is a nontrivial solution of (2.28). Then,

$$\lambda(\sin(t+1)\xi_k - 2\sin t\xi_k + \sin(t-1)\xi_k) + \sin t\xi_k = 0. \tag{2.29}$$

Hence, for any $k \in [1,n]$, $\lambda = \lambda_k = (4\sin^2(\xi_k/2))^{-1}$ is an eigenvalue of the linear operator K with the corresponding eigenfunction $\varphi_k(t) = \sin(t\xi_k)$. Since the linear operator K is identified with a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$, the set of eigenvalues of the linear operator K consists of the strictly decreasing finite sequence of $\lambda_k, k = 1,2,\ldots,n$. Obviously, the algebraic multiplicity of each eigenvalue λ_k of K is equal to 1. This completes the proof. □

Remark 2.12. When $\alpha = 0$, we see that BVP (1.1) is reduced to Dirichlet boundary value problem and $\lambda_k = (4\sin^2(k\pi/2(n+1)))^{-1}$, $k = 1,2,\ldots,n$. When $\alpha = 1$ and $m = n$, BVP (1.1) is reduced to the focal boundary value problem and $\lambda_k = (4\sin^2(2k-1)\pi/4n+2)^{-1}$, $k = 1,2,\ldots,n$.

Lemma 2.13. *Suppose that* (H_2) *holds, and* $u = (u_1,u_2,\ldots,u_n)^T \in P \setminus \{\theta\}$ *is a solution of BVP* (1.1). *Then* $u \in P^\circ$.

Proof. If $u(t) = 0$ for some $t \in [1,n]$, then

$$u(t+1) + u(t-1) = \Delta^2 u(t-1) = -f(t,u(t)) = 0. \tag{2.30}$$

So $u(t \pm 1) = 0$, and it follows that if u is zero somewhere in $[1,n]$, then it vanishes identically in $[1,n]$. □

Remark 2.14. Similarly to Lemma 2.13, we know also that if (H_2) holds and $u \in -P \setminus \{\theta\}$ is a solution of BVP (1.1), then $u \in -P^\circ$.

Lemma 2.15. *Suppose that* (H_2)–(H_4) *hold. Then the operator* A *is Fréchet differentiable at* θ *and* ∞, *where operator* A *is defined by* (2.20). *Moreover,* $A'(\theta) = \beta_0 K$ *and* $A'(\infty) = \beta_\infty K$.

Proof. By (H_3), for any $\varepsilon > 0$, there exist $\delta > 0$ such that $|f(t,x) - \beta_0 x| < \varepsilon|x|$ for any $0 < |x| < \delta$, $t \in [1,n]$. Hence, noticing that $f(t,0) = 0$ for any $t \in [1,n]$, we have

$$\left\| Au - A\theta - \beta_0 Ku \right\| = \left\| K(\mathrm{f}u - \beta_0 u) \right\| \le \|K\| \cdot \max_{t\in[1,n]} \left| f(t,u(t)) - \beta_0 u(t) \right| < \varepsilon \|K\| \cdot \|u\| \tag{2.31}$$

for any $u \in E$ with $0 < \|u\| < \delta$, where $\|K\| = \max_{t\in[1,n]} \sum_{k=1}^{n} |G(t,k)|$. Consequently,

$$\lim_{\|u\|\to 0} \frac{\left\| Au - A\theta - \beta_0 Ku \right\|}{\|u\|} = 0. \tag{2.32}$$

This means that the nonlinear operator A is Fréchet differentiable at θ, and $A'(\theta) = \beta_0 K$.

By (H_4), for any $\varepsilon > 0$, there exist $M > 0$ such that $|f(t,x) - \beta_\infty x| < \varepsilon|x|$ for any $|x| > M$, $t \in [1,n]$. Let $c = \max_{(t,x)\in[1,n]\times[-M,M]} |f(t,x) - \beta_\infty x|$. By the continuity of $f(t,x)$ with respect to x, we have $c < +\infty$. Then, for any $(t,x) \in [1,n] \times \mathbf{R}$, $|f(t,x) - \beta_\infty x| < \varepsilon|x| + c$. Thus

$$\left\| Au - \beta_\infty Ku \right\| \le \|K\| \cdot \max_{t\in[1,n]} \left| f(t,u(t)) - \beta_\infty u(t) \right| < \|K\| (\varepsilon \|u\| + c) \tag{2.33}$$

for any $u \in E$. Consequently,

$$\lim_{\|u\|\to \infty} \frac{\left\| Au - \beta_\infty Ku \right\|}{\|u\|} = 0, \tag{2.34}$$

which implies that operator A is Fréchet differentiable at ∞, and $A'(\infty) = \beta_\infty K$. The proof is completed. □

Lemma 2.16. *Let T be given in condition (H_5). Suppose that (H_1)–(H_4) hold. Then, $A(P) \subset P$, $A(-P) \subset -P$. Moreover, one has the following.*

(i) *There exists an $r_0 \in (0,T)$ such that for any $0 < r \leq r_0$,*

$$i(A, P \cap B(\theta, r), P) = 0, \qquad i(A, -P \cap B(\theta, r), -P) = 0. \tag{2.35}$$

(ii) *There exists an $R_0 > T$ such that for any $R \geq R_0$,*

$$i(A, P \cap B(\theta, R), P) = 0, \qquad i(A, -P \cap B(\theta, R), -P) = 0. \tag{2.36}$$

Proof. By (H_2) and the fact that $G(t,k)$ is positive on $[1,n]\times[1,n]$, we get that for any $t \in [1,n]$, $f(t,P) \subset P$, $f(t,-P) \subset -P$, and $K(P) \subset P$, $K(-P) \subset -P$. Then $A(P) \subset P$ and $A(-P) \subset -P$.

We only need to prove conclusion (i). The proof of conclusion (ii) is similar and will be omitted here. Let $\gamma_0 = \inf_{\|u\|=1}\|u - \beta_0 K u\|$. Condition (H_3) yields $\gamma_0 > 0$. It follows from (2.32) that there exists $r_0 \in (0,T)$ such that

$$\left\| Au - \beta_0 K u \right\| < \frac{1}{2}\gamma_0 \|u\|, \tag{2.37}$$

where $0 < \|u\| \leq r_0$. Setting $H(s,u) = sAu + (1-s)\beta_0 K u$, then $H : [0,1] \times E \to E$ is completely continuous. For any $s \in [0,1]$ and $0 < \|u\| \leq r_0$, we obtain that

$$\|u - H(s,u)\| \geq \left\| u - \beta_0 K u \right\| - s\left\| Au - \beta_0 K u \right\| \geq \gamma_0\|u\| - \frac{1}{2}\gamma_0\|u\| > 0. \tag{2.38}$$

According to the homotopy invariance of the fixed point index, for any $0 < r \leq r_0$, we have

$$i(A, P \cap B(\theta, r), P) = i(\beta_0 K, P \cap B(\theta, r), P), \tag{2.39}$$

$$i(A, -P \cap B(\theta, r), -P) = i(\beta_0 K, -P \cap B(\theta, r), -P). \tag{2.40}$$

Let $\varphi_1(t) = \sin(t\xi_1)$. Then $K\varphi_1 = \lambda_1 \varphi_1$ and $\varphi_1 \in P$ (see Lemma 2.11 and the proof of Lemma 2.9). We claim

$$u - \beta_0 K u \neq \sigma \varphi_1, \quad \forall u \in P \cap \partial B(\theta, r),\ \sigma \geq 0. \tag{2.41}$$

Indeed, we assume that there exist $u_1 \in P \cap \partial B(\theta, r)$ and $\sigma_1 \geq 0$ such that $u_1 - \beta_0 K u_1 = \sigma_1 \varphi_1$. Obviously, $u_1 = \beta_0 K u_1 + \sigma_1 \varphi_1 \geq \sigma_1 \varphi_1$. Since $\beta_0 \neq \lambda_k^{-1}, k = 1, 2, \ldots, n$, then $\sigma_1 > 0$. Set $\sigma_{\max} = \sup\{\sigma : u_1 \geq \sigma \varphi_1\}$. It is clear that $\sigma_1 \leq \sigma_{\max} < +\infty$ and $u_1 \geq \sigma_{\max}\varphi_1$. Then

$$u_1 = \beta_0 K u_1 + \sigma_1 \varphi_1 \geq \beta_0 K \sigma_{\max} \varphi_1 + \sigma_1 \varphi_1 = \left(\beta_0 \lambda_1 \sigma_{\max} + \sigma_1\right)\varphi_1. \tag{2.42}$$

Since $\beta_0 \lambda_1 > 1$, then $\beta_0 \lambda_1 \sigma_{\max} + \sigma_1 > \sigma_{\max}$, which contradicts with the definition of $\sigma_{\max}$. This proves (2.41).

It follows from Lemma 2.1 and (2.41) that

$$i(\beta_0 K, P \cap B(\theta, r), P) = 0. \tag{2.43}$$

Similarly to (2.43), we know also that

$$i(\beta_0 K, -P \cap B(\theta, r), -P) = 0. \tag{2.44}$$

By (2.39), (2.43), (2.40), and (2.44), we conclude

$$i(A, P \cap B(\theta, r), P) = 0, \qquad i(A, -P \cap B(\theta, r), -P) = 0. \tag{2.45}$$

□

3. Main Results

Now with the aid of the lemmas in Section 2, we are in position to state and prove our main results.

Theorem 3.1. *Assume that the conditions (H_1)–(H_5) hold. Then BVP* (1.1) *has at least two sign-changing solutions. Moreover, BVP* (1.1) *has at least two positive solutions and two negative solutions.*

Proof. From the proof of Lemma 2.7, we have

$$\begin{aligned}\sum_{k=1}^{n} G(t,k) &= \frac{t}{n+1-\alpha m}\sum_{k=1}^{n}(n+1-k) - \frac{\alpha t}{n+1-\alpha m}\sum_{k=1}^{m-1}(m-k) - \sum_{k=1}^{t-1}(t-k) \\ &= \frac{t}{n+1-\alpha m}\cdot\frac{n(n+1)}{2} - \frac{\alpha t}{n+1-\alpha m}\cdot\frac{m(m-1)}{2} - \frac{t(t-1)}{2} \\ &\le \frac{\left[(n+1)^2 - \alpha m^2\right]^2}{8(n+1-\alpha m)^2} = \rho.\end{aligned} \tag{3.1}$$

Since $G(t,k)$ is positive on $[1,n]\times[1,n]$, by (H_5), we have for any $u \in E$ with $\|u\| = T$,

$$\begin{aligned}|(Au)(t)| &= \left|\sum_{k=1}^{n} G(t,k) f(k,u(k))\right| \le \sum_{k=1}^{n} G(t,k)\left|f(k,u(k))\right| \\ &< \rho^{-1} T \sum_{k=1}^{n} G(t,k) \le T, \quad \forall t \in [1,n].\end{aligned} \tag{3.2}$$

This gives

$$\|Au\| < T = \|u\|. \tag{3.3}$$

By (3.3) and Lemmas 2.3 and 2.2, we have

$$\deg(I - A, B(\theta, T), \theta) = 1, \tag{3.4}$$

$$i(A, P \cap B(\theta, T), P) = 1, \tag{3.5}$$

$$i(A, -P \cap B(\theta, T), -P) = 1. \tag{3.6}$$

From (H_3) and Lemma 2.11, one has that the eigenvalues of the operator $A'(\theta) = \beta_0 K$ which are larger than 1 are

$$\beta_0 \lambda_1, \beta_0 \lambda_2, \dots, \beta_0 \lambda_{k_0}, \tag{3.7}$$

From (H_4) and Lemma 2.11, one has that the eigenvalues of the operator $A'(\infty) = \beta_\infty K$ which are larger than 1 are

$$\beta_\infty \lambda_1, \beta_\infty \lambda_2, \dots, \beta_\infty \lambda_{k_1}. \tag{3.8}$$

It follows from Lemmas 2.4 and 2.5 that there exist $0 < r_1 < r_0$ and $R_1 > R_0$ such that

$$\deg(I - A, B(\theta, r_1), \theta) = (-1)^{k_0} = 1, \tag{3.9}$$

$$\deg(I - A, B(\theta, R_1), \theta) = (-1)^{k_1} = 1, \tag{3.10}$$

where r_0 and R_0 are given in Lemma 2.16. Owing to Lemma 2.16, one has

$$i(A, P \cap B(\theta, r_1), P) = 0, \tag{3.11}$$

$$i(A, -P \cap B(\theta, r_1), -P) = 0, \tag{3.12}$$

$$i(A, P \cap B(\theta, R_1), P) = 0, \tag{3.13}$$

$$i(A, -P \cap B(\theta, R_1), -P) = 0. \tag{3.14}$$

According to the additivity of the fixed point index, by (3.5), (3.11), and (3.13), we have

$$i\Big(A, P \cap \Big(B(\theta, T) \setminus \overline{B(\theta, r_1)}\Big), P\Big) = i(A, P \cap B(\theta, T), P) - i(A, P \cap B(\theta, r_1), P) = 1 - 0 = 1, \tag{3.15}$$

$$i\Big(A, P \cap \Big(B(\theta, R_1) \setminus \overline{B(\theta, T)}\Big), P\Big) = i(A, P \cap B(\theta, R_1), P) - i(A, P \cap B(\theta, T), P) = 0 - 1 = -1. \tag{3.16}$$

Hence, the nonlinear operator A has at least two fixed points $u_1 \in P \cap (B(\theta, T) \setminus \overline{B(\theta, r_1)})$ and $u_2 \in P \cap (B(\theta, R_1) \setminus \overline{B(\theta, T)})$, respectively. Then, u_1 and u_2 are positive solutions of BVP (1.1).

Using again the additivity of the fixed point index, by (3.6), (3.12), and (3.14), we get

$$i\Big(A, -P\cap\Big(B(\theta,T)\setminus\overline{B(\theta,r_1)}\Big), -P\Big) = 1-0 = 1, \tag{3.17}$$

$$i\Big(A, -P\cap\Big(B(\theta,R_1)\setminus\overline{B(\theta,T)}\Big), -P\Big) = 0-1 = -1. \tag{3.18}$$

Hence, the nonlinear operator A has at least two fixed points $u_3 \in -P\cap(B(\theta,T)\setminus\overline{B(\theta,r_1)})$ and $u_4 \in -P\cap(B(\theta,R_1)\setminus\overline{B(\theta,T)})$, respectively. Then, u_3 and u_4 are negative solutions of BVP (1.1). Let

$$\begin{aligned}
\Gamma_1 &= \Big\{u\in P\cap\Big(B(\theta,T)\setminus\overline{B(\theta,r_1)}\Big) : Au=u\Big\},\\
\Gamma_2 &= \Big\{u\in P\cap\Big(B(\theta,R_1)\setminus\overline{B(\theta,T)}\Big) : Au=u\Big\},\\
\Gamma_3 &= \Big\{u\in -P\cap\Big(B(\theta,T)\setminus\overline{B(\theta,r_1)}\Big) : Au=u\Big\},\\
\Gamma_4 &= \Big\{u\in -P\cap\Big(B(\theta,R_1)\setminus\overline{B(\theta,T)}\Big) : Au=u\Big\}.
\end{aligned} \tag{3.19}$$

It follows from Lemmas 2.6, 2.13, Remark 2.14, and (3.15)–(3.18) that there exist open subsets O_1, O_2, O_3, and O_4 of E such that

$$\begin{aligned}
&\Gamma_1\subset O_1\subset P\cap\Big(B(\theta,T)\setminus\overline{B(\theta,r_1)}\Big), \qquad \Gamma_2\subset O_2\subset P\cap\Big(B(\theta,R_1)\setminus\overline{B(\theta,T)}\Big),\\
&\Gamma_3\subset O_3\in -P\cap\Big(B(\theta,T)\setminus\overline{B(\theta,r_1)}\Big), \qquad \Gamma_4\subset O_4\subset -P\cap\Big(B(\theta,R_1)\setminus\overline{B(\theta,T)}\Big),
\end{aligned} \tag{3.20}$$

$$\deg(I-A, O_1, \theta) = 1, \tag{3.21}$$

$$\deg(I-A, O_2, \theta) = -1, \tag{3.22}$$

$$\deg(I-A, O_3, \theta) = 1, \tag{3.23}$$

$$\deg(I-A, O_4, \theta) = -1. \tag{3.24}$$

By (3.4), (3.21), (3.23), (3.9), and the additivity of Leray-Schauder degree, we get

$$\deg\Big(I-A, B(\theta,T)\setminus\Big(\overline{O_1}\cup\overline{O_3}\cup\overline{B(\theta,r_1)}\Big), \theta\Big) = 1-1-1-1 = -2, \tag{3.25}$$

which implies that the nonlinear operator A has at least one fixed point $u_5 \in B(\theta,T)\setminus(\overline{O_1}\cup\overline{O_3}\cup\overline{B(\theta,r_1)})$.

Similarly, by (3.10), (3.22), (3.24), and (3.4), we get

$$\deg\Big(I-A, B(\theta,R_1)\setminus\Big(\overline{O_2}\cup\overline{O_4}\cup\overline{B(\theta,T)}\Big), \theta\Big) = 1+1+1-1 = 2, \tag{3.26}$$

which implies that the nonlinear operator A has at least one fixed point $u_6 \in B(\theta, R_1) \setminus (\overline{O_2} \cup \overline{O_4} \cup \overline{B(\theta, T)})$. Then, u_5 and u_6 are two distinct sign-changing solutions of BVP (1.1). Thus, the proof of Theorem 3.1 is finished. □

Theorem 3.2. *Assume that the conditions* (H_1)–(H_5) *hold, and that* $f(t,x) = -f(t,-x)$ *for* $t \in [1,n]$ *and* $x \in \mathbf{R}$. *Then BVP* (1.1) *has at least four sign-changing solutions. Moreover, BVP* (1.1) *has at least two positive solutions and two negative solutions.*

Proof. It follows from the proof of Theorem 3.1 that BVP (1.1) has at least six different nontrivial solutions $u_i (i = 1, 2, \ldots, 6)$ satisfying that

$$u_1, u_2 \in P^\circ, \qquad u_3, u_4 \in -P^\circ, \qquad u_5, u_6 \notin P \cup (-P), \qquad r_1 < \|u_5\| < \|T\| < \|u_6\| < R_1. \tag{3.27}$$

By the condition that $f(t,x) = -f(t,-x)$ for $t \in [1,n]$ and $x \in \mathbf{R}$, we know that $-u_5$ and $-u_6$ are also solutions of BVP (1.1). Let $u_7 = -u_5$, $u_8 = -u_6$, then $u_i (i = 1, 2, \ldots, 8)$ are different nontrivial solutions of BVP (1.1). The proof is completed. □

By the method used in the proof of Theorems 3.1 and 3.2, we can prove the following corollaries.

Corollary 3.3. *Assume that the conditions* (H_1)–(H_3) *and* (H_5) *or* (H_1), (H_2), (H_4), *and* (H_5) *hold. Then BVP* (1.1) *has at least one sign-changing solution. Moreover, BVP* (1.1) *has at least one positive solution and one negative solution.*

Corollary 3.4. *Assume that the conditions* (H_1)–(H_3) *and* (H_5) *or* (H_1), (H_2), (H_4), *and* (H_5) *hold, and that* $f(t,x) = -f(t,-x)$ *for* $x \in \mathbf{R}$ *and* $t \in [1,n]$. *Then BVP* (1.1) *has at least two sign-changing solutions. Moreover, BVP* (1.1) *has at least one positive solution and one negative solution.*

Next, we present a simple example to which Theorem 3.2 can be applied.

Example 3.5. Consider the second-order three-point discrete boundary value problem

$$\begin{aligned} &\Delta^2 u(t-1) + f(t, u(t)) = 0, \quad t \in [1,4], \\ &u(0) = 0, \qquad u(5) = u(3), \end{aligned} \tag{3.28}$$

where $n = 4$, $m = 3$, $\alpha = 1$, and

$$f(t,x) = \begin{cases} \dfrac{bx}{1+x^2}, & |x| \le 20, \\ \left(18 - \dfrac{2b}{401}\right)x + \dfrac{60b}{401} - 360, & 20 \le x \le 30, \\ \left(18 - \dfrac{2b}{401}\right)x + 360 - \dfrac{60b}{401}, & -30 \le x \le -20, \\ x\left(5 + \dfrac{1000}{100+x^2}\right), & |x| \ge 30, \end{cases} \tag{3.29}$$

$b \in (4\sin^2(3\pi/16), 4\sin^2(5\pi/16))$. Obviously, $\beta_0 = b, \beta_\infty = 5$, and $f(t,x) = -f(t,-x)$ for all $(t,x) \in [1,4] \times \mathbf{R}$.

From Lemma 2.11 and the proof of Lemma 2.9, we know that the set of eigenvalues of the linear operator K (see (2.19)) consists of the strictly decreasing finite sequence of λ_k, $k = 1, 2, 3, 4$, where $\lambda_k = (4\sin^2((2k-1)\pi/16))^{-1}$. Then the conditions (H_1)–(H_4) hold. Since $|f(t,x)| \leq (b/2) < (3/2)$ for all $(t,x) \in [1,4] \times [-12,12]$, then (H_5) holds with $T = 12$ and $\rho = 8$. Therefore, by Theorem 3.2, BVP (3.28) has at least four sign-changing solutions. Moreover, BVP (3.28) has at least two positive solutions and two negative solutions.

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