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Research Article

On a Max-Type Difference Inequality and Its Applications

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We prove a useful max-type difference inequality which can be applied in studying of some max-type difference equations and give an application of it in a recent problem from the research area. We also give a representation of solutions of the difference equation $x_n = \max\{x_{n-1}^{a_1}, \dots, x_{n-k}^{a_k}\}$.

1. Introduction

The investigation of max-type difference equations attracted some attention recently; see, for example, [1–20] and the references therein. In the beginning of the study of these equations the following difference equation was investigated:

$$x_n = \max\left\{\frac{A_n^{(1)}}{x_{n-1}}, \frac{A_n^{(2)}}{x_{n-2}}, \dots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k \in \mathbb{N}$, $A_n^{(i)}$, $i = 1, \dots, k$, are real sequences (mostly constant or periodic), and the initial values $x_{-1}, \dots, x_{-k}$ are different from zero (see, e.g., monograph [6] or paper [19] and the references therein).

The study of the next difference equation

$$x_n = \max\left\{B_n^{(0)}, B_n^{(1)} \frac{x_{n-p_1}^{r_1}}{x_{n-q_1}^{s_1}}, B_n^{(2)} \frac{x_{n-p_2}^{r_2}}{x_{n-q_2}^{s_2}}, \dots, B_n^{(k)} \frac{x_{n-p_k}^{r_k}}{x_{n-q_k}^{s_k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where p_i, q_i are natural numbers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_k$, $r_i, s_i \in \mathbb{R}_+ = [0, \infty)$, $i = 1, \ldots, k$, and $k \in \mathbb{N}$, was proposed by the first author in numerous talks; see, for example, [11, 13]. For some results in this direction see [1, 4, 5, 7, 8, 12, 14–18, 20].

A particular case of the difference equation

$$y_n = \max\left\{\frac{A}{y_{n-1}\cdots y_{n-m+1}}, \frac{1}{y_{n-m-1}\cdots y_{n-2m+1}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

arises naturally in certain models in automatic control (see [9]). By the change $x_n = y_n y_{n-1} \cdots y_{n-m+1}$ the equation is transformed into the equation

$$x_n = \max\left\{A, \frac{x_{n-1}}{x_{n-m}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

which is a special case of (1.2) and which is a natural prototype for the equation.

The following result, which extends the main result from the study in [18] was proved by the first author in [17] (see also [16]).

Theorem A. *Every positive solution to the difference equation*

$$x_n = \max\left\{\frac{A_1}{x_{n-1}^{\alpha_1}}, \frac{A_2}{x_{n-2}^{\alpha_2}}, \ldots, \frac{A_k}{x_{n-k}^{\alpha_k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $-1 < \alpha_i < 1$, $A_i \geq 0$, $i = 1, \ldots, k$, *converges to* $\max_{1\leq i\leq k}\{A_i^{1/(\alpha_i+1)}\}$.

Here we continue to study (1.5) by considering the cases when some of α_i's are equal to one. We also give a representation of well-defined solutions of the difference equation $x_n = \max\{x_{n-1}^{a_1}, \ldots, x_{n-k}^{a_k}\}$, where $a_i \in \mathbb{R}$, $i = 1, \ldots, k$.

2. Main Results

In this section we prove the main results of this note. Before this we formulate the following very useful auxiliary result which can be found in [10] and give a definition.

Lemma A. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of positive numbers which satisfy the inequality*

$$a_{n+k} \leq q \max\{a_{n+k-1}, a_{n+k-2}, \ldots, a_n\}, \quad \text{for } n \in \mathbb{N}, \tag{2.1}$$

where $q > 0$ *and* $k \in \mathbb{N}$ *are fixed. Then there exists an* $M > 0$ *such that*

$$a_n \leq M\left(\sqrt[k]{q}\right)^n, \quad n \in \mathbb{N}. \tag{2.2}$$

Definition 2.1. For a sequence $(x_n)_{n=-s}^{\infty}$, $s \in \mathbb{N}_0$, we say that it converges to zero geometrically if there is a $q \in [0,1)$ and $M > 0$ such that

$$|x_n| \le Mq^n, \tag{2.3}$$

for $n = -s, \dots, -1, 0, 1, \dots$.

Now we are in a position to formulate and prove the main results of this note.

Proposition 2.2. *Assume that $(a_n)_{n=-k}^{\infty}$ is a sequence of nonnegative numbers satisfying the difference inequality*

$$a_n \le \max\{\alpha_1 a_{n-1} - d_1, \dots, \alpha_k a_{n-k} - d_k\}, \quad n \in \mathbb{N}_0, \tag{2.4}$$

where $k \in \mathbb{N}$, $\alpha_i \in [0,1]$, $d_i \in \mathbb{R}_+$, $i \in \{1, \dots, k\}$, and if, for some i, $\alpha_i = 1$, then $d_i > 0$. Then the sequence a_n converges geometrically to zero as $n \to \infty$.

Proof. Let $\beta \in (0,1)$ be chosen such that

$$\max\{a_{-k}, \dots, a_{-1}\} \le \frac{c_m}{1-\beta}, \tag{2.5}$$

where

$$c_m = \min\left\{1, \min_{i:\alpha_i=1}\{d_i\}\right\}. \tag{2.6}$$

Then from (2.4) and using the fact that a_n are nonnegative numbers, we have that

$$a_n \le \max\left\{A \max_{i:\alpha_i \in [0,1)}\{a_{n-i}\}, \max_{j:\alpha_j=1}\{a_{n-j} - c_m\}\right\}, \tag{2.7}$$

where $A = \max_{i:\alpha_i \in [0,1)}\{\alpha_i\}$.

From (2.7), (2.5) and since $0 < \max\{A, \beta\} < 1$, we have that

$$\begin{aligned} a_0 &\le \max\left\{A \max_{i:\alpha_i \in [0,1)}\{a_{-i}\}, \max_{j:\alpha_j=1}\{a_{-j} - c_m\}\right\} \\ &\le \max\left\{\frac{Ac_m}{1-\beta}, \frac{c_m}{1-\beta} - c_m,\right\} \\ &= \max\left\{\frac{Ac_m}{1-\beta}, \frac{\beta c_m}{1-\beta}\right\} < \frac{c_m}{1-\beta}. \end{aligned} \tag{2.8}$$

Now assume that $a_n \leq c_m/(1-\beta)$, for $0 \leq n \leq n_0 - 1$. Then from (2.4) we get

$$\begin{aligned} a_{n_0} &\leq \max\left\{ A \max_{i:\alpha_i \in [0,1)} \{a_{n_0-i}\}, \max_{j:\alpha_j=1}\{a_{n_0-j} - c_m\}\right\} \\ &\leq \max\left\{\frac{Ac_m}{1-\beta}, \frac{c_m}{1-\beta} - c_m,\right\} \\ &= \max\left\{\frac{Ac_m}{1-\beta}, \frac{\beta c_m}{1-\beta}\right\} < \frac{c_m}{1-\beta}. \end{aligned} \tag{2.9}$$

Inequalities (2.8) and (2.9) along with the method of induction show that

$$0 \leq a_n \leq \frac{c_m}{1-\beta}, \quad \text{for } n \in \{-s, \dots, -1\} \cup \mathbb{N}_0. \tag{2.10}$$

Now note that from (2.10) we have that

$$a_n - c_m \leq \beta a_n, \quad \text{for } n \in \{-s, \dots, -1\} \cup \mathbb{N}_0. \tag{2.11}$$

From (2.7), (2.11) and the choice of c_m, it follows that for $n \in \mathbb{N}_0$

$$\begin{aligned} a_n &\leq \max\left\{ A \max_{i:\alpha_i \in [0,1)} \{a_{n-i}\}, \beta \max_{j:\alpha_j=1}\{a_{n-j}\}\right\} \\ &\leq \max\{A, \beta\} \max_{1 \leq i \leq k}\{a_{n-i}\}. \end{aligned} \tag{2.12}$$

Applying Lemma A in inequality (2.12) with $q = \max\{A, \beta\}$, the result follows. □

Remark 2.3. Note that the constant β in the proof of Proposition 2.2 depends on initial conditions of solutions to difference equation (2.4), so that this is not a uniform constant.

Lemma 2.4. *Consider the difference equation*

$$z_n = \min\{C_1 - \alpha_1 z_{n-1}, C_2 - \alpha_2 z_{n-2}, \dots, C_k - \alpha_k z_{n-k}\}, \quad n \in \mathbb{N}_0, \tag{2.13}$$

where $k \in \mathbb{N}$, $C_i \in \mathbb{R}_+$, $\alpha_j \in \mathbb{R}$, $i = 1, \dots, k$, *and there is* $i_0 \in \{1, \dots, k\}$ *such that* $C_{i_0} = 0$. *Then*

$$|z_n| \leq \max\{|\alpha_1||z_{n-1}| - C_1, |\alpha_2||z_{n-2}| - C_2, \dots, |\alpha_k||z_{n-k}| - C_k\}, \quad n \in \mathbb{N}_0. \tag{2.14}$$

Proof. If all terms in the right-hand side of (2.13) are nonnegative then clearly $0 \leq z_n \leq -\alpha_{i_0} z_{n-i_0}$, so that

$$|z_n| \leq |\alpha_{i_0}||z_{n-i_0}| = |\alpha_{i_0}||z_{n-i_0}| - C_{i_0}. \tag{2.15}$$

Otherwise, the set $S \subseteq \{1, \ldots, k\}$ of all indices for which the terms in (2.13) are negative is nonempty, so that $z_n = \min_{i \in S}\{C_i - \alpha_i z_{n-i}\} < 0$.

From this and since for such $i \in S$, $\alpha_i z_{n-i}$ must be positive, it follows that

$$|z_n| = \max_{i \in S}\{\alpha_i z_{n-i} - C_i\} = \max_{i \in S}\{|\alpha_i||z_{n-i}| - C_i\}. \tag{2.16}$$

From (2.15) and (2.16) inequality (2.14) easily follows. □

By Proposition 2.2 and Lemma 2.4 we obtain the following theorem.

Theorem 2.5. *Consider the difference equation*

$$x_n = \max\left\{\frac{A_1}{x_{n-1}^{\alpha_1}}, \frac{A_2}{x_{n-2}^{\alpha_2}}, \ldots, \frac{A_k}{x_{n-k}^{\alpha_k}}\right\}, \quad n \in \mathbb{N}_0, \tag{2.17}$$

where $k \in \mathbb{N}$, $0 \le A_i \le 1, -1 \le \alpha_i \le 1, -1 < \alpha_i A_i < 1$ for each $i \in \{1, \ldots, k\}$, and $A_i = 1$ for at least one $i \in \{1, \ldots, k\}$. Then every positive solution of (2.17) *converges to one.*

Proof. Taking the logarithm of (2.17) and using the change $y_n = -\ln x_n$, we obtain that

$$y_n = \min_{i: A_i \neq 0}\left\{\ln\frac{1}{A_i} - \alpha_i y_{n-i}\right\}, \quad n \in \mathbb{N}_0. \tag{2.18}$$

Now note that $\ln(1/A_i) \ge 0$ for those i such that $A_i \neq 0$, since $A_i \in (0, 1]$, and there is an $S_1 \subset \{1, \ldots, k\}$ such that $\ln(1/A_i) = 0$ when $i \in S_1$. By Lemma 2.4 we have that for every $n \in \mathbb{N}_0$

$$|y_n| \le \max_{i: A_i \neq 0}\left\{|\alpha_i||y_{n-i}| - \ln\frac{1}{A_i}\right\}. \tag{2.19}$$

From (2.19), noticing that if $|\alpha_i| = 1$ and $A_i \neq 0$, then $A_i \in (0, 1)$ so that $\ln(1/A_i) > 0$ and by applying Proposition 2.2 we obtain that $|y_n| \to 0$ as $n \to \infty$, from which it follows that $x_n = e^{-y_n} \to 1$ as $n \to \infty$, as desired. □

Remark 2.6. Recently Gelişken and Çinar in the paper: "On the global attractivity of a max-type difference equation," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 812674, 5 pages, 2009, have studied the asymptotic behavior to positive solutions of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^{\alpha}}\right\}, \quad n \in \mathbb{N}_0, \tag{2.20}$$

where $\alpha \in (0, 1)$ and $A > 0$. They claim that if $A \in (0, 1)$, then every positive solution to (2.20) converges to one. However the proof given there cannot be regarded as complete one. Namely, they first formulated the following lemma.

Lemma 2.7. *Let y_n be a solution to the difference equation*

$$y_n = \max\{1 - y_{n-1}, -\alpha y_{n-3}\}, \quad n \in \mathbb{N}_0. \tag{2.21}$$

Then for all $n \in \mathbb{N}_0$, the following inequality holds:

$$|y_n| \le \max\{|y_{n-1}| - 1, \alpha|y_{n-3}|\}. \tag{2.22}$$

Then they tried to show that $y_n \to 0$ as $n \to \infty$. Note that (2.21) is obtained by the change $x_n = A^{y_n}$ from (2.20), so that if it is proved that $y_n \to 0$ as $n \to \infty$ then $x_n \to 1$ as $n \to \infty$, from which the claim follows. In the beginning of the proof of the theorem they choose a number β such that $0 < |y_{n-1}| - 1 \le \beta|y_n|$, but do not say if these inequalities hold for all n or not, which is a bit confusing. Note that for different n the chosen number β can be different, which means that in this case β might be a function of n. Hence it is important that these inequalities hold for every $n \in \mathbb{N}_0 \cup \{-2, -1\}$, which was not proved. This motivated us to prove Proposition 2.2 which, among others, removes the gap.

Now we present a representation of solutions of a particular case of (1.5). The first author would like to express his sincere thanks to Professor L. Berg for a nice communication regarding this [2].

Theorem 2.8. *Consider the equation*

$$x_n = \max\{x_{n-1}^{a_1}, \ldots, x_{n-k}^{a_k}\}, \quad n \in \mathbb{N}_0, \tag{2.23}$$

where $k \in \mathbb{N}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, k$. Then every well-defined solution of equation (2.23) *has the following form:*

$$x_n = d_n^{\prod_{j=1}^{k} a_j^{i_n^{(j)}}}, \tag{2.24}$$

where

$$\left[\frac{n+k}{k}\right] \le i_n^{(1)} + \cdots + i_n^{(k)} \le n+1, \quad n \in \mathbb{N}_0, \tag{2.25}$$

$i_n^{(j)} \ge 0$, $j = 1, \ldots, k$, and where d_n is equal to one of the initial values $x_{-k}, \ldots, x_{-1}$.

Moreover, if $-1 < a_i < 1$, $i = 1, \ldots, k$, then $x_n \to 1$ as $n \to \infty$.

Proof. The case $k = 1$ is well known and simple. Just note that $x_n = x_{-1}^{a_1^{n+1}}$. Hence assume that $k \ge 2$. We prove the result by induction. For $n = 0$ we have

$$x_0 = \max\{x_{-1}^{a_1}, \ldots, x_{-k}^{a_k}\}. \tag{2.26}$$

Note that x_0 can be equal to one of the numbers $x_{-1}^{a_1},\dots,x_{-k}^{a_k}$ and that

$$x_{-i}^{a_i}=x_{-i}^{a_i\prod_{i\neq j}a_j^0},\quad i=1,\dots,k, \tag{2.27}$$

which is nothing but formula (2.24) in this case. From this we also have that

$$\left[\frac{0+k}{k}\right]=1=i_1^{(1)}+\cdots+i_1^{(k)}=1=1+0, \tag{2.28}$$

which is (2.25) in this case.

Now assume that we have proved (2.24) and (2.25) for $l\le n-1$. Then

$$\begin{aligned}x_n&=\max\{x_{n-1}^{a_1},\dots,x_{n-k}^{a_k}\}\\&=\max\left\{d_{n-1}^{\prod_{j=1}^k a_j^{i_{n-1}^{(j)}+\delta_1^j}},\dots,d_{n-k}^{\prod_{j=1}^k a_j^{i_{n-k}^{(j)}+\delta_k^j}}\right\},\end{aligned} \tag{2.29}$$

where δ_i^j is the Kronecker symbol and $[(l+k)/k]\le i_l^{(1)}+\cdots+i_l^{(k)}\le l$, for $l\le n-1$. Thus

$$i_{n-s}^{(1)}+\cdots+i_{n-s}^{(k)}+\delta_s^j\le n-s+1+1\le n+1, \tag{2.30}$$

for $s=1,\dots,k$ and

$$i_{n-s}^{(1)}+\cdots+i_{n-s}^{(k)}+\delta_s^j\ge\left[\frac{n-s+k}{k}\right]+1=\left[\frac{n+k-s+k}{k}\right]\ge\left[\frac{n+k}{k}\right], \tag{2.31}$$

$s=1,\dots,k$. Hence the first statement follows by induction.

Now assume that $\max_{1\le j\le k}\{|a_j|\}<1$. From this and (2.25) we have

$$\left|\prod_{j=1}^k a_j^{i_n^{(j)}}\right|\le\left(\max_{1\le j\le k}\{|a_j|\}\right)^{[(n+k)/k]}. \tag{2.32}$$

Inequality (2.32), the assumption $\max_{1\le j\le k}\{|a_j|\}<1$, and (2.24) imply that x_n tends to 1 as $n\to\infty$, finishing the proof of the theorem. □

Remark 2.9. Note that formula (2.24) holds for each value of parameters a_j, $j=1,\dots,k$, and for all solutions whose initial values are different from zero if one of these exponents is negative.

Remark 2.10. The second statement in Theorem 2.8 follows easily also from Lemma A.

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