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Research Article

Some Relations between Twisted (h, q)-Euler Numbers with Weight α and q-Bernstein Polynomials with Weight α

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By using fermionic p-adic q-integral on $\mathbb{Z}_p$, we give some interesting relationship between the twisted (h, q)-Euler numbers with weight α and the q-Bernstein polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $C_{p^n} = \{w \mid w^{p^n} = 1\}$ be the cyclic group of order p^n, and let

$$T_p = \lim_{n\to\infty} C_{p^n} = C_{p^\infty} = \cup_{n\geq 0} C_{p^n}, \tag{1.1}$$

(see [1–22]), be the locally constant space. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$. The p-adic absolute value is defined by $|x|_p = 1/p^r$, where $x = p^r s/t (r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s, t) = (p, s) = (p, t) = 1)$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$ as an indeterminate. The q-number is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \tag{1.2}$$

(see [1–22]). Note that $\lim_{q\to 1}[x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f \mid f : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \tag{1.3}$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.4}$$

(see [1–7]). From (1.4), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l), \tag{1.5}$$

where $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$.

For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, Kim defined q-Bernstein polynomials, which are different q-Bernstein polynomials of Phillips, as follows:

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \tag{1.6}$$

(see [5]). In [9], the p-adic extension of (1.6) is given by

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{q^{-1}}^{n-k}, \quad \text{where } x \in \mathbb{Z}_p,\ n,k \in \mathbb{Z}_+. \tag{1.7}$$

For $\alpha \in \mathbb{Z}$, $h \in \mathbb{Z}$, $w \in T_p$, and $q \in \mathbb{C}_p$ with $|1-q|_p \le 1$, twisted (h,q)-Euler numbers $E_{n,q,w}^{(h,\alpha)}$ with weight α are defined by

$$E_{n,q,w}^{(h,\alpha)} = \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_{q^\alpha}^n d\mu_{-q}(x). \tag{1.8}$$

In the special case, $x = 0$, $E_{n,q,w}^{(h,\alpha)}(0) = E_{n,q,w}^{(h,\alpha)}$ are called the n-th twisted (h,q)-Euler numbers with weight α.

In this paper, we investigate some relations between the q-Bernstein polynomials and the twisted (h,q)-Euler numbers with weight α. From these relations, we derive some interesting identities on the twisted (h,q)-Euler numbers and polynomials with weight α.

2. Twisted (h, q)-Euler Numbers and Polynomials with Weight α

By using p-adic q-integral and (1.8), we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p}\phi_w(x)q^{x(h-1)}[x]_{q^\alpha}^n d\mu_{-q}(x) &= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1}[x]_{q^\alpha}^n w^x q^{x(h-1)}(-q)^x \\ &= [2]_q\left(\frac{1}{1-q^\alpha}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+wq^{\alpha l+h}}.\end{aligned} \tag{2.1}$$

We set

$$F_{q,w}^{(h,\alpha)}(t) = \sum_{n=0}^{\infty}E_{n,q,w}^{(h,\alpha)}\frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}F_{q,w}^{(h,\alpha)}(t) &= \sum_{n=0}^{\infty}E_{n,q,w}^{(h,\alpha)}\frac{t^n}{n!} \\ &= [2]_q\sum_{n=0}^{\infty}\left(\left(\frac{1}{1-q^\alpha}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+wq^{\alpha l+h}}\right)\frac{t^n}{n!} \\ &= [2]_q\sum_{m=0}^{\infty}(-1)^m w^m q^{hm}e^{[m]_{q^\alpha}t}.\end{aligned} \tag{2.3}$$

Since $[x+y]_{q^\alpha} = [x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}$, we obtain

$$\begin{aligned}E_{n,q,w}^{(h,\alpha)}(x) &= \int_{\mathbb{Z}_p}\phi_w(y)q^{y(h-1)}[y+x]_{q^\alpha}t\,d\mu_{-q}(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}q^{\alpha x l}[x]_{q^\alpha}^{n-l}\int_{\mathbb{Z}_p}\phi_w(y)q^{y(h-1)}[y]_{q^\alpha}^l d\mu_{-q}(y) \\ &= \sum_{l=0}^{n}\binom{n}{l}q^{\alpha x l}[x]_{q^\alpha}^{n-l}E_{l,q,w}^{(h,\alpha)}.\end{aligned} \tag{2.4}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$ and $w \in T_p$, we have*

$$E_{n,q,w}^{(h,\alpha)}(x) = [2]_q\sum_{m=0}^{\infty}(-1)^m w^m q^{hm}[x+m]_{q^\alpha}^n. \tag{2.5}$$

Furthermore,

$$E_{n,q,w}^{(h,\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{\alpha x l} [x]_{q^\alpha}^{n-l} E_{l,q,w}^{(h,\alpha)} = \left([x]_{q^\alpha} + q^{\alpha x} E_{q,w}^{(h,\alpha)}\right)^n, \tag{2.6}$$

with usual convention about replacing $(E_{q,w}^{(h,\alpha)})^n$ *with* $E_{n,q,w}^{(h,\alpha)}$.

Let $F_{q,w}^{(h,\alpha)}(t,x) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h,\alpha)}(x) t^n/n!$. *Then we see that*

$$F_{q,w}^{(h,\alpha)}(t,x) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^{mh} e^{[x+m]_{q^\alpha} t}. \tag{2.7}$$

In the special case, $x = 0$, *let* $F_{q,w}^{(h,\alpha)}(t,0) = F_{q,w}^{(h,\alpha)}(t)$.

By (2.1), *we get*

$$E_{n,q^{-1},w^{-1}}^{(h,\alpha)}(1-x) = (-1)^n w q^{\alpha n+h-1} E_{n,q,w}^{(h,\alpha)}(x). \tag{2.8}$$

From (2.3) *and* (2.7), *we note that*

$$w q^h F_{q,w}^{(h,\alpha)}(t,1) + F_{q,w}^{(h,\alpha)}(t) = [2]_q. \tag{2.9}$$

By (2.9), *we get the following recurrence formula:*

$$E_{0,q,w}^{(h,\alpha)} = \frac{[2]_q}{1+q^h w}, \qquad q^h w E_{n,q,w}^{(h,\alpha)}(1) + E_{n,q,w}^{(h,\alpha)} = 0 \quad \text{if } n > 0. \tag{2.10}$$

By (2.10) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *For* $n \in \mathbb{Z}_+$ *and* $w \in T_p$, *we have*

$$E_{0,q,w}^{(h,\alpha)} = \frac{[2]_q}{1+q^h w}, \qquad q^h w \left(q^\alpha E_{q,w}^{(h,\alpha)} + 1\right)^n + E_{n,q,w}^{(h,\alpha)} = 0 \quad \text{if } n > 0, \tag{2.11}$$

with usual convention about replacing $(E_{q,w}^{(h,\alpha)})^n$ *with* $E_{n,q,w}^{(h,\alpha)}$.

By (2.4), *Theorem 2.1, and Theorem 2.2, we have*

$$q^{2h} w^2 E_{n,q,w}^{(h,\alpha)}(2) - \frac{[2]_q}{1+q^h w} q^{2h} w^2 - \frac{[2]_q}{1+q^h w} q^h w = q^{2h} w^2 \sum_{l=0}^{n} \binom{n}{l} q^{\alpha l} \left(q^\alpha E_{q,w}^{(h,\alpha)} + 1\right)^l - \frac{[2]_q}{1+q^h w} q^{2h} w^2 - \frac{[2]_q}{1+q^h w} q^h w$$

$$
\begin{aligned}
&= q^{2h}w^2\sum_{l=1}^{n}\binom{n}{l}q^{\alpha l}\left(q^{\alpha}E_{q,w}^{(h,\alpha)}+1\right)^{l}-\frac{[2]_q}{1+q^hw}q^hw\\
&= -q^hw\sum_{l=1}^{n}\binom{n}{l}q^{\alpha l}E_{l,q,w}^{(h,\alpha)}-\frac{[2]_q}{1+q^hw}q^hw\\
&= -q^hw\sum_{l=0}^{n}\binom{n}{l}q^{\alpha l}E_{l,q,w}^{(h,\alpha)}\\
&= -q^hwE_{n,q,w}^{(h,\alpha)}(1)=E_{n,q,w}^{(h,\alpha)} \quad \text{if } n>0.
\end{aligned}
\tag{2.12}
$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$, we have*

$$
E_{n,q,w}^{(h,\alpha)}(2)=\left(\frac{1}{q^{2h}w^2}\right)E_{n,q,w}^{(h,\alpha)}+\frac{[2]_q}{1+q^hw}+\left(\frac{1}{q^hw}\right)\frac{[2]_q}{1+q^hw}. \tag{2.13}
$$

By (2.8), *we see that*

$$
\begin{aligned}
q^{h-1}w\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}q^{(h-1)x}w^x\,d\mu_{-q}(x)&=(-1)^nq^{\alpha n+h-1}w\int_{\mathbb{Z}_p}[x-1]_{q^{\alpha}}^{n}q^{(h-1)x}w^x\,d\mu_{-q}(x)\\
&=(-1)^nq^{\alpha n+h-1}wE_{n,q,w}^{(h,\alpha)}(-1)=E_{n,q^{-1},w^{-1}}^{(h,\alpha)}(2).
\end{aligned}
\tag{2.14}
$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, we have*

$$
q^{h-1}w\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}q^{(h-1)x}w^x\,d\mu_{-q}(x)=E_{n,q^{-1},w^{-1}}^{(h,\alpha)}(2). \tag{2.15}
$$

Let $n \in \mathbb{N}$. By Theorems 2.3 and 2.4, we get

$$
\begin{aligned}
&q^{h-1}w\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n}q^{(h-1)x}w^x\,d\mu_{-q}(x)\\
&\quad=q^{2h}w^2E_{n,q^{-1},w^{-1}}^{(h,\alpha)}+q^{h-1}w\left(\frac{[2]_q}{1+q^hw}\right)+q^{2h-1}w^2\left(\frac{[2]_q}{1+q^hw}\right).
\end{aligned}
\tag{2.16}
$$

From (2.16), *we have*

$$
\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^{n}q^{(h-1)x}w^x\,d\mu_{-q}(x)=q^{h+1}wE_{n,q^{-1},w^{-1}}^{(h,\alpha)}+\left(\frac{[2]_q}{1+q^hw}\right)+q^hw\left(\frac{[2]_q}{1+q^hw}\right). \tag{2.17}
$$

Therefore, we obtain the following corollary.

Corollary 2.5. *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n} q^{(h-1)x} w^x d\mu_{-q}(x) = q^{h+1} w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q. \tag{2.18}$$

Kim defined the q-Bernstein polynomials with weight α of degree n as below.

For $x \in \mathbb{Z}_p$, the p-adic q-Bernstein polynomials with weight α of degree n are given by

$$B_{k,n}^{(\alpha)}(x,q) = \binom{n}{k} [x]_{q^{\alpha}}^{k} [1-x]_{q^{-\alpha}}^{n-k}, \text{ where } n,k \in \mathbb{Z}_+. \tag{2.19}$$

compare [5, 10, 22] By (2.19)*, we get the symmetry of q-Bernstein polynomials as follows:*

$$B_{k,n}^{(\alpha)}(x,q) = B_{n-k,n}^{(\alpha)}\left(1-x,q^{-1}\right), \tag{2.20}$$

see [8]. Thus, by Corollary 2.5, (2.19)*, and* (2.20)*, we see that*

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) q^{(h-1)x} w^x d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}\left(1-x,q^{-1}\right) q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n-l} q^{(h-1)x} w^x d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(q^{h+1} w E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q\right).\end{aligned} \tag{2.21}$$

For $n,k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q) q^{(h-1)x} w^x d\mu_{-q}(x) &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(q^{h+1} w E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q\right) \\ &= \begin{cases} q^{h+1} w E_{n,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases}\end{aligned} \tag{2.22}$$

Let us take the fermionic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials with weight α of degree n as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)q^{(h-1)x}w^x d\mu_{-q}(x) &= \binom{n}{k}\int_{\mathbb{Z}_p}[x]_q^k[1-x]_{q^{-\alpha}}^{n-k}q^{(h-1)x}w^x d\mu_{-q}(x)\\ &= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l E_{l+k,q,w}^{(h,\alpha)}.\end{aligned}\tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.6. *Let $n,k\in\mathbb{Z}_+$ with $n>k$. Then we have*

$$\int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(x,q)q^{(h-1)x}w^x d\mu_{-q}(x) = \begin{cases} q^{h+1}wE_{n,q^{-1},w^{-1}}^{(h,\alpha)}+[2]_q & \text{if } k=0,\\ q^{h+1}w\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k>0.\end{cases}\tag{2.24}$$

Moreover,

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l E_{l+k,q,w}^{(h,\alpha)} = \begin{cases} q^{h+1}wE_{n,q^{-1},w^{-1}}^{(h,\alpha)}+[2]_q & \text{if } k=0,\\ q^{h+1}w\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}E_{n-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k>0.\end{cases}\tag{2.25}$$

Let $n_1,n_2,k\in\mathbb{Z}_+$ with $n_1+n_2>2k$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x,q)B_{k,n_2}^{(\alpha)}(x,q)q^{(h-1)x}w^x d\mu_{-q}(x)\\ &\quad=\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n_1+n_2-l}q^{(h-1)x}w^x d\mu_{-q}(x)\\ &\quad=\binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(q^{h+1}wE_{n_1+n_2-l,q^{-1},w^{-1}}^{(h,\alpha)}+[2]_q\right)\\ &\quad=\begin{cases} q^{h+1}wE_{n_1+n_2,q^{-1},w^{-1}}^{(h,\alpha)}+[2]_q & \text{if } k=0,\\ \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\left(q^{h+1}wE_{n_1+n_2-l,q^{-1},w^{-1}}^{(h,\alpha)}+[2]_q\right) & \text{if } k\neq 0.\end{cases}\end{aligned}\tag{2.26}$$

Therefore, by (2.26), we obtain the following theorem.

Theorem 2.7. *For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x,q) B_{k,n_2}^{(\alpha)}(x,q) q^{(h-1)x} w^x d\mu_{-q}(x)$$
$$= \begin{cases} q^{h+1} w E_{n_1+n_2,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E_{n_1+n_2-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k \neq 0. \end{cases} \tag{2.27}$$

From the binomial theorem, we can derive the following equation:

$$\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(x,q) B_{k,n_2}^{(\alpha)}(x,q) q^{(h-1)x} w^x d\mu_{-q}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} \int_{\mathbb{Z}_p} [x]_q^{2k+l} q^{(h-1)x} w^x d\mu_{-q}(x) \tag{2.28}$$
$$= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E_{2k+l,q,w}^{(h,\alpha)}.$$

Thus, by (2.28) and Theorem 2.7, we obtain the following corollary.

Corollary 2.8. *Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have*

$$\sum_{l=0}^{n_1+n_2-2k} (-1)^l \binom{n_1+n_2-2k}{l} E_{2k+l,q}^{(h,\alpha)} = \begin{cases} q^{h+1} w E_{n_1+n_2,q^{-\alpha},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} E_{n_1+n_2-l,q^{-\alpha},w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases} \tag{2.29}$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for the q-Bernstein polynomials with weight α of degree n as follows:

$$\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(x,q) \cdots B_{k,n_s}^{(\alpha)}(x,q)}_{s\text{-times}} q^{(h-1)x} w^x d\mu_{-q}(x)$$
$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{q^{-\alpha}}^{n_1+\cdots+n_s-sk} q^{(h-1)x} w^x d\mu_{-q}(x)$$

$$= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\int_{\mathbb{Z}_p}[1-x]_{q^{-\alpha}}^{n_1+\cdots+n_s-l}q^{(h-1)x}w^x d\mu_{-q}(x)$$

$$= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left(q^{h+1}wE^{(h,\alpha)}_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}+[2]_q\right)$$

$$= \begin{cases} q^{h+1}wE^{(h,\alpha)}_{n_1+\cdots+n_s,q^{-1},w^{-1}}+[2]_q & \text{if } k=0, \\ q^{h+1}w\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}E^{(h,\alpha)}_{n_1+\cdots+n_s-l,q^{-1},w^{-1}} & \text{if } k>0. \end{cases} \tag{2.30}$$

Therefore, by (2.30), we obtain the following theorem.

Theorem 2.9. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we get*

$$\int_{\mathbb{Z}_p}\underbrace{B^{(\alpha)}_{k,n_1}(x,q)\cdots B^{(\alpha)}_{k,n_s}(x,q)}_{s\text{-times}}q^{(h-1)x}w^x d\mu_{-q}(x)$$

$$= \begin{cases} q^{h+1}wE^{(h,\alpha)}_{n_1+\cdots+n_s,q^{-1},w^{-1}}+[2]_q & \text{if } k=0, \\ q^{h+1}w\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}E^{(h,\alpha)}_{n_1+\cdots+n_s-l,q^{-1},w^{-1}} & \text{if } k>0. \end{cases} \tag{2.31}$$

By the definition of q-Bernstein polynomials with weight α and the binomial theorem, we easily get

$$\int_{\mathbb{Z}_p}\underbrace{B^{(\alpha)}_{k,n_1}(x,q)\cdots B^{(\alpha)}_{k,n_s}(x,q)}_{s\text{-times}}q^{(h-1)x}w^x d\mu_{-q}(x)$$

$$= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}\int_{\mathbb{Z}_p}[x]_q^{sk+l}q^{(h-1)x}w^x d\mu_{-q}(x)$$

$$= \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{n_1+\cdots+n_s-sk}(-1)^l\binom{n_1+\cdots+n_s-sk}{l}E^{(h,\alpha)}_{sk+l,q,w}. \tag{2.32}$$

Therefore, we have the following corollary.

Corollary 2.10. *For $w \in T_p$, $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$. Then we have*

$$\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^l \binom{n_1+\cdots+n_s-sk}{l} E_{sk+l,q,w}^{(h,\alpha)} = \begin{cases} q^{h+1} w E_{n_1+\cdots+n_s,q^{-1},w^{-1}}^{(h,\alpha)} + [2]_q & \text{if } k = 0, \\ q^{h+1} w \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} E_{n_1+\cdots+n_s-l,q^{-1},w^{-1}}^{(h,\alpha)} & \text{if } k > 0. \end{cases} \tag{2.33}$$

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