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Research Article

On Solution and Stability of a Two-Variable Functional Equations

Y. J. Cho,[1] R. Saadati,[2] S. Shabanian,[3] and S. M. Vaezpour[3]

[1] *Department of Mathematics Education and RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*
[2] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran 14778, Iran*
[3] *Department of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran*

Correspondence should be addressed to R. Saadati, rsaadati@eml.cc

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The main purpose of this paper is to investigate the stability of the functional equation $f(x+y, y+z) = 2f(x/2, y/2) + 2f(y/2, z/2)$ in normed spaces. The solutions of such functional equations are considered.

1. Introduction

The stability of functional equations was originated from a question of Ulam in 1940, concerning the stability of group homomorphism [1]. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors and there are many investigating results concerning this problem (see [2–12]).

Throughout this paper, we study the stability of the functional equation

$$f(x+y, y+z) = 2f\left(\frac{x}{2}, \frac{y}{2}\right) + 2f\left(\frac{y}{2}, \frac{z}{2}\right), \tag{1.1}$$

where X is a normed space, Y is a Banach space, and $f : X \times X \to Y$ is a mapping. The solutions of such functions are considered.

We recall that a mapping $f : X \times X \to Y$ is called *additive* if

$$f(x+z, y+w) = f(x,y) + f(z,w). \tag{1.2}$$

Throughout this paper X is a normed space and Y is a Banach space.

2. The General Solution of the Functional Equation (1.1)

Before proceeding the proof of the main result, we shall need the following two lemmas.

Lemma 2.1. *Let $f : X \times X \to Y$ be a mapping satisfying* (1.1) *and $f(-x,-y) = f(x,y)$ for all $x, y \in X$. Then f is zero.*

Proof. Suppose that $f : X \times X \to Y$ is a mapping satisfying (1.1) and $f(-x,-y) = f(x,y)$. It is clear that $f(0,0) = 0$. Letting $y = z = 0$ and replacing x by $2x$ in (1.1), we obtain

$$f(2x,0) = 2f(x,0). \tag{2.1}$$

Similarly, letting $x = y = 0$ and replacing z by $2x$ in (1.1), we obtain

$$f(0,2x) = 2f(0,x). \tag{2.2}$$

Putting $y = 0$ and replacing z by y in (1.1), we get

$$f(x,y) = 2f\left(\frac{x}{2},0\right) + 2f\left(0,\frac{y}{2}\right) = f(x,0) + f(0,y), \tag{2.3}$$

for all $x, y \in X$, where the last equality follows from (2.1) and (2.2). It follows from (2.3) and (1.1) that

$$\begin{aligned} f(x+y,0) + f(0,x-y) &= f(x+y,x-y) \\ &= 2f\left(\frac{y}{2},\frac{x}{2}\right) + 2f\left(\frac{x}{2},\frac{-y}{2}\right) \\ &= 2f\left(\frac{y}{2},0\right) + 2f\left(0,\frac{x}{2}\right) + 2f\left(\frac{x}{2},0\right) + 2f\left(0,\frac{-y}{2}\right) \\ &= f(y,0) + f(0,x) + f(x,0) + f(0,-y). \end{aligned} \tag{2.4}$$

Replacing x and y by $x/2$ and $x/2$ in (2.4), respectively, we obtain

$$f(x,0) = 2f\left(0,\frac{x}{2}\right) + 2f\left(\frac{x}{2},0\right). \tag{2.5}$$

Also, replacing y and x by $-x/2$ and $x/2$ in (2.4), respectively, we get

$$f(0,x) = 2f\left(\frac{x}{2},0\right) + 2f\left(0,\frac{x}{2}\right). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$f(x,0) = f(0,x). \tag{2.7}$$

Consequently, by (2.4), we obtain

$$f(x+y,0)+f(x-y,0)=2f(x,0)+2f(y,0). \tag{2.8}$$

Replacing y by x in (2.8), we get

$$f(2x,0)=4f(x,0). \tag{2.9}$$

By (2.1) and (2.9), we have

$$f(x,0)=0. \tag{2.10}$$

The equalities (2.7) and (2.10) imply that

$$f(0,x)=0. \tag{2.11}$$

Now, it follows from (2.3), (2.10), and (2.11) that

$$f(x,y)=0. \tag{2.12}$$

This completes the proof. □

Lemma 2.2. *Let $f : X \times X \to Y$ be a mapping satisfying* (1.1) *and $f(-x,-y)=-f(x,y)$ for all $x,y \in X$. Then f is additive.*

Proof. It is easy to show that $f(0,0)=0$. Putting $y=z=0$ and replacing x by $2x$ in (1.1), we obtain

$$f(2x,0)=2f(x,0). \tag{2.13}$$

Again, putting $x=y=0$ and replacing z by $2x$ in (1.1), we obtain

$$f(0,2x)=2f(0,x). \tag{2.14}$$

Putting $y=0$ and replacing z by y in (1.1), we get

$$f(x,y)=2f\left(\frac{x}{2},0\right)+2f\left(0,\frac{y}{2}\right)=f(x,0)+f(0,y), \tag{2.15}$$

for all $x, y \in X$, where the last equality follows from (2.13) and (2.14). It follows from (2.13), (2.14), and (2.15) that

$$\begin{aligned} f(x+y,0) - f(0,x-y) &= f(x+y,0) + f(0,y-x) \\ &= f(x+y,y-x) \\ &= f(x,y) + f(y,-x). \end{aligned} \tag{2.16}$$

Replacing y by $-y$ in (2.16), we obtain

$$f(x-y,0) - f(0,x+y) = f(x,-y) + f(-y,-x). \tag{2.17}$$

Also, we have

$$\begin{aligned} f(x+y,0) + f(0,x-y) &= f(x+y,x-y) \\ &= f(x,-y) + f(y,x). \end{aligned} \tag{2.18}$$

Replacing y by $-y$ in (2.18), we get

$$f(x-y,0) + f(0,x+y) = f(x,y) + f(-y,x). \tag{2.19}$$

Now, by (2.16), (2.17), (2.18), and (2.19), we have

$$f(x+y,0) - f(x-y,0) = 2f(y,0). \tag{2.20}$$

Replacing x with $x+y$ in the above equality, we get

$$f(x+2y,0) - f(x,0) = 2f(y,0). \tag{2.21}$$

Now, replacing y by $y/2$ in the previous equality, we obtain

$$f(x+y,0) = f(x,0) + f(y,0). \tag{2.22}$$

Similarly, one can prove that

$$f(0,z+w) = f(0,z) + f(0,w). \tag{2.23}$$

By (2.22) and (2.23), we conclude that

$$f(x+y,z+w) = f(x,z) + f(y,w), \tag{2.24}$$

which shows that f is additive. This completes the proof. □

Now, we are ready to present the general solution of (1.1).

Theorem 2.3. *Every mapping $f : X \times X \to Y$ satisfying* (1.1) *is additive.*

Proof. One can write

$$f(x,y) = f^e(x,y) + f^o(x,y), \tag{2.25}$$

where $f^e(x,y) := (f(x,y) + f(-x,-y))/2$ and $f^o := (f(x,y) - f(-x,-y))/2$. Since f satisfies (1.1), f^e and f^o satisfy (1.1). Further, we have $f^e(-x,-y) = f^e(x,y)$ and $f^o(-x,-y) = -f^o(x,y)$ for all $x, y \in X$. By Lemmas 2.1 and 2.2, $f^e(x,y)$ is zero and $f^o(x,y)$ is additive, respectively. It follows that f is additive. This completes the proof. □

3. The Stability of the Functional Equation (1.1)

Throughout this section, we prove the stability of the functional equation (1.1).

Theorem 3.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\left\| f(x+y, y+z) - 2f\left(\frac{x}{2}, \frac{y}{2}\right) - 2f\left(\frac{y}{2}, \frac{z}{2}\right) \right\| \le \phi(x,y,z), \tag{3.1}$$

where $\phi : X \times X \times X \to [0,\infty)$ satisfies $\sum_{i=0}^{\infty} 2^i \phi(x/2^i, y/2^i, z/2^i) < \infty$ for all $x, y, z \in X$. Then there exists a mapping $F : X \times X \to Y$ satisfying (1.1),

$$\begin{gathered} \|f(x,y) - F(x,y)\| \le \sum_{i=1}^{\infty} 2^i \left(\phi\left(\frac{x}{2^i}, 0, 0\right) + \phi\left(0, 0, \frac{y}{2^i}\right) \right) + \phi(x,0,y), \\ \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = F(x,y), \end{gathered} \tag{3.2}$$

for all $x, y \in X$.

Proof. Letting $x = y = z = 0$, we have $f(0,0) = 0$. Putting $y = z = 0$, we obtain

$$\left\| f(x,0) - 2f\left(\frac{x}{2}, 0\right) \right\| \le \phi(x,0,0). \tag{3.3}$$

By induction, we conclude that

$$\left\| f(x,0) - 2^n f\left(\frac{x}{2^n}, 0\right) \right\| \le \sum_{i=0}^{n-1} 2^i \phi\left(\frac{x}{2^i}, 0, 0\right). \tag{3.4}$$

Put $K_{m,n} = \|2^m f(x/2^m, 0) - 2^n f(x/2^n, 0)\|$. Then

$$\begin{aligned} K_{m,n} &\le K_{m,n+1} + K_{n+1,n} \\ &= K_{m,n+1} + 2^n \left\| 2f\left(\frac{x}{2^{n+1}}, 0\right) - f\left(\frac{x}{2^n}, 0\right) \right\| \\ &\le K_{m,n+1} + 2^n \phi\left(\frac{x}{2^n}, 0, 0\right), \end{aligned} \tag{3.5}$$

for all $m, n \in \mathbb{N}$ with $m > n$. By induction, we get

$$\left\| 2^m f\left(\frac{x}{2^m}, 0\right) - 2^n f\left(\frac{x}{2^n}, 0\right) \right\| \le \sum_{i=n}^{m-1} 2^i \phi\left(\frac{x}{2^i}, 0, 0\right) < \epsilon, \tag{3.6}$$

which implies that $\{2^n f(x/2^n, 0)\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, there exists $h(x)$ such that

$$\left\| 2^n f\left(\frac{x}{2^n}, 0\right) - h(x) \right\| \longrightarrow 0, \tag{3.7}$$

as $n \to \infty$ for all $x \in X$. Since

$$\left\| h(x) - 2h\left(\frac{x}{2}\right) \right\| \le \left\| h(x) - 2^n f\left(\frac{x}{2^n}, 0\right) \right\| + \left\| 2^n f\left(\frac{x}{2^n}, 0\right) - 2h\left(\frac{x}{2}\right) \right\| \longrightarrow 0, \tag{3.8}$$

as $n \to \infty$, $h(x) = 2h(x/2)$ for all $x \in X$.

Similarly, we can prove that there exists a mapping $g : X \to Y$ such that

$$\left\| 2^n f\left(0, \frac{x}{2^n}\right) - g(x) \right\| \longrightarrow 0, \tag{3.9}$$

as $n \to \infty$ and $g(x) = 2g(x/2)$ for all $x \in X$.

Now, if

$$d_n(x, y, z) = f\left(\frac{x+y}{2^n}, \frac{y+z}{2^n}\right) - 2f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) - 2f\left(\frac{y}{2^{n+1}}, \frac{z}{2^{n+1}}\right), \tag{3.10}$$

and $F_n = \|2^n f((x+y)/2^n, y/2^n) - h(x+y) - g(y)\|$, then we get

$$\begin{aligned} F_n &= \left\| 2^n d_n(x+y, 0, y) + 2^{n+1} f\left(\frac{x+y}{2^{n+1}}, 0\right) + 2^{n+1} f\left(0, \frac{y}{2^{n+1}}\right) - h(x+y) - g(y) \right\| \\ &\le \|2^n d_n(x+y, 0, y)\| + \left\| 2^{n+1} f\left(\frac{x+y}{2^{n+1}}, 0\right) - h(x+y) \right\| + \left\| 2^{n+1} f\left(0, \frac{y}{2^{n+1}}\right) - g(y) \right\| \end{aligned}$$

$$\begin{aligned}
&\le 2^n \|d_n(x+y,0,y)\| + \left\|2^{n+1} f\left(\frac{x+y}{2^{n+1}},0\right) - h(x+y)\right\| + \left\|2^{n+1} f\left(0,\frac{y}{2^{n+1}}\right) - g(y)\right\| \\
&\le 2^n \phi\left(\frac{x+y}{2^n},0,\frac{y}{2^n}\right) + \left\|2^{n+1} f\left(\frac{x+y}{2^{n+1}},0\right) - h(x+y)\right\| + \left\|2^{n+1} f\left(0,\frac{y}{2^{n+1}}\right) - g(y)\right\|,
\end{aligned} \tag{3.11}$$

for all $x, y \in X$. Thus we obtain

$$\lim_{n\to\infty} 2^n f\left(\frac{x+y}{2^n}, \frac{y}{2^n}\right) = h(x+y) + g(y), \tag{3.12}$$

for all $x, y \in X$. Let $G_n = \|2^n f(x/2^n, y/2^n) - h(x+y) - g(y) + h(y)\|$. Then we conclude that

$$\begin{aligned}
G_n &= \left\|-2^{n-1} d_{n-1}(x,y,0) + 2^{n-1} f\left(\frac{x+y}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 2^n f\left(\frac{y}{2^n},0\right) - h(x+y) - g(y) + h(y)\right\| \\
&\le \left\|2^{n-1} d_{n-1}(x,y,0)\right\| + \left\|2^{n-1} f\left(\frac{x+y}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - h(x+y) - g(y)\right\| \\
&\quad + \left\|2^n f\left(\frac{y}{2^n},0\right) - h(y)\right\| \\
&\le 2^{n-1}\phi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}},0\right) + \left\|2^{n-1} f\left(\frac{x+y}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - h(x+y) - g(y)\right\| \\
&\quad + \left\|2^n f\left(\frac{y}{2^n},0\right) - h(y)\right\|,
\end{aligned} \tag{3.13}$$

for all $x, y \in X$. It follows from (3.12) that

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = h(x+y) + g(y) - h(y), \tag{3.14}$$

for all $x, y \in X$. Put $H_n = \|2^n f(x/2^n, y/2^n) - h(x) - g(y)\|$. Since

$$\begin{aligned}
H_n &= \left\|2^n d_n(x,0,y) + 2^{n+1} f\left(\frac{x}{2^{n+1}},0\right) + 2^{n+1} f\left(0,\frac{y}{2^{n+1}}\right) - h(x) - g(y)\right\| \\
&\le \|2^n d_n(x,0,y)\| + \left\|2^{n+1} f\left(\frac{x}{2^{n+1}},0\right) - h(x)\right\| + \left\|2^{n+1} f\left(0,\frac{y}{2^{n+1}}\right) - g(y)\right\| \\
&\le 2^n \phi\left(\frac{x}{2^n},0,\frac{y}{2^n}\right) + \left\|2^{n+1} f\left(\frac{x}{2^{n+1}},0\right) - h(x)\right\| + \left\|2^{n+1} f\left(\frac{x}{2^{n+1}},0\right) - h(x)\right\| \\
&\longrightarrow 0,
\end{aligned} \tag{3.15}$$

as $n \to \infty$,

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = h(x) + g(y), \tag{3.16}$$

for all $x, y \in X$. It follows from (3.14) and (3.16) that

$$h(x+y) + g(y) - h(y) = h(x) + g(y), \tag{3.17}$$

for all $x, y \in X$. So $h(x+y) = h(x) + h(y)$ for all $x, y \in X$.

Similarly, we can show that

$$g(x+y) = g(x) + g(y), \tag{3.18}$$

for all $x, y \in X$. Let $F(x,y) = h(x) + g(y)$. Then we conclude that

$$\begin{aligned} F(x+y, y+z) &= h(x) + h(y) + g(y) + g(z) \\ &= h\left(\frac{x}{2}\right) + 2h\left(\frac{y}{2}\right) + 2g\left(\frac{y}{2}\right) + 2g\left(\frac{z}{2}\right) \\ &= 2F\left(\frac{x}{2}, \frac{y}{2}\right) + 2F\left(\frac{y}{2}, \frac{z}{2}\right), \end{aligned} \tag{3.19}$$

for all $x, y, z \in X$.

Now, letting $L_n = \|f(x,y) - 2^n f(x/2^n, 0) - 2^n f(0, y/2^n)\|$, we obtain

$$\|f(x,y) - F(x,y)\| \le \left\|f(x,y) - 2^n f\left(\frac{x}{2^n}, 0\right) - 2^n f\left(0, \frac{y}{2^n}\right)\right\| + L_n. \tag{3.20}$$

Since

$$\left\|2^n f\left(\frac{x}{2^n}, 0\right) + 2^n f\left(0, \frac{y}{2^n}\right) - F(x,y)\right\| \longrightarrow 0, \tag{3.21}$$

as $n \to \infty$, it is sufficient to show that

$$L_n \le \sum_{i=1}^{n-1} 2^i \left(\phi\left(\frac{x}{2^i}, 0, 0\right) + \phi\left(0, 0, \frac{y}{2^i}\right)\right) + \phi(x, 0, y). \tag{3.22}$$

We have

$$\begin{aligned} L_n \le{} & \left\|f(x,y) - 2f\left(\frac{x}{2}, 0\right) - 2f\left(0, \frac{y}{2}\right)\right\| + \left\|2f\left(\frac{x}{2}, 0\right) - 2^n f\left(\frac{x}{2^n}, 0\right)\right\| \\ & + \left\|2f\left(0, \frac{y}{2}\right) - 2^n f\left(0, \frac{y}{2^n}\right)\right\| \\ \le{} & \sum_{i=1}^{n-1} 2^i \left(\phi\left(\frac{x}{2^i}, 0, 0\right) + \phi\left(0, 0, \frac{y}{2^i}\right)\right) + \phi(x, 0, y). \end{aligned} \tag{3.23}$$

As $n \to \infty$, we have

$$\|f(x,y) - F(x,y)\| \le \sum_{i=1}^{\infty} 2^i \left(\phi\left(\frac{x}{2^i},0,0\right) + \phi\left(0,0,\frac{y}{2^i}\right)\right) + \phi(x,0,y). \tag{3.24}$$

This completes the proof. □

Corollary 3.2. *Suppose that* $f : X \times X \to Y$ *is a mapping such that*

$$\left\|f(x+y,y+z) - 2f\left(\frac{x}{2},\frac{y}{2}\right) - 2f\left(\frac{y}{2},\frac{z}{2}\right)\right\| \le \epsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right), \tag{3.25}$$

where $p > 1$ *and* $\epsilon > 0$. *Then there exists a mapping* $F : X \times X \to Y$ *satisfying* (1.1),

$$\begin{gathered} \|f(x,y) - F(x,y)\| \le \frac{2^p}{2^p - 2}\epsilon\left(\|x\|^p + \|y\|^p\right), \\ \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},\frac{y}{2^n}\right) = F(x,y), \end{gathered} \tag{3.26}$$

for all $x, y \in X$.

Corollary 3.3. *Suppose that* $f : X \times X \to Y$ *is a mapping such that*

$$\left\|f(x+y,y+z) - 2f\left(\frac{x}{2},\frac{y}{2}\right) - 2f\left(\frac{y}{2},\frac{z}{2}\right)\right\| \le \epsilon\left(\|x\|^p\|y\|^p + \|y\|^p\|z\|^p + \|z\|^p\|x\|^p\right), \tag{3.27}$$

where $p > 1/2$ *and* $\epsilon > 0$. *Then there exists a mapping* $F : X \times X \to Y$ *satisfying* (1.1),

$$\begin{gathered} \|f(x,y) - F(x,y)\| \le \epsilon\|x\|^p\|y\|^p, \\ \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},\frac{y}{2^n}\right) = F(x,y), \end{gathered} \tag{3.28}$$

for all $x, y \in X$.

Example 3.4. Let $X = Y = C[a,b]$ with the norm $\|x\| = \sup_{t\in[a,b]}|x(t)|$, where $b > a > 0$. Define $f : C[a,b] \times C[a,b] \to C[a,b]$ by

$$f(x,y)(t) = x(t) + ty(t)^2\ , \tag{3.29}$$

for all $t \in [a,b]$, and $\phi : X \times X \times X \to \mathbb{R}$ by

$$\phi(x,y,z) = b\left(\|x\| + \|y\|\right)^2. \tag{3.30}$$

It is not difficult to show that ϕ satisfies Theorem 3.1 since

$$\begin{aligned}
&\left\|f(x+y,y+z)-2f\left(\frac{x}{2},\frac{y}{2}\right)-2f\left(\frac{y}{2},\frac{z}{2}\right)\right\| \\
&\quad=\sup_{t\in[a,b]}\left|f(x+y,y+z)(t)-2f\left(\frac{x}{2},\frac{y}{2}\right)(t)-2f\left(\frac{y}{2},\frac{z}{2}\right)(t)\right| \\
&\quad=\sup_{t\in[a,b]}\left|\frac{t}{2}x(t)^2+\frac{t}{2}y(t)^2+2tx(t)y(t)\right| \qquad (3.31)\\
&\quad\le b\|x\|^2+b\|y\|^2+2b\|x\|\|y\|=b(\|x\|+\|y\|)^2 \\
&\quad=\phi(x,y,z),
\end{aligned}$$

for all $x,y,z\in X$. Theorem 3.1 implies that there exists a mapping $F: X\times X\to Y$ satisfying (1.1) and

$$\|f(x,y)-F(x,y)\|\le 2b\|x\|^2, \qquad (3.32)$$

for all $x,y\in X$.

Example 3.5. Let X be a normed space. Define the function $\phi_0 : X\times X\to[0,\infty)$ by

$$\phi_0(x,y)=\left(\sup_{f,g\in B_{X'}}\begin{vmatrix}f(x)&f(y)\\g(x)&g(y)\end{vmatrix}\right)^p, \qquad (3.33)$$

where $B_{X'}$ is the closed unit ball of the dual space X' and $p>1$. Then one can show that the function $\phi: X\times X\times X\to[0,\infty)$, defined by

$$\phi(x,y,z)=\phi_0(x,y)+\phi_0(y,z), \qquad (3.34)$$

satisfies Theorem 3.1.

Example 3.6. Define $\phi(x,y,z)=\ell^p\times\ell^p\times\ell^p\to[0,\infty)$ by

$$\phi(x,y,z)=\sum_{i=1}^{\infty}|\xi_i|^p+\sum_{i=1}^{\infty}|\eta_i|^p+\sum_{i=1}^{\infty}|\zeta_i|^p, \qquad (3.35)$$

where $x=(\xi_1,\xi_2,\ldots)$, $y=(\eta_1,\eta_2,\ldots)$, and $z=(\zeta_1,\zeta_2,\ldots)$. It is obvious that ϕ satisfies Theorem 3.1.

Theorem 3.7. *Suppose that $f: X\times X\to Y$ is a mapping and $\phi: X\times X\times X\to[0,\infty)$ is a function such that*

$$\left\|f(x+y,y+z)-2f\left(\frac{x}{2},\frac{y}{2}\right)-2f\left(\frac{y}{2},\frac{z}{2}\right)\right\|\le\phi(x,y,z), \qquad (3.36)$$

$\phi(0,0,0) = 0$ *and* $\sum_{i=1}^{\infty}(1/2^i)\phi(2^i x, 2^i y, 2^i z) < \infty$ *for all* $x, y, z \in X$*. Then there exists a mapping* $F : X \times X \to Y$ *satisfying* (1.1),

$$\|f(x,y) - F(x,y)\| \leq \sum_{i=2}^{\infty}\frac{1}{2^i}\left(\phi\left(2^i x,0,0\right) + \phi\left(0,0,2^i y\right)\right) + \phi(x,0,y),$$
$$\lim_{n\to\infty}\frac{1}{2^n}f(2^n x, 2^n y) = F(x,y), \tag{3.37}$$

for all $x, y \in X$*.*

Proof. It is clear that $f(0,0) = 0$. Now, letting $y = z = 0$, we obtain

$$\left\|\frac{1}{2}f(2x,0) - f(x,0)\right\| \leq \frac{1}{2}\phi(2x,0,0). \tag{3.38}$$

By induction, we have

$$\left\|\frac{1}{2^n}f(2^n x,0) - f(x,0)\right\| \leq \sum_{i=1}^{n}\frac{1}{2^i}\phi\left(2^i x,0,0\right). \tag{3.39}$$

It follows that, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n, m \in \mathbb{N}$ with $m > n > N$,

$$\left\|\frac{1}{2^m}f(2^m x,0) - \frac{1}{2^n}f(2^n x,0)\right\| \leq \sum_{i=n+1}^{m}\frac{1}{2^i}\phi\left(2^i x,0,0\right) < \epsilon. \tag{3.40}$$

Thus $\{(1/2^n)f(2^n x,0)\}$ is a Cauchy sequence in Y. The completeness of Y implies that there exists $h(x)$ such that

$$\lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^n x,0) - h(x)\right\| = 0. \tag{3.41}$$

Since

$$\left\|h(x) - \frac{1}{2}h(2x)\right\| \leq \left\|h(x) - \frac{1}{2^n}f(2^n x,0)\right\| + \left\|\frac{1}{2^n}f(2^n x,0) - \frac{1}{2}h(2x)\right\| \longrightarrow 0, \tag{3.42}$$

as $n \to \infty$, $h(x) = (1/2)h(2x)$.

Similarly, we conclude that there exists a mapping $g : X \to Y$ such that

$$\lim_{n\to\infty}\left\|\frac{1}{2^n}f(0,2^n z) - g(z)\right\| = 0, \tag{3.43}$$

and $g(z) = (1/2)g(2z)$. Let

$$d_n(x,y,z) = \frac{1}{2}f\left(2^{n+1}(x+y), 2^{n+1}(y+z)\right) - f(2^n x, 2^n y) - f(2^n y, 2^n z), \tag{3.44}$$

and $F_n = \|(1/2^{n+1}) f(2^{n+1}(x+y), 2^{n+1}y) - h(x+y) - g(y)\|$. Then we have

$$\begin{aligned}
F_n &= \left\| \frac{1}{2^n} d_n(x+y,0,y) + \frac{1}{2^n} f(2^n(x+y),0) + \frac{1}{2^n} f(0,2^n y) - h(x+y) - g(y) \right\| \\
&\le \left\| \frac{1}{2^n} d_n(x+y,0,y) \right\| + \left\| \frac{1}{2^n} f(2^n(x+y),0) - h(x+y) \right\| + \left\| \frac{1}{2^n} f(0,2^n y) - g(y) \right\| \\
&\le \frac{1}{2^n} \| d_n(x+y,0,y) \| + \left\| \frac{1}{2^n} f(2^n(x+y),0) - h(x+y) \right\| + \left\| \frac{1}{2^n} f(0,2^n y) - g(y) \right\| \\
&\le \frac{1}{2^n} \phi(2^n(x+y),0,2^n y) + \left\| \frac{1}{2^n} f(2^n(x+y),0) - h(x+y) \right\| + \left\| \frac{1}{2^n} f\left(0,2^{n+1} y\right) - g(y) \right\|,
\end{aligned} \tag{3.45}$$

for all $x, y \in X$. By hypothesis, (3.41) and (3.43), we get

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} f\left(2^{n+1}(x+y), 2^{n+1}y\right) = h(x+y) + g(y), \tag{3.46}$$

for all $x, y \in X$.

Now, let $L_n = \|(1/2^{n-1}) f(2^{n-1}x, 2^{n-1}y) - h(x+y) - g(y) + h(y)\|$. Then we obtain

$$\begin{aligned}
L_n &= \left\| -\frac{1}{2^{n-1}} d_{n-1}(x,y,0) + \frac{1}{2^n} f(2^n(x+y),2^n y) - \frac{1}{2^n} f(2^n y,0) - h(x+y) - g(y) + h(y) \right\| \\
&\le \left\| \frac{1}{2^{n-1}} d_{n-1}(x,y,0) \right\| + \left\| \frac{1}{2^n} f(2^n(x+y),2^n y) - h(x+y) - g(y) \right\| \\
&\quad + \left\| \frac{1}{2^{n-1}} f\left(2^{n-1}y,0\right) - h(y) \right\| \\
&\le \frac{1}{2^n} \phi(2^n x, 2^n y, 0) + \left\| \frac{1}{2^n} f\left(2^n(x+y),2^{n-1} y\right) - h(x+y) - g(y) \right\| \\
&\quad + \left\| \frac{1}{2^{n-1}} f\left(2^{n-1}y,0\right) - h(y) \right\|,
\end{aligned} \tag{3.47}$$

where the last inequality follows from (3.46). By (3.41) and (3.43), we conclude

$$\lim_{n\to\infty} \frac{1}{2^{n-1}} f\left(2^{n-1}x, 2^{n-1}y\right) = h(x+y) + g(y) - h(y), \tag{3.48}$$

for all $x, y \in X$. Put $M_n = \|(1/2^{n+1}) f(2^{n+1}x, 2^{n+1}y) - h(x) - g(y)\|$. Since

$$\begin{aligned}
M_n &= \left\| \frac{1}{2^n} d_n(x,0,y) + \frac{1}{2^n} f(2^n x,0) + \frac{1}{2^n} f(0,2^n y) - h(x) - g(y) \right\| \\
&\le \left\| \frac{1}{2^n} d_n(x,0,y) \right\| + \left\| \frac{1}{2^n} f(2^n x,0) - h(x) \right\| + \left\| \frac{1}{2^n} f(0,2^n y) - g(y) \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2^{n+1}}\phi\left(2^{n+1}x,0,2^{n+1}y\right)+\left\|\frac{1}{2^n}f(2^nx,0)-h(x)\right\| \\
&\quad+\left\|\frac{1}{2^n}f(0,2^ny)-g(y)\right\| \\
&\longrightarrow 0,
\end{aligned} \tag{3.49}$$

as $n \to \infty$, where the last inequality follows from (3.41) and (3.43). So we have

$$\lim_{n\to\infty}\frac{1}{2^{n+1}}f\left(2^{n+1}x,2^{n+1}y\right)=h(x)+g(y), \tag{3.50}$$

for all $x, y \in X$. By (3.48) and (3.50), we have

$$h(x+y)+g(y)-h(y)=h(x)+g(y), \tag{3.51}$$

for all $x, y \in X$. It follows that $h(x+y)=h(x)+h(y)$.

Similarly, we obtain $g(x+y)=g(x)+g(y)$.

Now, define the mapping $F : X\times X \to Y$ by $F(x,y)=h(x)+g(y)$. Then we have

$$\begin{aligned}
F(x+y,y+z)&=h(x)+h(y)+g(y)+g(z) \\
&=2h\left(\frac{x}{2}\right)+2h\left(\frac{y}{2}\right)+2g\left(\frac{y}{2}\right)+2g\left(\frac{z}{2}\right) \\
&=2F\left(\frac{x}{2},\frac{y}{2}\right)+2F\left(\frac{y}{2},\frac{z}{2}\right).
\end{aligned} \tag{3.52}$$

Now, we have

$$\begin{aligned}
\|f(x,y)-F(x,y)\| &\leq \left\|f(x,y)-\frac{1}{2^n}f(2^nx,0)-\frac{1}{2^n}f(0,2^ny)\right\| \\
&\quad+\left\|\frac{1}{2^n}f(2^nx,0)+\frac{1}{2^n}f(0,2^ny)-F(x,y)\right\|.
\end{aligned} \tag{3.53}$$

Let $H_n=\|f(x,y)-(1/2^n)f(2^nx,0)-(1/2^n)f(0,2^ny)\|$. Since we know that

$$\lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^nx,0)+\frac{1}{2^n}f(0,2^ny)-F(x,y)\right\|=0, \tag{3.54}$$

to prove (3.37), it is sufficient to show that

$$H_n \leq \sum_{i=2}^{n}\frac{1}{2^i}\left(\phi\left(2^ix,0,0\right)+\phi\left(0,0,2^iy\right)\right)+\phi(x,0,y), \tag{3.55}$$

for all $n \in \mathbb{N}$. We have

$$H_n \leq \left\| f(x,y) - \frac{1}{2}f(2x,0) - \frac{1}{2}f(0,2y) \right\| + \left\| \frac{1}{2}f(2x,0) - \frac{1}{2^n}f(2^n x,0) \right\| + \left\| \frac{1}{2}f(0,2y) - \frac{1}{2^n}f(0,2^n y) \right\| \leq \sum_{i=2}^{n} \frac{1}{2^i}\left(\phi\left(2^i x,0,0\right) + \phi\left(0,0,2^i y\right)\right) + \phi(x,0,y), \tag{3.56}$$

for all $x, y \in X$. As $n \to \infty$, we conclude

$$\|f(x,y) - F(x,y)\| \leq \sum_{i=2}^{\infty} \frac{1}{2^i}\left(\phi\left(2^i x,0,0\right) + \phi\left(0,0,2^i y\right)\right) + \phi(x,0,y), \tag{3.57}$$

for all $x, y \in X$. This completes the proof. □

Corollary 3.8. *Suppose that $\epsilon > 0$ and $f : X \times X \to Y$ is a mapping satisfying*

$$\left\| f(x+y, y+z) - 2f\left(\frac{x}{2}, \frac{y}{2}\right) - 2f\left(\frac{y}{2}, \frac{z}{2}\right) \right\| \leq \epsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right), \tag{3.58}$$

for some $p < 1$. Then there exists a mapping $F : X \times X \to Y$ satisfying (1.1),

$$\|f(x,y) - F(x,y)\| \leq \frac{2 - 2^p + 2^{2p-1}}{2 - 2^p}\epsilon\left(\|x\|^p + \|y\|^p\right), \qquad \lim_{n\to\infty} \frac{1}{2^n} f(2^n x, 2^n y) = F(x,y), \tag{3.59}$$

for all $x, y \in X$.

Example 3.9. (1) Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then one can show that the function $\phi : X \times X \times X \to \mathbb{R}$, defined by

$$\phi(x,y,z) = \left|\langle x,y\rangle\right|^p + \left|\langle y,z\rangle\right|^p \quad (p < 1), \tag{3.60}$$

satisfies Theorem 3.7.

(2) Suppose that $(X, \|\cdot\|)$ is a normed space. Then the function $\phi : X \times X \times X \to \mathbb{R}$, defined by $\phi(x,y,z) = \|x\|\|y\|\|z\|$, satisfies Theorem 3.7.

4. A Fixed Point Approach to the Stability

In this section, we apply the fixed point method to prove the stability of the functional equation (1.1).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ for all $x, y \in X$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We now introduce one of the fundamental results of fixed point theory.

Theorem 4.1 (see [13]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a contractive mapping with constant L. Then for each $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty, \tag{4.1}$$

for all $n \in \mathbb{N}$ or there exists an $n_0 \in \mathbb{N}$ such that

(a) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(b) *the sequence* $\{J^n x\}$ *converges to fixed point* x_0 *of* J;

(c) x_0 *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(d) $d(y, y_0) \leq (1/(1 - L)) d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [15–22]).

Theorem 4.2. *Suppose that X is a vector space and Y is a Banach space. Let $f : X \times X \to Y$ be a mapping and let $\phi : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

(a) $\|f(x + y, y + z) - 2f(x/2, y/2) - 2f(y/2, z/2)\| \leq \phi(x, y, z)$;

(b) $\phi(0, 0, 0) = 0$;

(c) *for all* $x, y \in X$,

$$\lim_{n \to \infty} \frac{1}{2^n} \phi(2^n x, 2^n y, 2^n z) = 0; \tag{4.2}$$

(d) *there exists a number* $L \in [0, 1)$ *such that*

$$\phi(x, y, \sigma(x, y)) \leq 2L\phi\left(\frac{x}{2}, \frac{y}{2}, \sigma\left(\frac{x}{2}, \frac{y}{2}\right)\right), \tag{4.3}$$

where $\sigma : X \times X \to X$ *is a mapping such that* $\sigma(x, 0) = 0$ *for all* $x \in X$.

Then there exists a unique mapping $F : X \times X \to Y$ satisfying (1.1) *and*

$$\|f(x, y) - F(x, y)\| \leq \frac{1}{2 - 2L} \phi(x, y, \sigma(x, y)), \tag{4.4}$$

for all $x, y \in X$.

Proof. Consider the set

$$\mathcal{S} = \{g : X \times X \longrightarrow Y\}, \tag{4.5}$$

and define the generalized metric on $\mathcal{S}$ by

$$d(g,h) = \inf\{C : \|g(x,y) - h(x,y)\| \le C\phi(x,y,\sigma(x,y)), \quad \forall(x,y) \in X \times X\}. \tag{4.6}$$

It is easy to show that $(\mathcal{S}, d)$ is complete.

Now, we consider the linear mapping $J : \mathcal{S} \to \mathcal{S}$ such that

$$Jg(x,y) = \frac{1}{2}g(2x,2y), \tag{4.7}$$

for all $x, y \in X$. The definition of J implies that

$$d(Jg, Jh) \le L\,d(g,h), \tag{4.8}$$

for all $g, h \in \mathcal{S}$. Replacing x by $2x$ and putting $y = z = 0$ in the first statement, we get

$$\|f(2x,0) - 2f(x,0)\| \le \phi(x,0,0) = \phi(2x,0,\sigma(2x,0)), \tag{4.9}$$

for all $x \in X$. So

$$\left\|f(x,0) - \frac{1}{2}f(2x,0)\right\| \le \frac{1}{2}\phi(2x,0,\sigma(2x,0)), \tag{4.10}$$

for all $x \in X$. Hence $d(f, Jf) \le 1/2$. By Theorem 4.1, there exists a mapping $F : X \times X \to Y$ such that F is a fixed point of J, that is,

$$F(2x,2y) = 2F(x,y), \tag{4.11}$$

for all $x, y \in X$. The mapping F is a unique fixed point of J in the set

$$\mathcal{S}' = \{g \in \mathcal{S} : d(f,g) < \infty\}. \tag{4.12}$$

This implies that there exists a $C < \infty$ such that

$$\|F(x,y) - f(x,y)\| \le C\phi(x,y,\sigma(x,y)), \tag{4.13}$$

for all $x, y \in X$. Also, we have

$$d(J^n f, F) \longrightarrow 0, \tag{4.14}$$

as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x, 2^n y) = F(x,y), \tag{4.15}$$

for all $x, y \in X$. By the third statement of Theorem 4.1, we have $d(f,F) \le (1/(1-L))d(f,Jf)$. This implies that

$$d(f,F) \le \frac{1}{2-2L}. \tag{4.16}$$

Thus, by the definition of $d(\cdot,\cdot)$, we conclude

$$\|f(x,y) - F(x,y)\| \le \frac{1}{2-2L}\phi(x,y,\sigma(x,y)), \tag{4.17}$$

for all $x, y \in X$. Let

$$D_F = F(x+y, y+z) - 2F\left(\frac{x}{2},\frac{y}{2}\right) - 2F\left(\frac{y}{2},\frac{z}{2}\right). \tag{4.18}$$

It follows from (4.15) that

$$\begin{aligned} \|D_F\| &= \lim_{n\to\infty}\frac{1}{2^n}\left\|f(2^n(x+y), 2^n(y+z)) - 2f\left(2^n\frac{x}{2}, 2^n\frac{y}{2}\right) - 2f\left(2^n\frac{y}{2}, 2^n\frac{z}{2}\right)\right\| \\ &\le \lim_{n\to\infty}\frac{1}{2^n}\phi(2^n x, 2^n y, 2^n z) = 0. \end{aligned} \tag{4.19}$$

So we have

$$F(x+y, y+z) = 2F\left(\frac{x}{2},\frac{y}{2}\right) + 2F\left(\frac{y}{2},\frac{z}{2}\right), \tag{4.20}$$

for all $x, y, z \in X$. So F satisfies (1.1). This completes the proof. □

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