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Research Article

Generalized Systems of Variational Inequalities and Projection Methods for Inverse-Strongly Monotone Mappings

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We introduce an iterative sequence for finding a common element of the set of fixed points of a nonexpansive mapping and the solutions of the variational inequality problem for three inverse-strongly monotone mappings. Under suitable conditions, some strong convergence theorems for approximating a common element of the above two sets are obtained. Moreover, using the above theorem, we also apply to find solutions of a general system of variational inequality and a zero of a maximal monotone operator in a real Hilbert space. As applications, at the end of the paper we utilize our results to study some convergence problem for strictly pseudocontractive mappings. Our results include the previous results as special cases extend and improve the results of Ceng et al., (2008) and many others.

1. Introduction

Variational inequalities are known to play a crucial role in mathematics as a unified framework for studying a large variety of problems arising, for instance, in structural analysis, engineering sciences and others.Roughly speaking, they can be recast as fixed-point problems, and most of the numerical methods related to this topic are based on projection methods. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, and let E be a nonempty, closed, convex subset of H. A mapping $A : E \to H$ is called α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in E \tag{1.1}$$

(see [1, 2]). It is obvious that every α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous. A mapping $S : E \to E$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.2}$$

We denote by $F(S)$ the set of fixed points of S and by P_E the metric projection of H onto E. Recall that the classical variational inequality, denoted by $VI(A, E)$, is to find an $x^* \in E$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in E. \tag{1.3}$$

The set of solutions of $VI(A, E)$ is denoted by Γ. The variational inequality has been widely studied in the literature; see, for example, [3–6] and the references therein.

For finding an element of $F(S) \cap \Gamma$, Takahashi and Toyoda [7] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_E (x_n - \lambda_n A x_n), \tag{1.4}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in E$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbf{R}^n$, Korpelevich (1976) [8] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in E, \\ y_n &= P_E(x_n - \lambda_n A x_n), \\ x_{n+1} &= P_E(x_{\mathrm{n}} - \lambda_n A y_n), \end{aligned} \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \in (0, 1/k)$. Many authors using extragradient method for approximating common fixed points and variational inequality problems (see also [9, 10]). Recently, Nadezhkina and Takahashi [11] and Zeng et al. [12] proposed some iterative schemes for finding elements in $F(S) \cap \Gamma$ by combining (1.4) and (1.5). Further, these iterative schemes are extended in Y. Yao and J. C. Yao [13] to develop a new iterative scheme for finding elements in $F(S) \cap \Gamma$.

Consider the following problem of finding $(x^*, y^*) \in E \times E$ such that (see cf. Ceng et al. [14]):

$$\begin{aligned} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in E, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in E, \end{aligned} \tag{1.6}$$

which is called *general system of variational inequalities* (GSVI), where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.6) reduces to finding $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in E, \end{aligned} \tag{1.7}$$

which is defined by [15, 16], and is called the new system of variational inequalities. Further, if $x^* = y^*$, then problem (1.7) reduces to the *classical variational inequality* $\mathrm{VI}(A, E)$, that is, find $x^* \in E$ such that $\langle Ax^*, x - x^* \rangle \geq 0$, for all $x \in E$.

We can characteristic problem, if $x^* \in F(S) \cap VI(A, E)$, then it follows that $x^* = Sx^* = P_E[x^* - \rho Ax^*]$, where $\rho > 0$ is a constant.

In 2008, Ceng et al. [14] introduced a relaxed extragradient method for finding solutions of problem (1.6). Let the mappings $A, B : E \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : E \to E$ be a nonexpansive mapping. Suppose $x_1 = u \in E$ and $\{x_n\}$ is generated by

$$\begin{aligned} y_n &= P_E(x_n - \mu Bx_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_E(y_n - \lambda_n \mathrm{A} y_n), \end{aligned} \tag{1.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. First, problem (1.6) is proven to be equivalent to a fixed point problem of a nonexpansive mapping.

In this paper, motivated by what is mentioned above, we consider generalized system of variational inequalities as follows.

Let E be a nonempty, closed, convex subset of a real Hilbert space H. Let $A, B, C : E \to H$ be three mappings. We consider the following problem of finding $(x^*, y^*, z^*) \in E \times E \times E$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \mu Bz^* + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \tau Cx^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in E, \end{aligned} \tag{1.9}$$

which is called a *general system of variational inequalities* where $\lambda > 0$, $\mu > 0$ and $\tau > 0$ are three constants.

In particular, if $A = B = C$, then problem (1.9) reduces to finding $(x^*, y^*, z^*) \in E \times E \times E$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \mu Az^* + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \tau Ax^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in E. \end{aligned} \tag{1.10}$$

Next, we consider some special classes of the GSVI problem (1.9) reduce to the following GSVI.

(i) If $\tau = 0$, then the GSVI problems (1.9) reduce to GSVI problem (1.6).

(ii) If $\tau = \mu = 0$, then the GSVI problems (1.9) reduce to classical variational inequality VI(A,E) problem.

The above system enters a class of more general problems which originated mainly from the Nash equilibrium points and was treated from a theoretical viewpoint in [17, 18]. Observe at the same time that, to construct a mathematical model which is as close as possible to a real complex problem, we often have to use constraints which can be expressed as one several subproblems of a general problem. These constrains can be given, for instance, by variational inequalities, by fixed point problems, or by problems of different types.

This paper deals with a relaxed extragradient approximation method for solving a system of variational inequalities over the fixed-point sets of nonexpansive mapping. Under classical conditions, we prove a strong convergence theorem for this method. Moreover, the proposed algorithm can be applied for instance to solving the classical variational inequality problems.

2. Preliminaries

Let E be a nonempty, closed, convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in E, denoted by $P_E x$, such that

$$\|x - P_E x\| \leq \|x - y\|, \quad \forall y \in E. \tag{2.1}$$

P_E is call the metric projection of H onto E.

Recall that, $P_E x$ is characterized by following properties: $P_E x \in E$ and

$$\begin{gathered} \langle x - P_E x, y - P_E x \rangle \leq 0, \\ \|x - y\|^2 \geq \|x - P_E x\|^2 + \|P_E x - y\|^2, \end{gathered} \tag{2.2}$$

for all $x \in H$ and $y \in E$.

Lemma 2.1 (see cf. Zhang et al. [19]). *The metric projection P_E has the following properties:*

(i) $P_E : H \to E$ *is nonexpansive;*

(ii) $P_E : H \to E$ *is firmly nonexpansive, that is,*

$$\|P_E x - P_E y\|^2 \leq \langle P_E x - P_E y, x - y \rangle, \quad \forall x, y \in H; \tag{2.3}$$

(iii) *for each* $x \in H$,

$$z = P_E(x) \Longleftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in E. \tag{2.4}$$

Lemma 2.2 (see Osilike and Igbokwe [20]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2. \quad (2.5)$$

Lemma 2.3 (see Suzuki [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.4 (see Xu [22]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \delta_n, \quad n \ge 0, \quad (2.6)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that

(i) $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

Lemma 2.5 (Goebel and Kirk [23]). *Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a nonempty, closed, convex subset E of a real Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in E converging weakly to some $x \in E$ (for short, $x_n \rightharpoonup x \in E$), and the sequence $\{(I-T)x_n\}$ converges strongly to some y (for short, $(I-T)x_n \to y$), it follows that $(I-T)x = y$. Here, I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \quad (2.7)$$

Remark 2.7. We also have that, for all $u, v \in E$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle \\ &\quad + \lambda^2\|Au - Av\|^2 \\ &\le \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \quad (2.8)$$

So, if $\lambda \le 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from E to H.

3. Main Results

In this section, we introduce an iterative precess by the relaxed extragradient approximation method for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of the variational inequality problem for three inverse-strongly monotone mappings in a real Hilbert space. We prove that the iterative sequence converges strongly to a common element of the above two sets.

In order to prove our main result, the following lemmas are needed.

Lemma 3.1. *For given $x^*, y^*, z^* \in E \times E \times E$, (x^*, y^*, z^*) is a solution of problem* (1.9) *if and only if x^* is a fixed point of the mapping $G : E \to E$ defined by*

$$\begin{aligned} G(x) = P_E\{&P_E[P_E(x-\tau Cx) - \mu BP_E(x-\tau Cx)] \\ &-\lambda AP_E[P_E(x-\tau Cx) - \mu BP_E(x-\tau Cx)]\}, \quad \forall x \in E, \end{aligned} \tag{3.1}$$

where $y^* = P_E(z^* - \mu Bz^*)$ and $z^* = P_E(x^* - \tau Cx^*)$.

Proof.

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \mu Bz^* + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle \tau Cx^* + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in E, \end{aligned} \tag{3.2}$$

$\Leftrightarrow$

$$\begin{aligned} &\langle (-y^* + \lambda Ay^*) + x^*, x - x^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle (-z^* + \mu Bz^*) + y^*, x - y^* \rangle \geq 0, \quad \forall x \in E, \\ &\langle (-x^* + \tau Cx^*) + z^*, x - z^* \rangle \geq 0, \quad \forall x \in E, \end{aligned} \tag{3.3}$$

$\Leftrightarrow$

$$\begin{aligned} &\langle (y^* - \lambda Ay^*) - x^*, x^* - x \rangle \geq 0, \quad \forall x \in E, \\ &\langle (z^* - \mu Bz^*) - y^*, y^* - x \rangle \geq 0, \quad \forall x \in E, \\ &\langle (x^* - \tau Cx^*) - z^*, z^* - x \rangle \geq 0, \quad \forall x \in E, \end{aligned} \tag{3.4}$$

$\Leftrightarrow$

$$\begin{aligned} x^* &= P_E(y^* - \lambda Ay^*), \\ y^* &= P_E(z^* - \mu Bz^*), \\ z^* &= P_E(x^* - \tau Cx^*), \end{aligned} \tag{3.5}$$

$\Leftrightarrow x^* = P_E[P_E(z^* - \mu Bz^*) - \lambda AP_E(z^* - \mu Bz^*)]$.

Thus,

$$x^* = P_E\{P_E[P_E(x^* - \tau Cx^*) - \mu BP_E(x^* - \tau Cx^*)] - \lambda AP_E[P_E(x^* - \tau Cx^*) - \mu BP_E(x^* - \tau Cx^*)]\}. \tag{3.6}$$

□

Lemma 3.2. *The mapping G defined by Lemma 3.1 is nonexpansive mappings.*

Proof. For all $x, y \in E$,

$$\begin{aligned}
\|G(x) - G(y)\| = \|&P_E\{P_E[P_E(x - \tau Cx) - \mu BP_E(x - \tau Cx)] \\
&\quad -\lambda AP_E[P_E(x - \tau Cx) - \mu BP_E(x - \tau Cx)]\} \\
&- P_E\{P_E[P_E(y - \tau Cy) - \mu BP_E(y - \tau Cy)] \\
&\quad -\lambda AP_E[P_E(y - \tau Cy) - \mu BP_E(y - \tau Cy)]\} \\
\le \|&[P_E(x - \tau Cx) - \mu BP_E(x - \tau Cx)] \\
&- \lambda AP_E[P_E(x - \tau Cx) - \mu BP_E(x - \tau Cx)] \\
&- [P_E(y - \tau Cy) - \mu BP_E(y - \tau Cy)] \\
&-\lambda AP_E[P_E(y - \tau Cy) - \mu BP_E(y - \tau Cy)]\| \\
= \|&(I - \lambda A)[P_E(x - \tau Cx) - \mu BP_E(x - \tau Cx)] \\
&-(I - \lambda A)[P_E(y - \tau Cy) - \mu BP_E(y - \tau Cy)]\| \\
\le \|&[P_E(x - \tau Cx) - \mu BP_E(x - \tau Cx)] - [P_E(y - \tau Cy) - \mu BP_E(y - \tau Cy)]\| \\
= \|&(I - \mu B)[P_E(x - \tau Cx)] - (I - \mu B)[P_E(y - \tau Cy)]\| \\
\le \|&P_E(x - \tau Cx) - P_E(y - \tau Cy)\| \\
\le \|&(x - \tau Cx) - (y - \tau Cy)\| \\
= \|&(I - \tau C)(x) - (I - \tau C)(y)\| \\
\le \|&x - y\|.
\end{aligned} \tag{3.7}$$

This shows that $G : E \to E$ is a nonexpansive mapping. □

Throughout this paper, the set of fixed points of the mapping G is denoted by Υ. Now, we are ready to proof our main results in this paper.

Theorem 3.3. *Let E be a nonempty, closed, convex subset of a real Hilbert space H. Let the mapping $A, B, C : E \to H$ be α-inverse-strongly monotone, β-inverse-strongly monotone, and γ-inverse-strongly monotone, respectively. Let S be a nonexpansive mapping of E into itself such that $F(S) \cap \Upsilon \neq \emptyset$. Let f be a contraction of H into itself and given $x_1 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{aligned} z_n &= P_E(x_n - \tau C x_n), \\ y_n &= P_E(z_n - \mu B z_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_E(y_n - \lambda A y_n), \quad n \geq 0, \end{aligned} \tag{3.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, $\tau \in (0, 2\gamma)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ converges strongly to $\overline{x} \in F(S) \cap \Gamma$, where $\overline{x} = P_{F(S)\cap\Gamma} f(\overline{x})$ and $(\overline{x}, \overline{y}, \overline{z})$ is a solution of problem (1.9), where

$$\begin{aligned} \overline{y} &= P_E(\overline{z} - \mu B \overline{z}), \\ \overline{z} &= P_E(\overline{x} - \tau C \overline{x}). \end{aligned} \tag{3.9}$$

Proof. Let $x^* \in F(S) \cap \Gamma$. Then, $x^* = Sx^*$ and $x^* = Gx^*$, that is,

$$x^* = P_E\{P_E[P_E(x^* - \tau C x^*) - \mu B P_E(x^* - \tau C x^*)] - \lambda A P_E[P_E(x^* - \tau C x^*) - \mu B P_E(x^* - \tau C x^*)]\}. \tag{3.10}$$

Put $x^* = P_E(y^* - \lambda A y^*)$ and $t_n = P_E(y_n - \lambda A y_n)$. Then, $x^* = P_E[P_E(z^* - \mu B z^*) - \lambda A P_E(z^* - \mu B z^*)]$ implies that $y^* = P_E(z^* - \mu B z^*)$, where $z^* = P_E(x^* - \tau C x^*)$. Since $I - \lambda A$, $I - \mu B$ and $I - \tau C$ are nonexpansive mappings. We obtain that

$$\begin{aligned} \|t_n - x^*\| &= \|P_E(y_n - \lambda A y_n) - x^*\| \\ &= \|P_E(y_n - \lambda A y_n) - P_E(y^* - \lambda A y^*)\| \\ &\leq \|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\| \\ &= \|(I - \lambda A) y_n - (I - \lambda A) y^*\| \\ &\leq \|y_n - y^*\| \\ &= \|y_n - P_E(z^* - \mu B z^*)\| \\ &= \|P_E(z_n - \mu B z_n) - P_E(z^* - \mu B z^*)\| \end{aligned} \tag{3.11}$$

$$\leq \left\| (I-\mu B)z_n - (I-\mu B)z^* \right\| \leq \|z_n - z^*\|, \tag{3.12}$$

$$\begin{aligned} \|z_n - z^*\| &= \|P_E(x_n - \tau C x_n) - P_E(x^* - \tau C x^*)\| \\ &\leq \|(x_n - \tau C x_n) - (x^* - \tau C x^*)\| \\ &= \|(I-\tau C)x_n - (I-\tau C)x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.12), we have

$$\|t_n - x^*\| \leq \|x_n - x^*\|, \tag{3.14}$$

and by (3.11) we also have

$$\left\| y_n - y^* \right\| \leq \|x_n - x^*\|. \tag{3.15}$$

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n$ and by Lemma 2.2, we compute

$$\begin{aligned} \|x_{n+1} - x^*\| &= \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^* \right\| \\ &= \left\| \alpha_n \left(f(x_n) - x^* \right) + \beta_n (x_n - x^*) + \gamma_n (S t_n - x^*) \right\| \\ &\leq \alpha_n \left\| f(x_n) - x^* \right\| + \beta_n \|x_n - x^*\| + \gamma_n \|S t_n - x^*\| \\ &\leq \alpha_n \left\| f(x_n) - x^* \right\| + \beta_n \|x_n - x^*\| + \gamma_n \|t_n - x^*\| \\ &\leq \alpha_n \left\| f(x_n) - x^* \right\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= \alpha_n \left\| f(x_n) - x^* \right\| + (1-\alpha_n) \|x_n - x^*\| \\ &= \alpha_n \left\| f(x_n) - f(x^*) + f(x^*) - x^* \right\| + (1-\alpha_n) \|x_n - x^*\| \\ &\leq \alpha_n \left\| f(x_n) - f(x^*) \right\| + \alpha_n \left\| f(x^*) - x^* \right\| + (1-\alpha_n) \|x_n - x^*\| \\ &\leq \alpha_n k \|x_n - x^*\| + \alpha_n \left\| f(x^*) - x^* \right\| + (1-\alpha_n) \|x_n - x^*\| \end{aligned}$$

$$
\begin{aligned}
&= (\alpha_n k + (1-\alpha_n))\|x_n - x^*\| + \alpha_n \left\| f(x^*) - x^* \right\| \\
&= (1 - \alpha_n(1-k))\|x_n - x^*\| + \alpha_n \left\| f(x^*) - x^* \right\| \\
&= (1 - \alpha_n(1-k))\ \|x_n - x^*\| + \alpha_n (1-k) \frac{\left\| f(x^*) - x^* \right\|}{(1-k)}.
\end{aligned}
\tag{3.16}
$$

By induction, we get

$$
\|x_{n+1} - x^*\| \le M, \tag{3.17}
$$

where $M = \max\{\|x_0 - x^*\| + (1/(1-k))\|f(x^*) - x^*\|\},\ n \ge 0$. Therefore, $\{x_n\}$ is bounded. Consequently, by (3.11), (3.12) and (3.13), the sequences $\{t_n\}, \{St_n\}, \{y_n\}, \{Ay_n\}, \{z_n\}, \{Bz_n\}$, $\{Cx_n\}$, and $\{f(x_n)\}$ are also bounded. Also, we observe that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \|P_E(x_{n+1} - \tau C x_{n+1})\ - P_E(x_n - \tau C x_n)\| \\
&\le \|(I - \tau C)x_{n+1} - (I - \tau C)x_n\| \\
&\le \|x_{n+1} - x_n\|,
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\|t_{n+1} - t_n\| &= \left\| P_E(y_{n+1} - \lambda A y_{n+1}) - P_E(y_n - \lambda A y_n) \right\| \\
&\le \left\| (y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n) \right\| \\
&= \left\| (I - \lambda A) y_{n+1} - (I - \lambda A) y_n \right\| \\
&\le \left\| y_{n+1} - y_n \right\| \\
&= \left\| P_E(z_{n+1} - \mu B z_{n+1}) - P_E(z_n - \mu B z_n) \right\| \\
&\le \|z_{n+1} - z_n\| \\
&\le \|x_{n+1} - x_n\|.
\end{aligned}
\tag{3.19}
$$

Let $x_{n+1} = (1-\beta_n)w_n + \beta_n x_n$. Thus, we get

$$
w_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} = \frac{\alpha_n f(x_n) + \gamma_n S P_C(y_n - \lambda_n A y_n)}{1-\beta_n} = \frac{\alpha_n u + \gamma_n S t_n}{1-\beta_n} \tag{3.20}
$$

It follows that

$$
\begin{aligned}
w_{n+1}-w_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}St_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n St_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}f(x_{n+1})}{1-\beta_{n+1}}+\frac{\gamma_{n+1}St_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}+\frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)}{1-\beta_n}-\frac{\gamma_n St_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)+\frac{\gamma_{n+1}St_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n St_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)\\
&\quad+\frac{\gamma_{n+1}St_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_{n+1}St_n}{1-\beta_{n+1}}+\frac{\gamma_{n+1}St_n}{1-\beta_{n+1}}-\frac{\gamma_n St_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1}-St_n)+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)St_n\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)\\
&\quad+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)St_n+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1}-St_n)\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)\left(f(x_n)+St_n\right)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(St_{n+1}-St_n).
\end{aligned}
\tag{3.21}
$$

Combining (3.19) and (3.21), we obtain

$$
\begin{aligned}
\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|\left\|f(x_{n+1})-f(x_n)\right\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\left\|f(x_n)+St_n\right\|\\
&\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right|\|St_{n+1}-St_n\|-\|x_{n+1}-x_n\|\\
&\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|k\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\left\|f(x_n)+St_n\right\|\\
&\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right|\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|\\
&\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|k\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\left\|f(x_n)+St_n\right\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right|\|x_{n+1}-x_n\|-\|x_{n+1}-x_n\| \\
&=\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|k\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)+St_n\| \\
&+\left|\frac{\gamma_{n+1}-1+\beta_{n+1}}{1-\beta_{n+1}}\right|\|x_{n+1}-x_n\| \\
&=\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|k\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)+St_n\| \\
&+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right|\|x_{n+1}-x_n\|.
\end{aligned}
\tag{3.22}
$$

This together with (i), (ii), and (iii) implies that

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.23}$$

Hence, by Lemma 2.3, we have

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{3.24}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.25}$$

From (3.18) and (3.19), we also have $\|z_{n+1}-z_n\|\to 0\|t_{n+1}-t_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$ as $n\to\infty$. Since

$$x_{n+1}-x_n=\alpha_n f(x_n)+\beta_n x_n+\gamma_n St_n-x_n=\alpha_n\big(f(x_n)-x_n\big)+\gamma_n(St_n-x_n), \tag{3.26}$$

it follows by (ii) and (3.25) that

$$\lim_{n\to\infty}\|x_n-St_n\|=0. \tag{3.27}$$

Since $x^* \in F(S) \cap \Gamma$, from (3.15) and Lemma 2.2, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|St_n - x^*\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|P_E(y_n - \lambda A y_n) - P_E(y^* - \lambda A y^*)\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|(y_n - \lambda A y_n) - (y^* - \lambda A y^*)\right\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|(y_n - y^*) - \lambda (A y_n - A y^*)\right\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \left[\|y_n - y^*\|^2 - 2\lambda \langle y_n - y^*, A y_n - A y^* \rangle + \lambda^2 \left\|A y_n - A y^*\right\|^2\right] \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \left[\left\|y_n - y^*\right\|^2 - 2\lambda\alpha \left\|A y_n - A y^*\right\|^2 + \lambda^2 \left\|A y_n - A y^*\right\|^2\right] \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left[\left\|y_n - y^*\right\|^2 + \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2\right] \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left[\|x_n - x^*\|^2 + \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2\right] \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + (\beta_n + \gamma_n) \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 + \gamma_n \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2.
\end{aligned} \tag{3.28}$$

Therefore, we have

$$\begin{aligned}
-\gamma_n \lambda(\lambda - 2\alpha) \left\|A y_n - A y^*\right\|^2 &\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad \times (\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_n - x_{n+1}\|.
\end{aligned} \tag{3.29}$$

From (ii), (iii), and $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$, we get $\|A y_n - A y^*\| \to 0$ as $n \to \infty$.

Since $x^* \in F(S) \cap \Upsilon$, from (3.11) and Lemma 2.2, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|y_n - y^*\right\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|P_E(z_n - \mu B z_n) - P_E(z^* - \mu B z^*)\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|(z_n - \mu B z_n) - (z^* - \mu B z^*)\right\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left\|(z_n - z^*) - (\mu B z_n - \mu B z^*)\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left[\|z_n - z^*\|^2 + \mu(\mu - 2\beta)\|B z_n - B z^*\|^2\right] \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 + \gamma_n \mu(\mu - 2\beta)\|B z_n - B z^*\|^2.
\end{aligned} \tag{3.30}$$

Thus, we also have

$$\begin{aligned}
-\gamma_n \mu(\mu - 2\beta)\|B z_n - B z^*\|^2 &\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= \alpha_n \left\|f(x_n) - x^*\right\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad \times (\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|.
\end{aligned} \tag{3.31}$$

By again (ii), (iii), and (3.25), we also get $\|B z_n - B z^*\| \to 0$ as $n \to \infty$.

Let $x^* \in F(S) \cap \Upsilon$; again from (3.12), (3.13) and Lemma 2.2, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S t_n - x^*\right\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|t_n - x^*\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|z_n - z^*\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|(x_n - \tau C x_n) - (x^* - \tau C x^*)\|^2 \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left[\|x_n - x^*\|^2 + \tau(\tau - 2\gamma)\|C x_n - C x^*\|^2\right] \\
&\le \alpha_n \left\|f(x_n) - x^*\right\|^2 + \|x_n - x^*\|^2 + \gamma_n \tau(\tau - 2\gamma)\|C x_n - C x^*\|^2.
\end{aligned} \tag{3.32}$$

Again, we have

$$
\begin{aligned}
-\gamma_n\tau(\tau-2\gamma)\|Cx_n-Cx^*\|^2 &\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&= \alpha_n\left\|f(x_n)-x^*\right\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\
&\quad\times(\|x_n-x^*\|-\|x_{n+1}-x^*\|)\\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|.
\end{aligned}
\tag{3.33}
$$

Similarly again by (ii), (iii), and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, and from (3.33), we also that $\|Cx_n-Cx^*\|\to 0$.

On the other hand, we compute that

$$
\begin{aligned}
\|z_n-z^*\|^2 &= \|P_E(x_n-\tau Cx_n)-P_E(x^*-\tau Cx^*)\|^2\\
&\le \langle (x_n-\tau Cx_n)-(x^*-\tau Cx^*), P_E(x_n-\tau Cx_n)-P_E(x^*-\tau Cx^*)\rangle\\
&= \langle (x_n-\tau Cx_n)-(x^*-\tau Cx^*), z_n-z^*\rangle\\
&= \frac{1}{2}\Big[\|(x_n-\tau Cx_n)-(x^*-\tau Cx^*)\|^2+\|z_n-z^*\|^2\\
&\qquad -\|(x_n-\tau Cx_n)-(x^*-\tau Cx^*)-(z_n-z^*)\|^2\Big]\\
&= \frac{1}{2}\Big[\|(I-\tau C)x_n-(I-\tau C)x^*\|^2+\|z_n-z^*\|^2\\
&\qquad -\|(x_n-\tau Cx_n)-(x^*-\tau Cx^*)-(z_n-z^*)\|^2\Big]\\
&\le \frac{1}{2}\Big[\|x_n-x^*\|^2+\|z_n-z^*\|^2-\|(x_n-z_n)-\tau(Cx_n-Cx^*)-(x^*-z^*)\|^2\Big]\\
&= \frac{1}{2}\Big[\|x_n-x^*\|^2+\|z_n-z^*\|^2-\|[(x_n-z_n)-(x^*-z^*)]-\tau(Cx_n-Cx^*)\|^2\Big]\\
&= \frac{1}{2}\Big[\|x_n-x^*\|^2+\|z_n-z^*\|^2-\|(x_n-z_n)-(x^*-z^*)\|^2\\
&\qquad +2\tau\langle (x_n-z_n)-(x^*-z^*), Cx_n-Cx^*\rangle-\tau^2\|Cx_n-Cx^*\|^2\Big].
\end{aligned}
\tag{3.34}
$$

So, we obtain

$$
\begin{aligned}
\|z_n-z^*\|^2 &\le \|x_n-x^*\|^2-\|(x_n-z_n)-(x^*-z^*)\|^2\\
&\quad +2\tau\langle (x_n-z_n)-(x^*-z^*), Cx_n-Cx^*\rangle-\tau^2\|Cx_n-Cx^*\|^2.
\end{aligned}
\tag{3.35}
$$

Hence, by (3.12), it follows that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n St_n-x^*\right\|^2\\
&\le \alpha_n\left\|f(x_n)-x^*\right\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|St_n-x^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n k\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|t_n - x^*\|^2 \\
&\le \alpha_n k\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|z_n - z^*\|^2 \\
&\le \alpha_n k\|x_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 - \gamma_n\|(x_n - z_n) - (x^* - z^*)\|^2 \\
&\quad + 2\tau\gamma_n\langle (x_n - z_n) - (x^* - z^*), Cx_n - Cx^*\rangle - \tau^2\gamma_n\|Cx_n - Cx^*\|^2 \\
&= \alpha_n k\|x_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - \gamma_n\|(x_n - z_n) - (x^* - z^*)\|^2 \\
&\quad + 2\tau\gamma_n\langle (x_n - z_n) - (x^* - z^*), Cx_n - Cx^*\rangle - \tau^2\gamma_n\|Cx_n - Cx^*\|^2 \\
&\le \alpha_n k\|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|(x_n - z_n) - (x^* - z^*)\|^2 \\
&\quad + 2\tau\gamma_n\langle (x_n - z_n) - (x^* - z^*), Cx_n - Cx^*\rangle \\
&\le \alpha_n k\|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n\|(x_n - z_n) - (x^* - z^*)\|^2 \\
&\quad + 2\tau\gamma_n\|(x_n - z_n) - (x^* - z^*)\|\|Cx_n - Cx^*\|,
\end{aligned}
\tag{3.36}
$$

which implies that

$$
\begin{aligned}
\gamma_n\|(x_n - z_n) - (x^* - z^*)\|^2 &\le \alpha_n k\|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\tau\gamma_n\|(x_n - z_n) - (x^* - z^*)\|\|Cx_n - Cx^*\| \\
&\le \alpha_n k\|x_{n+1} - x^*\|^2 + 2\gamma_n\tau\|(x_n - z_n) - (x^* - z^*)\|\|Cx_n - Cx^*\| \\
&\quad + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned}
\tag{3.37}
$$

By (ii), (iii), $\|x_n - x_{n+1}\| \to 0$, and $\|Cx_n - Cx^*\| \to 0$ as $n \to \infty$, from (3.37) and we get $\|(x_n - z_n) - (x^* - z^*)\| \to 0$ as $n \to \infty$. Now, observe that

$$
\begin{aligned}
\|(z_n - t_n) + (x^* - z^*)\|^2 &= \left\|z_n - P_E(y_n - \lambda Ay_n) + P_E(y^* - \lambda Ay^*) - z^*\right\|^2 \\
&= \Big\|z_n - P_E(y_n - \lambda Ay_n) + P_E(y^* - \lambda Ay^*) \\
&\quad -z^* + \mu Bz_n - \mu Bz_n + \mu Bz^* - \mu Bz^*\Big\|^2 \\
&= \Big\|z_n - \mu Bz_n - (z^* - \mu Bz^*) \\
&\quad -\left[P_E(y_n - \lambda Ay_n) - P_E(y^* - \lambda Ay^*)\right] + \mu(Bz_n - Bz^*)\Big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) - \left[P_E\left(y_n - \lambda A y_n\right) - P_E\left(y^* - \lambda A y^*\right)\right] \right\|^2 \\
&\quad + 2\mu \langle B z_n - B z^*, z_n - \mu B z_n - \left(z^* - \mu B z^*\right) \\
&\qquad\qquad - \left[P_E\left(y_n - \lambda A y_n\right) - P_E\left(y^* - \lambda A y^*\right)\right] + \mu\left(B z_n - B z^*\right) \rangle \\
&= \left\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) - \left[P_E\left(y_n - \lambda A y_n\right) - P_E\left(y^* - \lambda A y^*\right)\right] \right\|^2 \\
&\quad + 2\mu \langle B z_n - B z^*, \left(z_n - t_n\right) + \left(x^* - z^*\right) \rangle \\
&\leq \left\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) \right\|^2 \\
&\quad - \left\| P_E\left(y_n - \lambda_n A y_n\right) - P_E\left(y^* - \lambda_n A y^*\right) \right\|^2 \\
&\quad + 2\mu \| B z_n - B z^* \| \| \left(z_n - t_n\right) + \left(x^* - z^*\right) \| \\
&\leq \left\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) \right\|^2 \\
&\quad - \left\| S P_E\left(y_n - \lambda_n A y_n\right) - S P_E\left(y^* - \lambda_n A y^*\right) \right\|^2 \\
&\quad + 2\mu \| B z_n - B z^* \| \| \left(z_n - t_n\right) + \left(x^* - z^*\right) \| \\
&= \left\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) \right\|^2 - \| S t_n - S x^* \|^2 \\
&\quad + 2\mu \| B z_n - B z^* \| \| \left(z_n - t_n\right) + \left(x^* - z^*\right) \| \\
&\leq \left\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) - \left(S t_n - x^*\right) \right\| \\
&\quad \times \left(\| z_n - \mu B z_n - \left(z^* - \mu B z^*\right) \| + \| S t_n - x^* \| \right) \\
&\quad + 2\mu \| B z_n - B z^* \| \| \left(z_n - t_n\right) + \left(x^* - z^*\right) \|.
\end{aligned}
\tag{3.38}
$$

Since $\|St_n - x_n\| \to 0$, $\|(x_n - z_n) - (x^* - z^*)\| \to 0$, and $\|Bz_n - Bz^*\| \to 0$, as $n \to \infty$, it follows that

$$\|(z_n - t_n) + (x^* - z^*)\| \to 0, \quad \text{as } n \to \infty. \tag{3.39}$$

Since

$$\|St_n - t_n\| \leq \|St_n - x_n\| + \|(x_n - z_n) - (x^* - z^*)\| + \|(z_n - t_n) + (x^* - z^*)\|, \tag{3.40}$$

we obtain

$$\lim_{n \to \infty} \|St_n - t_n\| = 0. \tag{3.41}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, x_n-\overline{x}\rangle \le 0, \tag{3.42}$$

where $\overline{x}=P_{F(S)\cap\Upsilon}f(\overline{x})$.

Indeed, since $\{t_n\}$ and $\{St_n\}$ are two bounded sequence in E, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that t_{n_i} of t_n such that $t_{n_i}\rightharpoonup z\in E$ and

$$\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, St_n-\overline{x}\rangle = \lim_{i\to\infty}\langle f(\overline{x})-\overline{x}, St_{n_i}-\overline{x}\rangle. \tag{3.43}$$

Since $\lim_{n\to\infty}\|St_n-t_n\|=0$, we obtain $St_{n_i}\rightharpoonup z$ as $i\to\infty$. Now, we claim that $z\in F(S)\cap\Upsilon$. First by Lemma 2.5, it is easy to see that $z\in F(S)$.

Since $\|St_n-t_n\|\to 0$, $\|St_n-x_n\|\to 0$, and

$$\begin{aligned}\|t_n-x_n\| &= \|t_n-St_n+St_n-x_n\|\\ &\le \|t_n-St_n\|+\|St_n-x_n\|\\ &= \|St_n-t_n\|+\|St_n-x_n\|,\end{aligned} \tag{3.44}$$

we conclude that $\|t_n-x_n\|\to 0$ as $n\to\infty$. Furthermore, by Lemma 3.2 that G is nonexpansive, then

$$\begin{aligned}\|t_n-G(t_n)\| &= \|G(x_n)-G(t_n)\|\\ &\le \|x_n-t_n\|.\end{aligned} \tag{3.45}$$

Thus $\lim_{n\to\infty}\|t_n-G(t_n)\|=0$. According to Lemma 2.5, we obtain $z\in\Upsilon$. Therefore, there holds $z\in F(S)\cap\Upsilon$.

On the other hand, it follows from (2.2) that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, x_n-\overline{x}\rangle &= \limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, St_n-\overline{x}\rangle\\ &= \lim_{n\to\infty}\langle f(\overline{x})-\overline{x}, St_{n_i}-\overline{x}\rangle\\ &= \langle f(\overline{x})-\overline{x}, z-\overline{x}\rangle\\ &\le 0.\end{aligned} \tag{3.46}$$

Finally, we show that $x_n \to \overline{x}$, and by (3.14) that

$$\begin{aligned}
\|x_{n+1}-\overline{x}\|^2 &= \left\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n St_n-\overline{x}\right\|^2 \\
&\le \left\|\beta_n(x_n-\overline{x})+\gamma_n(St_n-\overline{x})\right\|^2+2\alpha_n\left\langle f(x_n)-\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&\le \left\|\beta_n(x_n-\overline{x})+\gamma_n(St_n-\overline{x})\right\|^2+2\alpha_n\left\langle f(x_n)-f(\overline{x}),x_{n+1}-\overline{x}\right\rangle \\
&\quad +2\alpha_n\left\langle f(\overline{x})-\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&\le \left[\beta_n\|x_n-\overline{x}\|^2+\gamma_n\|St_n-\overline{x}\|^2\right]+2\alpha_n\left\|f(x_n)-f(\overline{x})\right\|\|x_{n+1}-\overline{x}\| \\
&\quad +2\alpha_n\left\langle f(\overline{x})-\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&\le \left[\beta_n\|x_n-\overline{x}\|^2+\gamma_n\|t_n-\overline{x}\|^2\right]+2\alpha_n\alpha\|x_n-\overline{x}\|\|x_{n+1}-\overline{x}\| \\
&\quad +2\alpha_n\left\langle f(\overline{x})-\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&\le (1-\alpha_n)^2\|x_n-\overline{x}\|^2+\alpha_n\alpha\left(\|x_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2\right) \\
&\quad +2\alpha_n\left\langle f(\overline{x})-\overline{x},x_{n+1}-\overline{x}\right\rangle
\end{aligned} \tag{3.47}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-\overline{x}\|^2 &\le \left(1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}\right)\|x_n-\overline{x}\|^2+\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-\overline{x}\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\left\langle f(\overline{x})-\overline{x},x_{n+1}-\overline{x}\right\rangle \\
&:= (1-\sigma_n)\|x_n-\overline{x}\|^2+\delta_n, \quad n\ge 0,
\end{aligned} \tag{3.48}$$

where $\sigma_n = (2(1-\alpha)\alpha_n)/1-\alpha\alpha_n$ and $\delta_n = \alpha_n^2/(1-\alpha\alpha_n)\|x_n-\overline{x}\|^2+2\alpha_n/(1-\alpha\alpha_n)\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle$. Therefore, by (3.46) and Lemma 2.4, we get that $\{x_n\}$ converges to $\overline{x}$, where $\overline{x} = P_{F(S)\cap\Upsilon}f(\overline{x})$. This completes the proof. □

Setting $A = B = C$, we obtain the following corollary.

Corollary 3.4. *Let E be a nonempty, closed, convex subset of a real Hilbert space H. Let the mapping $A : E \to H$ be α-inverse-strongly monotone. Let S be a nonexpansive mapping of E into itself such that $F(S)\cap\Upsilon\neq\emptyset$. Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{aligned}
z_n &= P_E(x_n-\tau Ax_n), \\
y_n &= P_E(z_n-\mu Az_n), \\
x_{n+1} &= \alpha_n f(x_n)+\beta_n x_n+\gamma_n SP_E(y_n-\lambda Ay_n), \quad n\ge 1,
\end{aligned} \tag{3.49}$$

where $\lambda,\mu,\tau \in (0,2\alpha)$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ *converges strongly to* $\overline{x} \in F(S) \cap \Upsilon$, *where* $\overline{x} = P_{F(S)\cap\Upsilon} f(\overline{x})$ *and* $(\overline{x}, \overline{y}, \overline{z})$ *is a solution of problem* (1.10), *where*

$$\begin{aligned} \overline{y} &= P_E(\overline{z} - \mu A \overline{z}), \\ \overline{z} &= P_E(\overline{x} - \tau A \overline{x}). \end{aligned} \tag{3.50}$$

Setting $A \equiv B \equiv 0$ (the zero operators), we obtain the following corollary for solving the foxed points problem and the classical variational inequality problems.

Corollary 3.5. *Let* E *be a nonempty, closed, convex subset of a real Hilbert space* H. *Let the mapping* $A : E \to H$ *be* α*-inverse-strongly monotone. Let* S *be a nonexpansive mapping of* E *into itself such that* $F(S) \cap VI(A,E) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and given* $x_0 \in H$ *arbitrarily and* $\{x_n\}$ *is generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_E(x_n - \lambda A x_n), \quad n \geq 1, \tag{3.51}$$

where $\lambda \in (0, 2\alpha)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ converges strongly to $\overline{x} \in F(S) \cap VI(A,E)$, where $\overline{x} = P_{F(S)\cap VI(A,E)} f(\overline{x})$.

4. Some Applications

We recall that a mapping $T : E \to E$ is called strictly pseudocontractive if there exists some k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in E. \tag{4.1}$$

For recent convergence result for strictly pseudocontractive mappings, put $A = I - T$. Then, we have

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2. \tag{4.2}$$

On the other hand,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle. \tag{4.3}$$

Hence, we have

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-k}{2}\|Ax-Ay\|^{2}. \tag{4.4}$$

Consequently, if $T : E \rightarrow E$ is a strictly pseudocontractive mapping with constant k, then the mapping $A = I - T$ is $(1-k)/2$-inverse-strongly monotone.

Setting $A = I - T$, $B = I - V$, and $C = I - W$, we obtain the following corollary.

Theorem 4.1. *Let E be a nonempty, closed, convex subset of a real Hilbert space H. Let T, V, W be strictly pseudocontractive mappings with constant k of C into itself, and let S be a nonexpansive mapping of E into itself such that $F(S) \cap \Upsilon \neq \emptyset$. Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{gathered} z_n = (I-\tau)x_n + \tau W x_n, \\ y_n = (I-\mu)z_n + \mu V z_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S\big((1-\lambda)y_n + \lambda T y_n\big), \quad n \geq 1, \end{gathered} \tag{4.5}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\tau \in (0, 2\gamma)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\overline{x} \in F(S) \cap \Upsilon$, where $\overline{x} = P_{F(S)\cap\Upsilon} f(\overline{x})$ and $(\overline{x}, \overline{y}, \overline{z})$ is a solution of problem (1.9), *where*

$$\begin{gathered} \overline{y} = P_E(\overline{z} - \mu B\overline{z}), \\ \overline{z} = P_E(\overline{x} - \tau C\overline{x}). \end{gathered} \tag{4.6}$$

Proof. Since $A = I - T$, $B = I - V$, and $C = I - W$, we have

$$\begin{gathered} P_E(x_n - \tau C x_n) = (I-\tau)x_n + \tau W x_n, \\ P_E(y_n - \lambda A y_n) = (I-\lambda)y_n + \lambda T y_n, \\ P_E(z_n - \mu B z_n) = (I-\mu)z_n + \mu V z_n. \end{gathered} \tag{4.7}$$

Thus, the conclusion follows immediately from Theorem 3.3. □

If $f(x) = x_0$, for all $x \in E$, and $T = V = W$ in Theorem 4.1, we obtain the following corollary.

Corollary 4.2. *Let E be a nonempty, closed, convex subset of a real Hilbert space H. Let T be strictly pseudocontractive mappings with constant k of C into itself, and let S be a nonexpansive mapping of E into itself such that $F(S) \cap \Upsilon \neq \emptyset$. Given $x_0 \in H$ arbitrarily and $\{x_n\}$ is generated by*

$$\begin{aligned} z_n &= (I-\tau)x_n + \tau T x_n, \\ y_n &= (I-\mu)z_n + \mu T z_n, \\ x_{n+1} &= \alpha_n x_0 + \beta_n x_n + \gamma_n S\big((1-\lambda)y_n + \lambda T y_n\big), \quad n \geq 1, \end{aligned} \tag{4.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\tau \in (0, 2\gamma)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ such that

$$\begin{gathered} \alpha_n + \beta_n + \gamma_n = 1, \\ \lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty, \\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \end{gathered} \tag{4.9}$$

Then, $\{x_n\}$ converges strongly to $\overline{x} \in F(S) \cap \Upsilon$, where $\overline{x} = P_{F(S)\cap\Upsilon}\overline{x}$ and $(\overline{x}, \overline{y}, \overline{z})$ is a solution of problem (1.10), *where*

$$\begin{aligned} \overline{y} &= P_E(\overline{z} - \mu A \overline{z}) \\ \overline{z} &= P_E(\overline{x} - \tau A \overline{x}). \end{aligned} \tag{4.10}$$

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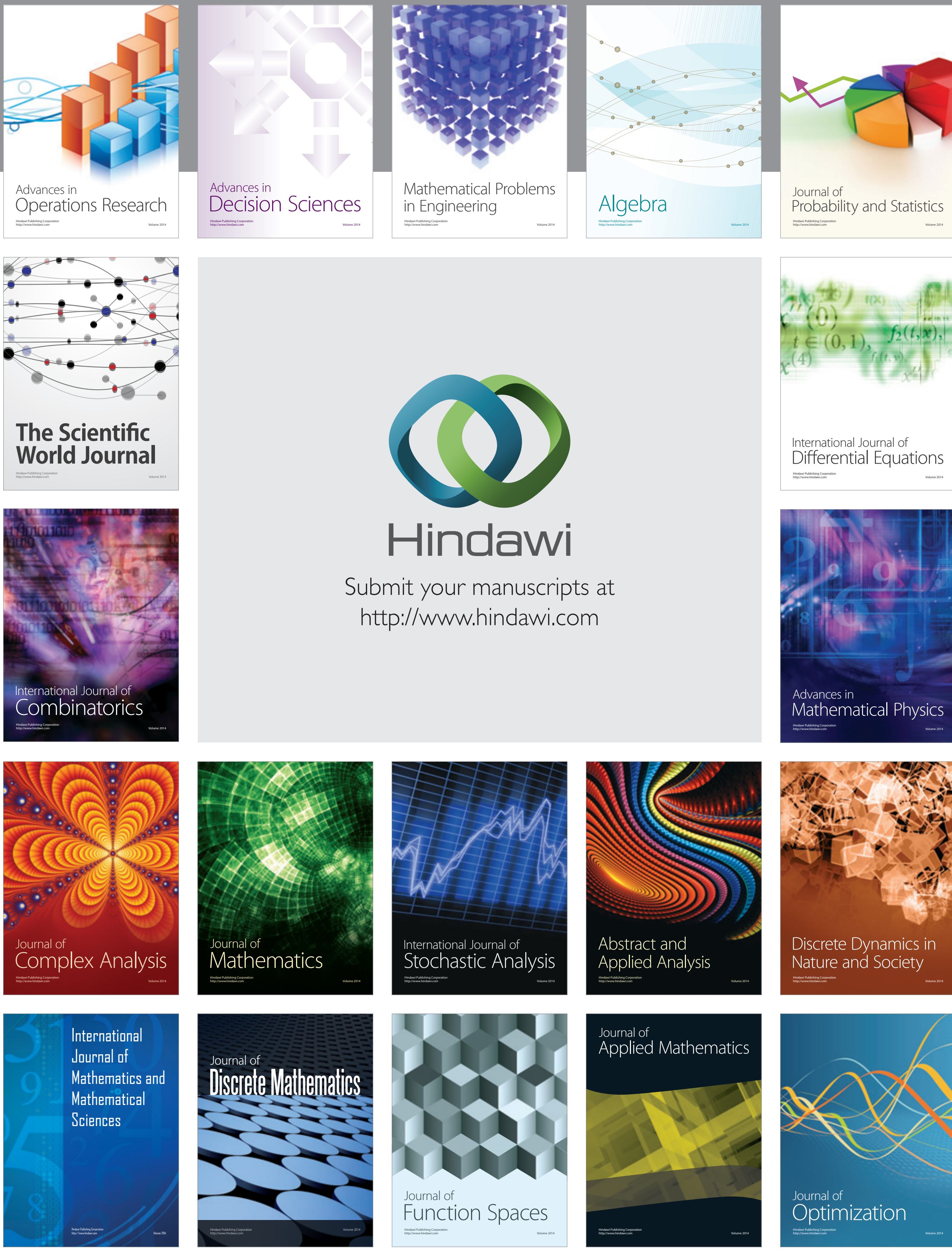

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