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Research Article

Stochastic Functional Differential Equation under Regime Switching

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We discuss stochastic functional differential equation under regime switching $dx(t) = f(x_t, r(t), t)dt + q(r(t))x(t)dW_1(t) + \sigma(r(t))|x(t)|^{\beta}x(t)dW_2(t)$. We obtain unique global solution of this system without the linear growth condition; furthermore, we prove its asymptotic ultimate boundedness. Using the ergodic property of the Markov chain, we give the sufficient condition of almost surely exponentially stable of this system.

1. Introduction

Recently, many papers devoted their attention to the hybrid system, they concerned that how to change if the system undergoes the environmental noise and the regime switching. For the detailed understanding of this subject, [1] is good reference.

In this paper we will consider the following stochastic functional equation:

$$dx(t) = f(x_t, r(t), t)dt + q(r(t))x(t)dW_1(t) + \sigma(r(t))|x(t)|^{\beta}x(t)dW_2(t). \tag{1.1}$$

The switching between these N regimes is governed by a Markovian chain $r(t)$ on the state space $\mathbb{S} = \{1, 2, \ldots, N\}$. $x_t \in C([-\tau, 0]; R^n)$ is defined by $x_t(\theta) = x(t+\theta)$; $\theta \in [-\tau, 0]$. $C([-\tau, 0]; R^n)$ denote the family of continuous functions from $[-\tau, 0]$ to R^n, which is a Banach space with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. $f : C([-\tau, 0]; R^n) \times \mathbb{S} \times R_+ \to R^n$ satisfies local Lipschitz condition as follows.

Assumption A. For each integer $k \geq 1, \ldots$, there is a positive number H_k such that

$$\left|f(\varphi_1, k) - f(\varphi_2, k)\right| \leq H_k\|\varphi_1 - \varphi_2\| \tag{1.2}$$

for all $t \geq 0$ and those $\varphi_1, \varphi_2 \in C([-\tau, 0]; R^n)$ with $\|\varphi_1\| \vee \|\varphi_2\| \leq k$.

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $W_i(t)$ $(i = 1, 2)$, $t \geq 0$, be the standard Brownian motion defined on this probability space. We also denote by $R^n_+ = \{x \in R^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$. Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{uv})_{N\times N}$ given by

$$P\{r(t+\delta) = v \mid r(t) = u\} = \begin{cases} \gamma_{uv}\delta + o(\delta), & \text{if } u \neq v, \\ 1 + \gamma_{uv}\delta + o(\delta), & \text{if } u = v, \end{cases} \tag{1.3}$$

where $\delta > 0$. Here γ_{uv} is the transition rate from u to v and $\gamma_{uv} \geq 0$ if $u \neq v$ while

$$\gamma_{uu} = -\sum_{v\neq u} \gamma_{uv}. \tag{1.4}$$

We assume that the Markov chain $r(\cdot)$ is independent on the Brownian motion $W_i(\cdot), i = 1, 2$; furthermore, W_1 and W_2 are independent.

In addition, throughout this paper, let $C^{2,1}(R^n \times [-\tau, \infty) \times \mathbb{S}; \overline{R}_+)$ denote the family of all positive real-valued functions $V(x, t, k)$ on $R^n \times [-\tau, \infty) \times \mathbb{S}$ which are continuously twice differentiable in x and once in t. If for the following equation

$$dx(t) = f(x_t, r(t), t)dt + g(x, r(t), t)dB(t), \tag{1.5}$$

there exists $V \in C^{2,1}(R^n \times [-\tau, \infty) \times \mathbb{S}; R_+)$, define an operator ℓV from $C([-\tau, 0]; R^n) \times R_+ \times \mathbb{S}$ to R by

$$\begin{aligned} \ell V(\varphi, t, k) = {} & V_t(\varphi(0), t, k) + V_x(\varphi(0), t, k) f(\varphi, k, t) \\ & + \frac{1}{2}\text{trace}\left[g^T(\varphi(0), k, t) V_{xx}(\varphi(0), k, t) g(\varphi(0), k, t)\right] \\ & + \sum_{l=1}^{N} \gamma_{kl} V(\varphi(0), t, l), \end{aligned} \tag{1.6}$$

where

$$\begin{gathered} V_t(x, t, k) = \frac{\partial V(x, t, k)}{\partial t}, \qquad V_x(x, t, k) = \left(\frac{\partial V(x, t, k)}{\partial x_1}, \dots, \frac{\partial V(x, t, k)}{\partial x_n}\right), \\ V_{xx}(x, t, k) = \left(\frac{\partial V^2(x, t, k)}{\partial x_i \partial x_j}\right)_{n\times n}. \end{gathered} \tag{1.7}$$

Here we should emphasize that [1, Page 305] the operator ℓV (thought as a single notation rather than ℓ acting on V) is defined on $C([-\tau, 0]; R^n) \times R_+ \times \mathbb{S}$ although V is defined on $R^n \times [-\tau, \infty) \times \mathbb{S}$.

2. Global Solution

Firstly, in this paper, we are concerned about that the existence of global solution of stochastic functional differential equation (1.1).

In order to have a global solution for any given initial data for a stochastic functional equation, it is usually required to satisfy the local Lipschitz condition and the linear growth condition [1, 2]. In addition, as a generation of linear condition, it is also mentioned in [3, 4] with one-sided linear growth condition. The authors improve the results using polynomial growth condition in [5, 6]. After that, these conditions were mentioned under regime systems [7–9].

Replacing the linear growth condition or the one-sided linear growth condition, we impose the so-called polynomial growth condition on the function f for (1.1).

Assumption B. For each $i \in \mathbb{S}$, there exist nonnegative constants $\alpha, \kappa_i, \overline{\kappa}_i, \gamma, K$ and probability measures μ, on $[-\tau, 0]$ such that

$$\langle \phi(0), f(\varphi, i, t) \rangle \le |\varphi(0)|^2 \left[\kappa_i |\varphi(0)|^\alpha + \overline{\kappa}_i \int_{-\tau}^{0} |\varphi(\theta)|^\alpha d\mu(\theta) + \gamma \right] + K|\phi(0)| \tag{2.1}$$

for any $\varphi \in C([-\tau, 0]; R^n)$.

Theorem 2.1. *Under the conditions of Assumptions A and B, if $2\beta > \alpha$, and $\sigma(i) \neq 0$ for $i = 1, 2, \ldots, n$, there almost surely exists a unique globally solution $x(t)$ to* (1.1) *on $t \ge -\tau$ for any given initial data $\varphi \in C([-\tau, 0]; R^n)$.*

Proof. Since the coefficients of (1.1) are locally Lipschitz, there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time. In order to prove this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficiently large such that $1/m_0 < \min_{-\tau \le \theta \le 0} |\xi(\theta)| < \max_{-\tau \le \theta \le 0} |\xi(\theta)| < m_0$. For each $m \ge m_0$, we define the stopping time

$$\tau_m = \inf\left\{ t \in [-\tau, \tau_e) : x_i(t) \notin \left(\frac{1}{m}, m \right) \text{ for some } i = 1, 2, \ldots, n \right\}. \tag{2.2}$$

Clearly τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m \to \infty} \tau_m$, if we can obtain that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. for all $t \ge 0$. That is, to complete the proof, also equivalent to prove that, for any $t > 0, P(\tau_m \le t) \to 0$ as $m \to \infty$. If this conclusion is false, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \le T\} > \varepsilon. \tag{2.3}$$

So there exists an integer $m_1 \ge m_0$ such that

$$P\{\tau_m \le T\} > \varepsilon \quad \forall m \ge m_1. \tag{2.4}$$

To prove the conclusion that we desired, for any $p \in (0,1/2)$, define a C^2-function: $R^n \times \mathbb{S} \rightarrow \overline{R}_+$ by

$$V(x,k) = c(k)\left[1 + |x|^2\right]^p, \tag{2.5}$$

where $\{c(k), 1 \le k \le N\}$ is positive constant sequence. Applying the generalized Itô formula,

$$\begin{aligned} dV(x,k) &= d\left[c(k)\left(1 + |x|^2\right)^p\right] \\ &= \ell V(\varphi,k)dt + 2c(k)p\left(1 + |x|^2\right)^{p-1}\left[q(k)|x|^2 dW_1(t) + \sigma(k)|x|^{\beta+2} dW_2(t)\right], \end{aligned} \tag{2.6}$$

where ℓV is computed as

$$\begin{aligned} \ell V(\varphi,k) = pc(k)\left(1 + |x|^2\right)^{p-2}&\left[2\left(1 + |x|^2\right)\langle x, f(\varphi,k,t)\rangle + q^2(k)|x|^2\right. \\ &\left. +\sigma^2(k)|x|^{2\beta+2} + (2p-1)\left(q^2(k)|x|^4 + \sigma^2(k)|x|^{2\beta+4}\right)\right] \\ &+ \sum_{l=1}^{N} \gamma_{kl} V(\varphi(0),l). \end{aligned} \tag{2.7}$$

Let $\widetilde{q} = \max_{k,l\in\mathbb{S}}\{c(l)/c(k)\}$. For any $k, l \in \mathbb{S}$, we get

$$V(x,l) = c(l)\left(1 + |x|^2\right)^p \le \widetilde{q}c(k)\left(1 + |x|^2\right)^p = \widetilde{q}V(x,k). \tag{2.8}$$

Therefore,

$$\sum_{l=1}^{N} \gamma_{kl} V(x,l) \le \widetilde{q} \sum_{l=1}^{N} |\gamma_{kl}| V(x,k). \tag{2.9}$$

According to Assumption B, the first term in (2.7)

$$\begin{aligned} &pc(k)\left(1 + |x|^2\right)^{p-2}\left[2\left(1 + |x|^2\right)\langle x, f(\varphi,k,t)\rangle\right] \\ &\quad\le 2c(k)p\left(1 + |x|^2\right)^{p-1}\left[\kappa_k |x|^{\alpha+2} + \gamma |x|^2 + K|x|\right] \\ &\qquad + 2c(k)\overline{\kappa}_k p\left(1 + |x|^2\right)^{p-1} |x|^2 \int_{-\tau}^{0} |x(t+\theta)|^{\alpha} d\mu(\theta). \end{aligned} \tag{2.10}$$

By the Young inequality and noting that $p \in (0, 1/2)$, it is obvious that

$$
\begin{aligned}
&2c(k)\overline{\kappa}_k p\left(1+|x|^2\right)^{p-1}|x|^2\int_{-\tau}^{0}|x(t+\theta)|^{\alpha}d\mu(\theta)\\
&\quad\le 2c(k)\overline{\kappa}_k p\left(1+|x|^{2p}\right)\int_{-\tau}^{0}|x(t+\theta)|^{\alpha}d\mu(\theta)\\
&\quad\le 2c(k)\overline{\kappa}_k p\int_{-\tau}^{0}|x(t+\theta)|^{\alpha}d\mu(\theta)+\frac{4c(k)\overline{\kappa}_k p^2}{\alpha+2p}|x|^{\alpha+2p}\\
&\qquad+\frac{2c(k)\alpha\overline{\kappa}_k p}{\alpha+2p}\int_{-\tau}^{0}|x(t+\theta)|^{\alpha+2p}d\mu(\theta)\\
&\quad\le 2c(k)\overline{\kappa}_k p\left(|x|^{\alpha}+|x|^{\alpha+2p}\right)+2c(k)\overline{\kappa}_k p\left\{\int_{-\tau}^{0}|x(t+\theta)|^{\alpha}d\mu(\theta)-|x|^{\alpha}\right\}\\
&\qquad+\frac{2c(k)\alpha\overline{\kappa}_k p}{\alpha+2p}\left\{\int_{-\tau}^{0}|x(t+\theta)|^{\alpha+2p}d\mu(\theta)-|x|^{\alpha+2p}\right\},
\end{aligned}
\tag{2.11}
$$

where the first inequality we have used the elementary inequality: for any $a, b \ge 0$ and $r \in (0,1)$, $(a+b)^r \le a^r + b^r$. Therefore we have

$$
\begin{aligned}
&d\left[c(k)\left(1+|x|^2\right)^p\right]\\
&\quad\le\left\{H_k(x)+2c(k)\overline{\kappa}_k p\left[\int_{-\tau}^{0}|x(t+\theta)|^{\alpha}d\mu(\theta)-|x|^{\alpha}\right]\right.\\
&\qquad\left.+\frac{2c(k)\alpha\overline{\kappa}_k p}{\alpha+2p}\left[\int_{-\tau}^{0}|x(t+\theta)|^{\alpha+2p}d\mu(\theta)-|x|^{\alpha+2p}\right]\right\}dt\\
&\qquad+2c(k)p\left(1+|x|^2\right)^{p-1}\left[q(k)|x|^2dW_1(t)+\sigma(k)|x|^{\beta+2}dW_2(t)\right],
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
H_k(x)={}&2c(k)\overline{\kappa}_k p\left(|x|^{\alpha}+|x|^{\alpha+2p}\right)\\
&+pc(k)\left(1+|x|^2\right)^{p-2}\left[q^2(k)|x|^2+\sigma^2(k)|x|^{2\beta+2}\right.\\
&\qquad\left.+(2p-1)\left(q^2(k)|x|^4+\sigma^2(k)|x|^{2\beta+4}\right)\right]\\
&+2c(k)p(1+|x|^2)^{p-1}\left[\kappa_k|x|^{\alpha+2}+\gamma|x|^2+K|x|\right]+\sum_{l=1}^{N}\gamma_{kl}V(\varphi(0),l).
\end{aligned}
\tag{2.13}
$$

Using the elementary inequality: for any $a, b \geq 0$ and $r \geq 1$, $(a+b)^r \leq 2^{r-1}(a^r + b^r)$, we obtain that $(1+|x|^2)^{2-p} \leq 2^{1-p}[1+|x|^{4-2p}]$, also we have

$$\left(1+|x|^2\right)^{2-p}\sum_{l=1}^{N}\gamma_{kl}V(x,l) \leq \tilde{q}\sum_{l=1}^{N}|\gamma_{kl}|2^{1-p}\left(1+|x|^{4-2p}\right)\left(1+|x|^{2p}\right). \tag{2.14}$$

Therefore,

$$\begin{aligned}
H_k(x) &\leq \left(1+|x|^2\right)^{2-p}H_k(x) \\
&\leq 2^{2-p}c(k)\overline{\kappa}_k p\left[1+|x|^{4-2p}\right]\left(|x|^{\alpha}+|x|^{\alpha+2p}\right) \\
&\quad + pc(k)\Big[q^2(k)|x|^2+\sigma^2(k)|x|^{2\beta+2} \\
&\qquad\qquad + (2p-1)\left(q^2(k)|x|^4+\sigma^2(k)|x|^{2\beta+4}\right)\Big] \\
&\quad + 2c(k)p\left(1+|x|^2\right)\left[\kappa_k|x|^{\alpha+2}+\gamma|x|^2+K|x|\right] \\
&\quad + \tilde{q}\sum_{l=1}^{N}|\gamma_{kl}|2^{1-p}\left(1+|x|^{4-2p}\right)\left(1+|x|^{2p}\right).
\end{aligned} \tag{2.15}$$

Noting that $p \in (0, 1/2)$ and $2\beta > \alpha \geq 0$, by the boundedness property of polynomial functions, there exists a positive constant M_k such that $H_k(x) \leq (1+|x|^2)^{2-p}H_k(x) \leq M_k$. Taking expectation from two sides of (2.6) leads to

$$\mathbb{E}V(x(\tau_m \wedge T), r(\tau_m \wedge T)) = \mathbb{E}V(x(0), r(0)) + \mathbb{E}\int_0^{\tau_m \wedge T} \ell V(x_s, r(s))ds, \tag{2.16}$$

and from (2.12) and (2.15), we have

$$\begin{aligned}
\mathbb{E}V(x(\tau_m \wedge T), r(\tau_m \wedge T)) &\leq \mathbb{E}V(x(0), r(0)) + M_k T \\
&\quad + 2\check{c}\overline{\kappa}_k p\mathbb{E}\left\{\int_0^{\tau_m \wedge T}\left[\int_{-\tau}^{0}|x(t+\theta)|^{\alpha}d\mu(\theta) - |x|^{\alpha}\right]ds\right\} \\
&\quad + \frac{2\check{c}\alpha\overline{\kappa}_k p}{\alpha+2p}\mathbb{E}\left\{\int_0^{\tau_m \wedge T}\left[\int_{-\tau}^{0}|x(t+\theta)|^{\alpha+2p}d\mu(\theta) - |x|^{\alpha+2p}\right]ds\right\},
\end{aligned} \tag{2.17}$$

where we denote $\check{c} = \max_{k\in\mathbb{S}, 1\leq i\leq n} c_i(k)$.

By the Fubini theorem and a substitution technique, we may compute that

$$\begin{aligned}
&\int_0^{\tau_m\wedge T}\int_{-\tau}^0 |x(s+\theta)|^{\alpha+2p}d\mu(\theta)ds-\int_0^{\tau_m\wedge T}|x(s)|^{\alpha+2p}ds\\
&\quad=\int_{-\tau}^0 d\mu(\theta)\int_{\theta}^{\tau_m\wedge T+\theta}|x(s)|^{\alpha+2p}ds-\int_0^{\tau_m\wedge T}|x(s)|^{\alpha+2p}ds\\
&\quad\le\int_{-\tau}^0 d\mu(\theta)\int_{-\tau}^{\tau_m\wedge T}|x(s)|^{\alpha+2p}ds-\int_0^{\tau_m\wedge T}|x(s)|^{\alpha+2p}ds\\
&\quad\le\int_{-\tau}^0|\xi(s)|^{\alpha+2p}ds.
\end{aligned}\tag{2.18}$$

Similarly,

$$\int_0^{\tau_m\wedge T}\int_{-\tau}^0 |x(s+\theta)|^{\alpha}d\mu(\theta)ds-\int_0^{\tau_m\wedge T}|x(s)|^{\alpha}ds\le\int_{-\tau}^0|\xi(s)|^{\alpha}ds.\tag{2.19}$$

Therefore we rewrite (2.17) into

$$\begin{aligned}
\mathbb{E}V(x(\tau_m\wedge T),r(\tau_m\wedge T))&\le V(\xi(0),r(0))+MT\\
&\quad+2\check{c}\overline{\kappa}_k p\int_{-\tau}^0\mathbb{E}|\xi(s)|^{\alpha}ds+\frac{2\check{c}\alpha\overline{\kappa}_k p}{\alpha+2p}\int_{-\tau}^0\mathbb{E}|\xi(s)|^{\alpha+2p}ds\\
&:=K_T,
\end{aligned}\tag{2.20}$$

where K_T is bounded and K_T is independent of m.

By the definition of $\tau_m, x(\tau_m)=m$ or $1/m$, so let $\Omega_m=\{\tau_m\le T\}$ for $m\ge m_1$ and by (2.6), noting that for every $\omega\in\Omega_m$, there is some m such that $x_m(\tau_m,\omega)$ equals either m or $1/m$ hence

$$\begin{aligned}
&P(\tau_m\le T)\min_{i\in\mathbb{S}}\left\{c(i)\left(1+m^2\right)^p\wedge c(i)\left(1+\frac{1}{m^2}\right)^p\right\}\\
&\quad\le P(\tau_m\le T)V(x(\tau_m),i)\\
&\quad\le\mathbb{E}\left[I_{\{\tau_m\le T\}}V(x(\tau_m\wedge T),r(\tau_m\wedge T))\right]\\
&\quad\le\mathbb{E}V(x(\tau_m\wedge T),r(\tau_m\wedge T))\le K_T.
\end{aligned}\tag{2.21}$$

Letting $m\to\infty$ implies that

$$\limsup_{m\to\infty}P(\tau_m\le T)=0.\tag{2.22}$$

So we must obtain $\tau_\infty=\infty$ a.s., as required. The proof is complete. □

3. Asymptotic Boundedness

Theorem 2.1 shows that the solution of SDE (1.1) exists globally and will not explode under some reasonable conditions. In the study of stochastic system, stochastically ultimate boundedness is more important topic comparing with nonexplosion of the solution, which means that the solution of this system will survive under finite boundedness in the future. Here we examine the $2p$th moment boundedness.

Lemma 3.1. *Under the conditions of Theorem 2.1, for any $p \in (0, 1/2)$, there exists a constant K_p independent on the initial data such that the global solution $x(t)$ of SDE* (1.1) *has the property that*

$$\limsup_{t\to\infty} \mathbb{E}\left(|x(t)|^{2p}\right) \le K_p. \tag{3.1}$$

Proof. First, Theorem 2.1 indicates that the solution $x(t)$ of (1.1) almost surely remain in R^n for all $t \ge -\tau$ with probability 1.

Applying the Itô formula to $e^{\varepsilon t}V(x,k)$ and taking expectation yields

$$\mathbb{E}V(x,k) = e^{-\varepsilon t}V(\xi(0), r(0)) + e^{-\varepsilon t}\mathbb{E}\int_0^t e^{\varepsilon s}[\ell V(x_s,k) + \varepsilon V(x(s),k)]ds. \tag{3.2}$$

Here $\ell V(\varphi,k)$ is defined as before (2.7).

Now we consider the function

$$\begin{aligned}
\Phi_k(x) = {} & 2c(k)\overline{\kappa}_k p\left(e^{\varepsilon\tau}|x|^{\alpha} + \frac{\alpha e^{\varepsilon\tau} + 2p}{\alpha + 2p}|x|^{\alpha+2p}\right) \\
& + pc(k)\left(1+|x|^2\right)^{p-2}\Big[q^2(k)|x|^2 + \sigma^2(k)|x|^{2\beta+2} \\
& \qquad + (2p-1)\left(q^2(k)|x|^4 + \sigma^2(k)|x|^{2\beta+4}\right)\Big] \\
& + 2c(k)p\left(1+|x|^2\right)^{p-1}\left[\kappa_k|x|^{\alpha+2} + \gamma|x|^2 + K|x|\right] \\
& + \left[\check{c}\tilde{q}\sum_{l=1}^{N}|\gamma_{kl}| + \varepsilon\check{c}\right]\left(1+|x|^{2p}\right).
\end{aligned} \tag{3.3}$$

Similar to the proof of Theorem 2.1, then we know (3.3) is upper bounded; there exists constant ψ_k such that $\Phi_k(x) \le (1+|x|^2)^{2-p}\Phi_k(x) \le \psi_k \le \Psi := \max_{1\le k\le N}\psi_k$; therefore, (3.2) implies that

$$\begin{aligned}
\mathbb{E}V(x,k) \le {} & e^{-\varepsilon t}V(\xi(0), r(0)) \\
& + e^{-\varepsilon t}\mathbb{E}\int_0^t e^{\varepsilon s}\Bigg[\Psi + 2c(k)\overline{\kappa}_k p\left(\int_{-\tau}^0 |x(s+\theta)|^{\alpha}d\mu(\theta) - e^{\varepsilon\tau}|x(s)|^{\alpha}\right) \\
& \qquad + \frac{2c(k)\alpha\overline{\kappa}_k p}{\alpha+2p}\left(\int_{-\tau}^0 |x(s+\theta)|^{\alpha+2p}d\mu(\theta) - e^{\varepsilon\tau}|x(s)|^{\alpha+2p}\right)\Bigg]ds.
\end{aligned} \tag{3.4}$$

We have the following calculus transformation:

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}\int_{-\tau}^0 |x(s+\theta)|^\alpha d\mu(\theta)ds - \int_0^t e^{\varepsilon(s+\tau)}|x(s)|^\alpha ds \\
&\quad = \int_{-\tau}^0 d\mu(\theta)\int_\theta^{t+\theta} e^{\varepsilon(s-\theta)}|x(s+\theta)|^\alpha ds - \int_0^t e^{\varepsilon(s+\tau)}|x(s)|^\alpha ds \\
&\quad \le \int_{-\tau}^0 d\mu(\theta)\int_{-\tau}^t e^{\varepsilon(s+\tau)}|x(s)|^\alpha ds - \int_0^t e^{\varepsilon(s+\tau)}|x(s)|^\alpha ds \\
&\quad \le e^{\varepsilon\tau}\int_{-\tau}^0 e^{\varepsilon\theta}|\xi(\theta)|^\alpha d\theta < \infty.
\end{aligned} \tag{3.5}$$

Similarly,

$$\int_0^t\int_{-\tau}^0 e^{\varepsilon s}|x|^{\alpha+2p}(s+\theta)d\mu(\theta)ds - \int_0^t e^{\varepsilon(s+\tau)}|x|^{\alpha+2p}(s)ds \le e^{\varepsilon\tau}\int_{-\tau}^0 e^{\varepsilon\theta}\xi_i^{\alpha+2p}(\theta)d\theta < \infty. \tag{3.6}$$

We therefore have from (3.4)

$$\begin{aligned}
\mathbb{E}V(x,k) \le{}& e^{-\varepsilon t}V(\xi(0),r(0)) + \Psi\varepsilon^{-1}\left(1-e^{-\varepsilon t}\right) \\
&+ 2\check{c}\kappa_k p e^{-\varepsilon(t-\tau)}\mathbb{E}\int_{-\tau}^0 e^{\varepsilon\theta}|\xi(\theta)|^\alpha d\theta \\
&+ e^{-\varepsilon(t-\tau)}\frac{2\check{c}\alpha\kappa_k p}{\alpha+2p}\mathbb{E}\int_{-\tau}^0 e^{\varepsilon\theta}|\xi(\theta)|^{\alpha+2p}d\theta.
\end{aligned} \tag{3.7}$$

Clearly,

$$\limsup_{t\to\infty}\mathbb{E}V(x,k) \le \Psi\varepsilon^{-1}; \tag{3.8}$$

denote $\widehat{c} = \min\{c_i(k) : 1 \le i \le n, k \in \mathbb{S}\}$; therefore,

$$\widehat{c}\limsup_{t\to\infty}\mathbb{E}|x|^{2p} \le \limsup_{t\to\infty}\mathbb{E}\left[c(k)\left(1+|x|^2\right)^p\right] \le \limsup_{t\to\infty}\mathbb{E}V(x,k), \tag{3.9}$$

which yields

$$\limsup_{t\to\infty}\mathbb{E}|x|^{2p} \le \frac{\Psi}{\widehat{c}\varepsilon} =: K_p. \tag{3.10}$$

This means that the solution is bounded in the $2p$th moment; the stochastically ultimate boundedness will follow directly. □

Definition 3.2. The solutions $x(t)$ of SDE (1.1) are called stochastically ultimately bounded, if for any $\epsilon \in (0,1)$, there is a positive constant $\chi(=\chi(\epsilon))$, such that the solution of SDE (1.1) with any positive initial value has the property that

$$\limsup_{t\to+\infty} P\{|x(t)| > \chi\} < \epsilon. \tag{3.11}$$

Theorem 3.3. *The solution of* (1.1) *is stochastically ultimately bounded under the condition Lemma 3.1; that is, for any $\epsilon \in (0,1)$, there is a positive constant $\chi(=\chi(\epsilon))$, such that for any positive initial value the solution of Lemma 3.1 has the property that*

$$\limsup_{t\to+\infty} P\{|x(t)| > \chi\} < \epsilon. \tag{3.12}$$

Proof. This can be easily verified by Chebyshev's inequality and Lemma 3.1 by choosing $\chi = (K_p/\epsilon)^{1/p}$ sufficiently large because of the following

$$\lim\sup_{t\to+\infty} P(|x(t)| > \chi) \le \frac{\limsup_{t\to+\infty}\mathbb{E}\left[|x|^p\right]}{\chi^p}. \tag{3.13}$$

□

4. Stabilization of Noise

From Sections 2 and 3, we know that under the condition $\sigma(i)\neq 0$ and $2\beta > \alpha$, the Brownian noise $\sigma(i)|x(t)|^{\beta}x(t)dW_1(t)$ can suppress the potential explosion of the solution and guarantee this global solution to be bounded in the sense of the $2p$th moment. Clearly, the boundedness results are also dependent only on the choice of β under the condition $\sigma(i)\neq 0$ and independent of $q(i)$. This implies that the noise $W_1(t)$ plays no role to guarantee existence and boundedness of the global solution to (1.1). This section is devoted to consider the effect of noise $q(i)x(t)dW_1(t)$, we will show that the system (1.1) is exponential stability if for some sufficiently large $q(i)$.

For the purpose of stability study, we impose the following the general polynomial growth condition:

Assumption C. For each $i \in \mathbb{S}$, there exist nonnegative constants $\alpha, \kappa_i, \overline{\kappa}_i, \gamma,$ and K and probability measures μ, on $[-\tau, 0]$ such that

$$\left|f(\varphi,i,t)\right| \le \left|\varphi(0)\right|\left[\kappa_i\left|\varphi(0)\right|^{\alpha} + \overline{\kappa}_i\int_{-\tau}^{0}\left|\varphi(\theta)\right|^{\alpha}d\mu(\theta) + \gamma\right] \tag{4.1}$$

for any $\varphi \in C([-\tau,0];R^n)$.

Clearly, Assumption C is stronger than the one-sided polynomial growth condition Assumption B. Therefore, Theorems 2.1 and Lemma 3.1 still hold under Assumption C.

In [10, Page 165], for a given nonlinear SDE with Markovian switching

$$dx(t) = f(x(t),r(t),t)dt + g(x(t),r(t),t)dB(t), \tag{4.2}$$

any solution starting form a nonzero state will remain to be non-zero. But for the system (1.1)

$$dx(t) = f(x_t, r(t), t)dt + q(r(t))x(t)dW_1(t) + \sigma(r(t))|x(t)|^{\beta}x(t)dW_2(t), \tag{4.3}$$

the drift coefficient is a functional, so we will prove this non-zero property under Assumption C.

Lemma 4.1. *Let $x(t)$ be the global solution of* (1.1). *Under Assumption C, if $2\beta > \alpha$ and $\sigma(i) \neq 0$, for any non-zero initial data $x(0) \neq 0$*

$$P(x(t) \neq 0) = 1, \quad t \geq 0; \tag{4.4}$$

that is, almost all the sample path of any solution starting from a non-zero state will never reach the origin.

Proof. For any initial data $\xi \in C([-\tau, 0]; R^n)$ satisfying $x(0) \neq 0$, for sufficiently large positive number i_0, such that $|x(0)| > 1/i_0$. For each integer $i \geq i_0$, define the stopping time

$$\rho_i = \inf\left\{t \geq 0 : |x(t)| \leq \frac{1}{i}\right\}. \tag{4.5}$$

Clearly, ρ_i is increasing as $i \to \infty$ and $\rho_i \to \rho_\infty$ a.s. If we can show that $\rho_\infty = \infty$ a.s., the desired result $P(x(t) \neq 0) = 1$ on $t \geq 0$ follows. This is equivalent to proving that, for any $t > 0, P(\rho_i \leq t) \to 0$ as $i \to \infty$.

To prove this statement, define a C^2-function

$$V_1(x, k) = c(k)\left[\sqrt{x} - \frac{1}{2}\log x\right], \tag{4.6}$$

where $V_1(\cdot) > 0$ and $V_1(0^+) = \infty$. Applying the Itô formula and taking the expectation yield

$$\mathbb{E}V_1(\left|x(t \wedge \rho_i), r(t \wedge \rho_i)\right|) = \mathbb{E}V_1(|x(0, r(0))|) + \mathbb{E}\int_0^{t\wedge\rho_i} \ell V_1(x_s, r(s))ds, \tag{4.7}$$

where ℓV_1 is defined as

$$\begin{aligned} \ell V_1(\varphi, k) = &\frac{1}{2}c(k)\left(|\varphi(0)|^{-3/2} - |\varphi(0)|^{-2}\right)\varphi^T(0)f(\varphi, t, k) \\ &- \frac{\sigma^2(k)c(k)}{8}\left(|\varphi(0)|^{2\beta+(1/2)} - 2|\varphi(0)|^{2\beta}\right) - \frac{q^2(k)c(k)}{8}\left(|\varphi(0)|^{1/2} - 2\right) \\ &+ \sum_{l=1}^{N}\gamma_{kl}V(\varphi(0), l), \end{aligned} \tag{4.8}$$

for any $\varphi \in C([-\tau, 0]; R^n)$. By Assumption C and the Young inequality, the first term of (4.8) will be written as

$$
\begin{aligned}
&\left(|\varphi(0)|^{-3/2} - |\varphi(0)|^{-2}\right)\varphi^T(0) f(\varphi, t, k) \\
&\quad \le \left(|\varphi(0)|^{-1/2} + |\varphi(0)|^{-1}\right)|f(\varphi, t, k)| \\
&\quad \le \left(|\varphi(0)|^{1/2} + 1\right)\left[\kappa_k|\varphi(0)|^{\alpha} + \overline{\kappa}_k \int_{-\tau}^{0} |\varphi(\theta)|^{\alpha} d\mu(\theta) + \gamma\right] \\
&\quad \le (\kappa_k + \overline{\kappa}_k)\left(|\varphi(0)|^{\alpha+1/2} + |\varphi(0)|^{\alpha}\right) + \gamma|\varphi(0)|^{1/2} + \gamma \\
&\qquad + \frac{\alpha\overline{\kappa}_k}{\alpha + (1/2)}\left[\int_{-\tau}^{0} |\varphi(0)|^{\alpha+(1/2)} d\mu(\theta) - |\varphi(0)|^{\alpha+(1/2)}\right] \\
&\qquad + \overline{\kappa}_k\left[\int_{-\tau}^{0} |\varphi(0)|^{\alpha} d\mu(\theta) - |\varphi(0)|^{\alpha}\right].
\end{aligned}
\tag{4.9}
$$

Substituting (4.9) into (4.8) gives

$$
\begin{aligned}
\ell V_1(\varphi, k) \le \widehat{H}_k(\varphi(0)) &+ \frac{c(k)\alpha\overline{\kappa}_k}{\alpha + (1/2)}\left[\int_{-\tau}^{0} |\varphi(0)|^{\alpha+(1/2)} d\mu(\theta) - |\varphi(0)|^{\alpha+(1/2)}\right] \\
&+ c(k)\overline{\kappa}_k\left[\int_{-\tau}^{0} |\varphi(0)|^{\alpha} d\mu(\theta) - |\varphi(0)|^{\alpha}\right],
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
\widehat{H}_k(x) \le &-\frac{\sigma^2(k)c(k)}{8}\left(|x|^{2\beta+(1/2)} - 2|x|^{2\beta}\right) - \frac{q^2(k)c(k)}{8}\left(|x|^{1/2} - 2\right) \\
&+ c(k)(\kappa_k + \overline{\kappa}_k)\left(|x|^{\alpha+(1/2)} + |x|^{\alpha}\right) + \gamma|x|^{1/2} + \gamma \\
&+ \tilde{q}\sum_{l=1}^{N}|\gamma_{kl}|c(k)\left(|x|^{1/2} - \log|x|\right) \\
\le &-\frac{\sigma^2(k)c(k)}{8}\left(|x|^{2\beta+(1/2)} - 2|x|^{2\beta}\right) - \frac{q^2(k)c(k)}{8}\left(|x|^{1/2} - 2\right) - \tilde{q}\sum_{l=1}^{N}|\gamma_{kl}|c(k)\log|x| \\
&+ c(k)(\kappa_k + \overline{\kappa}_k)\left(|x|^{\alpha+1/2} + |x|^{\alpha}\right) + \gamma|x|^{1/2} + \gamma \\
&+ \tilde{q}\sum_{l=1}^{N}|\gamma_{kl}|c(k)|x|^{1/2}.
\end{aligned}
\tag{4.11}
$$

Noting (2.9) and $2\beta > \alpha$, $\sigma(k) \neq 0$, using the boundedness property of polynomial functions, there exists a constant $\widehat{M}_k$ such that $\widehat{H}_k(x) \leq \widehat{M}_k \leq \widehat{M} =: \max_{1\leq k\leq N}\{\widehat{M}_k\}$.

Furthermore, we may estimate that

$$\int_0^{t\wedge\rho_i}\left[\int_{-\tau}^0 |x(s+\theta)|^{\alpha+(1/2)}d\mu(\theta) - |x(s)|^{\alpha+(1/2)}\right]ds \leq \int_{-\tau}^0 |\xi(s)|^{\alpha+(1/2)}ds,$$
$$\int_0^{t\wedge\rho_i}\left[\int_{-\tau}^0 |x(s+\theta)|^{\alpha}d\mu(\theta) - |x(s)|^{\alpha}\right]ds \leq \int_{-\tau}^0 |\xi(s)|^{\alpha}ds. \tag{4.12}$$

It therefore follows that

$$\begin{aligned}\mathbb{E}V_1(|x(t\wedge\rho_i), r(t\wedge\rho_i)|) &\leq \mathbb{E}V_1(|x(0,r(0))|) + \widehat{M}t \\ &\quad + \int_{-\tau}^0 |\xi(s)|^{\alpha+(1/2)}ds + \int_{-\tau}^0 |\xi(s)|^{\alpha}ds \\ &=: \widehat{K}_t.\end{aligned} \tag{4.13}$$

We know that

$$\begin{aligned}\frac{1}{2}\log i\,\mathbb{P}(\rho_i \leq t) &\leq \mathbb{P}(\rho_i \leq t)\left(\sqrt{\frac{1}{i}} - \frac{1}{2}\log\frac{1}{i}\right) \\ &\leq \mathbb{E}\left[I_{\{\rho_i\leq t\}}\right]V_1(|x(\rho_i \leq t)|) \leq \mathbb{E}V_1(|x(\rho_i \leq t)|) \\ &\leq \widehat{K}_t,\end{aligned} \tag{4.14}$$

which implies that

$$\limsup_{i\to\infty}\mathbb{P}(\rho_i \leq t) \leq \lim_{i\to\infty}\frac{2\widehat{K}_t}{\log i} = 0, \tag{4.15}$$

as required. The proof is completed. □

This lemma shows that almost all the sample path of any solution of (1.1) starting from a non-zero state will never reach the origin. Because of this nice property, the Lyapunov functions we can choose need not be imposed globally but only in a deleted neighborhood of the origin.

Especially, the hybrid system always switch from any regime to another regime, so it is reasonable to assume that the Markov chain $r(t)$ is irreducible. It means to the condition that irreducible Markov chain has a unique stationary probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in R^{1\times N}$ which can be determined by solving the following linear equation $\pi\Gamma = 0$ subject to $\sum_{k=1}^N \pi_k = 1$ and $\pi_k > 0$ for any $k \in \mathbb{S}$, where Γ is generator $\Gamma = (\gamma_{\mathrm{uv}})_{N\times N}$.

Theorem 4.2. *Suppose the Markov chain $r(t)$ is irreducible, under Assumption A and C, if for $\delta \in (0,1), k \in \mathbb{S}, \sigma(k) \neq 0$ and $2\beta > \alpha$, the solution $x(t)$ of SDE* (1.1) *with any initial data $\xi \in C([-\tau,0];R^n)$ satisfying $x(0) \neq 0$ has the property*

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq \sum_{j=1}^{N} \pi_j \left[\phi_j - \frac{q_j^2}{2}\right] \quad a.s., \tag{4.16}$$

where

$$\phi(k) = \max_{x>0}\left\{ -\frac{\sigma^2(k)}{2}|x|^{2\beta} + (\kappa_k + \overline{\kappa}_k)|x|^{\alpha} + \gamma \right\}. \tag{4.17}$$

In particular, the nonlinear hybrid system (1.1) *is almost surely exponentially stable if*

$$\sum_{j=1}^{N} \pi_j \left[\phi_j - \frac{q_j^2}{2}\right] < 0. \tag{4.18}$$

Proof. By Theorem 2.1 and Lemma 4.1, (1.1) almost surely admits a global solution $x(t)$ for all $t \geq 0$ and $x(t) \neq 0$ almost surely. Applying the Itô formula to the function $\log|x(t)|$ leads to

$$\begin{aligned}\log|x(t)| = {} & \log|x(0)| \\ & + \int_0^t \left\{ |x(t)|^{-2}\langle x(s), f(x_s, r(s), s)\rangle - \frac{1}{2}\left[\sigma(r(s))^2|x(s)|^{2\beta} + q^2(r(s))\right] \right\} ds \\ & + \int_0^t q(r(s))dW_1(s) + \sigma(r(s))|x(s)|^{\beta} dW_2(s).\end{aligned} \tag{4.19}$$

Define $M(t) = \int_0^t \sigma(r(s))|x(s)|^{\beta} dW_2(s)$; clearly $M(t)$ is a continuous local martingale with the quadratic variation

$$\langle M(t), M(t)\rangle = \int_0^t \sigma^2(r(s))|x(s)|^{2\beta} ds. \tag{4.20}$$

For any $\delta \in (0,1)$, choose $\vartheta > 0$ such that $\delta\vartheta > 1$ and each positive integer $n > 0$; the exponential martingale inequality yields

$$\mathbb{P}\left\{ \sup_{0\leq t\leq n} \left[M(t) - \frac{\delta}{2}\int_0^t \sigma^2(r(s))|x(s)|^{2\beta} ds \right] \geq \log n^{\vartheta} \right\} \leq \frac{1}{n^{\delta\vartheta}}. \tag{4.21}$$

Since $\sum_{n=1}^{\infty} n^{-\delta\vartheta} < \infty$, by the Borel-Cantelli lemma, there exists an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, there exists an integer $n(\omega)$, when $n > n(\omega)$ and $n-1 \leq t \leq n$,

$$M(t) \leq \frac{\delta}{2}\int_0^t \sigma^2(r(s))|x(s)|^{2\beta} ds + \vartheta\delta \log(t+1). \tag{4.22}$$

This, together with Assumption C, denote $\widetilde{q} =: \max_{k\in\mathbb{S}}\{q(k)\}$ and $\widetilde{\overline{\kappa}} =: \max_{k\in\mathbb{S}}\{\overline{\kappa}(k)\}$; noting the definition of (4.17), we therefore have

$$\begin{aligned}\log|x(t)| \le{}& \log|x(0)| + \int_0^t\left[-\frac{\sigma^2(r(s))(1-\delta)}{2}|x(s)|^{2\beta} + \kappa(r(s))|x(s)|^{\alpha}\right.\\&\left.+\overline{\kappa}(r(s))\int_{-\tau}^{0}|x(s+\theta)|^{\alpha}d\mu(\theta) + \gamma - \frac{q^2(r(s))}{2}\right]ds\\&+ \widetilde{q}W_1(t) + \vartheta\delta\log(t+1)\\\le{}& \log|x(0)| + \int_0^t\left[\phi_\delta(r(s)) - \frac{q^2(r(s))}{2}\right]ds + \widetilde{\overline{\kappa}}\int_{-\tau}^{0}|\xi(s)|^{\alpha}ds\\&+ \widetilde{q}W_1(t) + \vartheta\delta\log(t+1),\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}\phi_\delta(k) &= -\frac{\sigma^2(k)(1-\delta)}{2}|x|^{2\beta} + (\kappa(k) + \overline{\kappa}(k))|x|^{\alpha} + \gamma\\\int_0^t\int_{-\tau}^{0}|x(s+\theta)|^{\alpha}d\mu(\theta)ds &- \int_0^t|x(s)|^{\alpha}ds \le \int_{-\tau}^{0}|\xi(s)|^{\alpha}ds.\end{aligned} \tag{4.24}$$

Applying the strong law of large number [2, Page 12] to the Brownian motion, we therefore have

$$\lim_{t\to\infty}\frac{W_1(t)}{t} = 0 \quad \text{a.s..} \tag{4.25}$$

Moreover, letting $\delta \to 0$, by the ergodic property of the Markov chain, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left[\phi(r(s)) - \frac{q^2(r(s))}{2}\right]ds = \sum_{j=1}^{N}\pi_j\left[\phi_j - \frac{q_j^2}{2}\right] \quad \text{a.s..} \tag{4.26}$$

Combined (4.25) and (4.26), it follows from (4.23)

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le \sum_{j=1}^{N}\pi_j\left[\phi_j - \frac{q_j^2}{2}\right] \quad \text{a.s..} \tag{4.27}$$

Thus the assertion (4.16) follows.

Clearly, if

$$\sum_{j=1}^{N}\pi_j\left[\phi_j - \frac{q_j^2}{2}\right] < 0, \tag{4.28}$$

system (1.1) is almost surely exponentially stable; the proof is completed. □

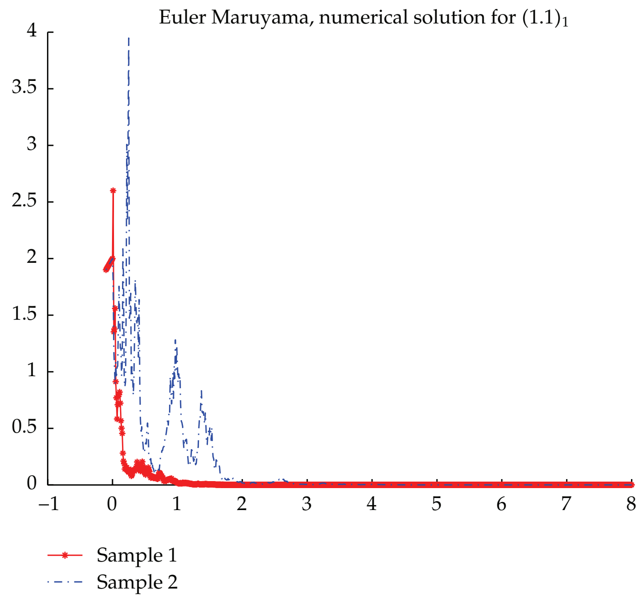

Figure 1: We choose $h = a = b = 1/4$, $q = 2$, $\sigma = 1$ in $(1.1)_1$

Remark. This results is generation of Theorem 4.2 in [6]. The author consider functional differential equation:

$$dx(t) = f(x_t, t)dt + qx(t)dW_1(t) + \sigma|x(t)|^{\beta}x(t)dW_2(t); \tag{4.29}$$

for example,

$$dx(t) = x(t)\left[h + ax(t) + b\int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right]dt + qx(t)dW_1(t) + \sigma x^2(t)dW_2(t), \qquad (1.1)_1$$

with $\beta = 1$, $\alpha = 1$, $\kappa = |a|$, $\overline{\kappa} = |b|$, $\gamma = |h|$, $\sigma = 1$,

$$\phi = \max_{x>0}\left\{-\frac{1}{2}x^2 + (|a| + |b|)x + |h|\right\} = |h| + \frac{1}{2}(|a| + |b|)^2; \tag{4.30}$$

hence, we choose $q^2/2 > |h| + 1/2(|a| + |b|)^2$ satisfying $\phi - (q^2/2) < 0$; the stochastic functional system $(1.1)_1$ is almost surely exponentially stable. We can observe the numerical simulation in Figure 1.

Example 4.3. Let us assume that the Markov chain $r(t)$ is on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{pmatrix}, \tag{4.31}$$

where $\gamma_{12} > 0$ and $\gamma_{21} > 0$. It is easy to see that the Markov chain has its stationary probability distribution $\pi = (\pi_1, \pi_2)$ given by

$$\pi_1 = \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}}, \qquad \pi_2 = \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}}. \tag{4.32}$$

As pointed out in Section,we may regard SDE (1.1) as the result of the following two equations:

$$dx(t) = f(x_t, 1, t)dt + q(1)x(t)dW_1(t) + \sigma(1)|x(t)|^{\beta}x(t)dW_2(t), \qquad (1.1)_{2_1}$$

$$dx(t) = f(x_t, 2, t)dt + q(2)x(t)dW_1(t) + \sigma(2)|x(t)|^{\beta}x(t)dW_2(t). \qquad (1.1)_{2_2}$$

Noting that $\sum_{i=1}^{n} \pi_i[\phi(i) - (q^2(i)/2)]$ has the form

$$\sum_{i=1}^{2} \pi_i \left[\phi(i) - \frac{q^2(i)}{2}\right] = \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}} \left[\phi(1) - \frac{q^2(1)}{2}\right] + \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}} \left[\phi(2) - \frac{q^2(2)}{2}\right], \tag{4.33}$$

that is, for a given state 1, $(1.1)_{2_1}$ may be written as

$$dx(t) = x(t)\left[1 + x(t) + \int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right]dt + x(t)dW_1(t) + x^2(t)dW_2(t), \tag{4.34}$$

with $x(0) = 1$ when $t \geq 0$. Applied the condition (4.17) with $q(1) = 1, \sigma(1) = \kappa(1) = \overline{\kappa}(1) = \gamma = \alpha = \beta = 1$ to the system (4.34) yields

$$\phi(1) = \max_{x>0}\left\{-\frac{1}{2}x^2 + 2x + 1\right\} = 3, \tag{4.35}$$

satisfying $\phi(1) - (q^2(1)/2) > 0$, which shows that the trajectory of (4.34) will not satisfied the conditions of Theorem 4.2 in [6] although it has global solution.

However, as the result of Markovian switching, the overall behavior, that is SDE (1.1) will be almost surely exponentially stable as long as

$$\phi(2) - \frac{q^2(2)}{2} < 0, \qquad \sum_{i=1}^{2} \pi_i \left[\phi(i) - \frac{q^2(i)}{2}\right] < 0, \tag{4.36}$$

namely, the transition rate γ_{21} from $(1.1)_{2_2}$ to $(1.1)_{2_1}$ is less than the transition rate γ_{12} from $(1.1)_{2_1}$ to $(1.1)_{2_2}$, which ensure that

$$\frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}} \left[\phi(1) - \frac{q^2(1)}{2}\right] + \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}} \left[\phi(2) - \frac{q^2(2)}{2}\right] < 0. \tag{4.37}$$

Example 4.4. Consider another stochastic differential equation with Markovian switching, where $r(t)$ is a Markov chain taking values in $S = \{1, 2, 3\}$. Here subsystem of (1.1) is writtern as three different equations:

$$dx(t) = x(t)\left[\frac{1}{4} + \frac{1}{4}x(t) + \frac{1}{4}\int_{-\tau}^{0} x(t+\theta)d\mu(\theta)\right]dt + 2x(t)dW_1(t) + x^2(t)dW_2(t), \qquad (1.1)_{3_1}$$

where $q(1) = 2, \sigma(1) = 1, \kappa(1) = \overline{\kappa}(1) = 1/4, \gamma = 1/4, \alpha = \beta = 1$,

$$\phi(1) = \max_{x>0}\left\{-\frac{1}{2}x^2 + \left(\frac{1}{4} + \frac{1}{4}\right)x + \frac{1}{4}\right\} = \frac{3}{8}, \quad \phi(1) - \frac{1}{2}q^2(1) = -\frac{13}{8} < 0;$$

$$dx(t) = x(t)\left[1 + x^2(t) + \int_{-\tau}^{0} x^2(t+\theta)d\mu(\theta)\right]dt + 2x(t)dW_1(t) + x^3(t)dW_2(t), \qquad (1.1)_{3_2}$$

where $q(2) = 2$, $\sigma(2) = \kappa(2) = \overline{\kappa}(2) = \gamma = 1$, $\alpha = \beta = 2$,

$$\phi(2) = \max_{x>0}\left\{-\frac{1}{2}x^4 + 2x^2 + 1\right\} = 3, \qquad \phi(2) - \frac{1}{2}q^2(2) = 1 > 0;$$

$$dx(t) = x(t)\left[1 - 2x^2(t) - 6\int_{-\tau}^{0} x^2(t+\theta)d\mu(\theta)\right]dt + 4x(t)dW_1(t) + 2x^3(t)dW_2(t), \qquad (1.1)_{3_3}$$

where $q(3) = 4$, $\sigma(3) = \kappa(3) = 2$, $\overline{\kappa}(3) = 6$, $\gamma = 1$, $\alpha = \beta = 2$,

$$\phi(3) = \max_{x>0}\left\{-2x^4 + 8x^2 + 1\right\} = 9, \qquad (4.38)$$

we compute

$$\phi(3) - \frac{1}{2}q^2(3) = 1 > 0. \qquad (4.39)$$

Case 1. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \qquad (4.40)$$

By solving the linear equation $\pi\Gamma = 0$ subject to $\sum_{k=1}^{N} \pi_k = 1$ and $\pi_k > 0$ for any $k \in \mathbb{S}$, we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{7}{15}, \frac{1}{5}, \frac{1}{3}\right). \qquad (4.41)$$

Then

$$\sum_{i=1}^{3} \pi_i \left[\phi(i) - \frac{1}{2} q^2(i) \right] = -\frac{27}{120} < 0. \tag{4.42}$$

Case 2. Suppose the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -5 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{pmatrix}. \tag{4.43}$$

By solving the linear equation $\pi\Gamma = 0$ subject to $\sum_{k=1}^{N} \pi_k = 1$ and $\pi_k > 0$ for any $k \in \mathbb{S}$, we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right). \tag{4.44}$$

Then

$$\sum_{i=1}^{3} \pi_i \left[\phi(i) - \frac{1}{2} q^2(i) \right] = \frac{11}{32} > 0. \tag{4.45}$$

Therefore, by Theorems 4.2, System (1.1) is almost surely exponentially stable in Case 1.

We can see the impact of the Markov chain $r(t)$. The distribution $(\pi = (\pi_1, \pi_2, \ldots, \pi_n))$ of $r(t)$ plays a very important role, which, combined with $\sum_{i=1}^{n} \pi_i [\phi(i) - (1/2) q^2(i)] < 0$, determine that system (1.1) is almost surely exponentially stable. If $r(t)$ spends enough time in the "good" states (the state where $\phi(i) - (q^2(i)/2) < 0$ for some i), even if there exist some "bad" states (the states where $\phi(i) - (q^2(i)/2) > 0$ for some i), the system (1.1) will still be almost surely exponentially stable.

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