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Research Article

Linear Multistep Methods for Impulsive Differential Equations

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This paper deals with the convergence and stability of linear multistep methods for impulsive differential equations. Numerical experiments demonstrate that both the mid-point rule and two-step BDF method are of order $p = 0$ when applied to impulsive differential equations. An improved linear multistep method is proposed. Convergence and stability conditions of the improved methods are given in the paper. Numerical experiments are given in the end to illustrate the conclusion.

1. Introduction

Impulsive differential equations provide a natural framework for mathematical modeling in ecology, population dynamic, optimal control, and so on. The studies focus on the theory of impulsive differential equations initiated in [1, 2]. In recent years many researches on the theory of impulsive differential equations are published (see [3–7]). And the numerical properties of impulsive differential equations begin to attract the authors' interest (see [8, 9]). But there are still few papers focus on the numerical properties of linear multistep methods for impulsive differential equations. In this paper, we will study the convergence and stability of linear multistep methods.

This paper focuses on the numerical solutions of impulsive differential equations as follows

$$\begin{aligned} x'(t) &= f(t,x), \quad t > 0, \quad t \neq \tau_d, \quad d \in N, \\ \Delta x &= I_k(x), \quad t = \tau_d, \\ x(t_0^+) &= x_0, \end{aligned} \tag{1.1}$$

where $f : R^+ \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ and $\tau_d < \tau_{d+1}$ with $\lim_{k\rightarrow+\infty}\tau_k = +\infty$. We assume $\Delta x = x(t+0) - x(t)$, where $x(t+0)$ is the right limit of $x(t)$.

In this paper, we consider the following equation:

$$\begin{aligned} x'(t) &= ax, \quad t > 0, \ t \neq d, \\ \Delta x &= bx, \quad t = d, \\ x(0^+) &= x_0, \end{aligned} \tag{1.2}$$

where $a, b, x_0 \in \mathbb{C}$, $b \neq -1$, $d \in N$.

Remark 1.1. If $b = -1$, then we obtain that $x(t) \equiv 0$ for $t > 1$. Therefore we omit this case in the paper.

Definition 1.2 (see [4]). $x(t)$ is said to be the solution of (1.2), if

(1) $\lim_{t\rightarrow 0^+} x(t) = x_0$;

(2) $x(t)$ is differentiable and $x'(t) = ax(t)$ for $t \in (0, +\infty), t \neq d, d \in N$;

(3) $x(t)$ is left continuous in $(0, +\infty)$ and $x(d^+) = (1 + b)x(d), d \in N$.

Theorem 1.3 (see [8]). *Equation* (1.2) *has a unique solution in* $(0, +\infty)$

$$x(t) = \begin{cases} x_0 e^{at}(1+b)^{[t]}, & t > 0, \quad t \neq d, \quad d \in N, \\ x_0 e^{ad}(1+b)^{d-1}, & t = d, \end{cases} \tag{1.3}$$

where $[\cdot]$ *denotes the greatest integer function towards minus infinity.*

2. Linear Multistep Methods

2.1. Linear Multistep Methods for ODEs

The standard form of linear multistep methods can be defined by

$$\sum_{i=0}^{k} \alpha_i x_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \tag{2.1}$$

where α_i and β_i are constants subject to the conditions:

$$\alpha_k = 1, \qquad |\alpha_0| + |\beta_0| \neq 0, \tag{2.2}$$

and $f_{n+i} := f(t_{n+i}, x_{n+i})$, $i = 0, 1, \ldots, k$.

Theorem 2.1 (see [10]). *The linear multistep methods* (2.1) *are convergent of order* p *for ODEs if and only if the following conditions are satisfied:*

$$\begin{gathered}\sum_{i=0}^{k}\alpha_i = 0,\\ \sum_{i=0}^{k}\alpha_i i^q = q\sum_{i=0}^{k}\beta_i i^{q-1} \quad \text{for } q = 1,\dots,p,\\ \sum_{i=0}^{k}\alpha_i i^{p+1} \neq (p+1)\sum_{i=0}^{k}\beta_i i^p.\end{gathered} \tag{2.3}$$

2.2. Linear Multistep Methods for Impulsive Differential Equations

Let $h = 1/m$ be a given stepsize with integer m. In this subsection, we consider the case when $m \geq k$. The application of the linear multistep methods (2.1) in case of (1.2) yields

$$\begin{aligned}x_{0,0} &= x_0,\\ x_{wm,l} &= -\sum_{i=0}^{k-1}\frac{\alpha_i - ha\beta_i}{\alpha_k - ha\beta_k}x_{wm,l+i-k}, \quad w = 0,1,\dots,\ l = k,\dots,m,\\ x_{(w+1)m,0} &= (1+b)x_{wm,m},\\ x_{(w+1)m,l} &= -\sum_{i=0}^{k-l-1}\frac{\alpha_i - ha\beta_i}{\alpha_k - ha\beta_k}x_{wm,m+l+i-k}\\ &\quad -\sum_{i=k-l}^{k-1}\frac{\alpha_i - ha\beta_i}{\alpha_k - ha\beta_k}x_{(w+1)m,l+i-k}, \quad l = 1,\dots,\ k-1,\end{aligned} \tag{2.4}$$

where $x_{wm,l}$ is an approximation of $x(t_{wm+l})$, and $x_{wm,0}$ denotes an approximation of $x(w^+)$. Here, we assume that the other starting value besides x_0, that is, $x_{0,1},\dots,x_{0,k-1}$, has been calculated by a one-step method of order 2.

Remark 2.2. As a special case, when $k = 1$ the corresponding consistent process (2.4) takes the form:

$$\begin{aligned}x_{0,0} &= x_0,\\ x_{wm,l} &= \left(1 + \frac{ha}{1 - ha\beta_1}\right)x_{wm,l-1}, \quad l = 1,\dots,m,\ w = 0,1,\dots,\\ x_{(w+1)m,0} &= (1+b)x_{wm,m},\end{aligned} \tag{2.5}$$

which is consistent with process (2.2) in [9].

Table 1: The explicit Euler method for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	5.025613608106683e + 003	1.283179151329931e − 001
20	2.739728003516164e + 003	6.621084007886913e − 002
40	1.433580942834451e + 003	3.361075658131119e − 002
80	7.337102544190930e + 002	1.693073363701148e − 002
160	3.712172178908295e + 002	8.496581963094774e − 003
320	1.867162056329107e + 002	4.256079227844101e − 003
Ratio	1.988136041178199e + 000	1.996340177952628e + 000

Remark 2.3. When $k = 2$, the corresponding process (2.4) takes the form:

$$\begin{aligned} x_{0,0} &= x_0, \\ x_{wm,l} &= -\frac{\alpha_0 - ha\beta_0}{\alpha_2 - ha\beta_2} x_{wm,l-2} - \frac{\alpha_1 - ha\beta_1}{\alpha_2 - ha\beta_2} x_{wm,l-1}, \quad w = 0,1,\ldots, \quad l = 2,\ldots,m, \\ x_{(w+1)m,0} &= (1+b)x_{wm,m}, \\ x_{(w+1)m,1} &= -\frac{\alpha_0 - ha\beta_0}{\alpha_2 - ha\beta_2} x_{wm,m-1} - \frac{\alpha_1 - ha\beta_1}{\alpha_2 - ha\beta_2} x_{(w+1)m,0}, \end{aligned} \tag{2.6}$$

where we assume that $x_{0,1}$ has been calculated by a one-step method of order $p \geq 2$.

In order to test the convergence, we consider the following equations:

$$\begin{aligned} x'(t) &= x, \quad t > 0, \ t \neq d, \\ \Delta x &= 2x, \quad t = d, \\ x(0^+) &= 2, \end{aligned} \tag{2.7}$$

$$\begin{aligned} x'(t) &= -x, \quad t > 0, \ t \neq d, \\ \Delta x &= 2x, \quad t = d, \\ x(0^+) &= 1. \end{aligned} \tag{2.8}$$

We use the process (2.5) in case of $\beta_1 = 0$ (i.e., the explicit Euler method) and the process (2.6) in case of the mid-point rule and 2-step BDF methods to get numerical solutions at $t = 5$, where the corresponding analytic solution can be calculated by Theorem 1.3. We have listed the absolute errors and the ratio of the errors of the case $m = 160$ over that $m = 320$ in the following tables.

We can conclude from Table 1 that the explicit Euler method is of order 1 which means that the process (2.4) is defined reasonable. Tables 2 and 3 imply that both methods are of order 0, when applied to the given impulsive differential equations.

Table 2: The mid-point rule for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	1.734606333614340e + 004	2.611636966771883e + 002
20	1.785542792421782e + 004	2.362243809005279e + 002
40	1.811021056819620e + 004	2.241871891262895e + 002
80	1.823713002067653e + 004	2.182840114478902e + 002
160	1.830041179075918e + 004	2.153620314650900e + 002
320	1.833200086044313e + 004	2.139085286186221e + 002
Ratio	9.982768345951743e − 001	1.006794973794894e + 000

Table 3: The 2-step BDF methods for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	4.051557443305367e + 004	1.585857685173612e + 000
20	4.538988757794288e + 004	1.365280016628375e + 000
40	4.842123370314762e + 004	1.268609538412996e + 000
80	5.011596302177681e + 004	1.223047705616427e + 000
160	5.101279019662662e + 004	1.200895589468623e + 000
320	5.147422493980224e + 004	1.189969480734175e + 000
Ratio	9.910356155198985e − 001	1.009181839460040e + 000

3. The Improved Linear Multistep Methods

In this section, we will consider the improved linear multistep methods:

$$\sum_{i=0}^{k} \widehat{\alpha}_i^n x_{n+i-k} = h \sum_{i=0}^{k} \widehat{\beta}_i^n f(t_{n+i-k}, x_{n+i-k}), \tag{3.1}$$

where $\widehat{\alpha}_i^n = \widehat{\alpha}_i^n(n,i)$, $\widehat{\beta}_i^n = \widehat{\beta}_i^n(n,i)$ and $h = 1/m$. The application of method (3.1) in case of (1.2), yields

$$\sum_{i=0}^{k} \widehat{\alpha}_i^n x_{n+i-k} = h \sum_{i=0}^{k} a \widehat{\beta}_i^n x_{n+i-k}. \tag{3.2}$$

In the rest section of this section, we will propose a convergence condition of the method (3.1) for (1.2). Firstly we give a definition about the residual of (3.2), which is essentially the local truncation error.

Definition 3.1. Assume that $x(t)$ is the analytic solution of (1.2). Then the residual of process (3.2) is defined by

$$R_n = \sum_{i=0}^{k} \widehat{\alpha}_i^n x(t_{n+i-k}) - h \sum_{i=0}^{k} a \widehat{\beta}_i^n x(t_{n+i-k}). \tag{3.3}$$

Definition 3.2. The improved linear multistep methods (3.2) is said to be of order p for (1.2), if the residual defined by (3.3) satisfies $R_n = O(h^{p+1})$ for arbitrary n.

The following theorem gives a condition under which the improved linear multistep methods can preserve their original order for ODEs when applied to (1.2). Without loss of generality, we assume $\{t_{n+i-k}\} = 0$ for $i = i_0, i_1, \ldots, i_c, 0 \leq i_c < k$, where $\{t\}$ denotes the fractional part of t.

Theorem 3.3. *Assume* (2.3) *holds, and there exists a function* $\mu_n = \mu(n)$ *together with a constant* λ *such that* $\widehat{\alpha}_i^n$ *and* $\widehat{\beta}_i^n$ *in* (3.2) *satisfy*

$$\begin{aligned} &\sum_{i=0, i\neq i_1,\ldots,i_c}^{k} \left(\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n\right) e^{at_{n+i-k}} (1+b)^{[t_{n+i-k}]} \\ &\quad + \sum_{j=1}^{c} \left(\widehat{\alpha}_{i_j}^n - ha\widehat{\beta}_{i_j}^n\right) e^{at_{n+i_j-k}} (1+b)^{[t_{n+i_j-k}]-1} \\ &= \mu_n \sum_{i=0}^{k} (\alpha_i - h\lambda\beta_i) e^{\lambda t_{n+i-k}}. \end{aligned} \tag{3.4}$$

Then, the improved linear multistep methods (3.2) *are of order* p *for* (1.2).

Proof. It follows from Definition 3.1 that

$$R_n = \sum_{i=0}^{k} \widehat{\alpha}_i^n x(t_{n+i-k}) - h \sum_{i=0}^{k} \widehat{\beta}_i^n a x(t_{n+i-k}) = \sum_{i=0}^{k} \left(\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n\right) x(t_{n+i-k}). \tag{3.5}$$

By Theorem 1.3, we have

$$x(t_{n+i-k}) = \begin{cases} x_0 e^{at_{n+i-k}} (1+b)^{[t_{n+i-k}]}, & \{t_{n+i-k}\} \neq 0, \\ x_0 e^{at_{n+i-k}} (1+b)^{t_{n+i-k}-1}, & \{t_{n+i-k}\} = 0. \end{cases} \tag{3.6}$$

Therefore,

$$\begin{aligned} R_n &= x_0 \sum_{i=0, i\neq i_1,\ldots,i_c}^{k} \left(\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n\right) e^{at_{n+i-k}} (1+b)^{[t_{n+i-k}]} \\ &\quad + x_0 \sum_{j=1}^{c} \left(\widehat{\alpha}_{i_j}^n - ha\widehat{\beta}_{i_j}^n\right) e^{at_{n+i_j-k}} (1+b)^{t_{n+i_j-k}-1} \\ &= \mu_n x_0 \sum_{i=0}^{k} (\alpha_i - h\lambda\beta_i) e^{\lambda t_{n+i-k}} \\ &= \mu_n x_0 \sum_{i=0}^{k} (\alpha_i - h\lambda\beta_i) e^{\lambda h(n+i-k)}. \end{aligned} \tag{3.7}$$

We can express the residual as a power series in h: collecting terms in R_n to obtain

$$R_n = \mu_n x_0 \left(r_0 + r_1 h + \cdots + r_q h^q + \cdots\right). \tag{3.8}$$

Then,

$$\begin{aligned}
r_q &= \sum_{i=0}^{k}\alpha_i \frac{\lambda^q (n+i-k)^q}{q!} - \sum_{i=0}^{k}\lambda\beta_i \frac{\lambda^{q-1}(n+i-k)^{q-1}}{(q-1)!} \\
&= \sum_{i=0}^{k}\alpha_i \frac{\lambda^q (n+i-k)^q}{q!} - \sum_{i=0}^{k}\beta_i \frac{\lambda^{q}(n+i-k)^{q-1}}{(q-1)!} \\
&= \sum_{i=0}^{k}\alpha_i \frac{\lambda^q}{q!}\sum_{j=0}^{q}\frac{q!}{j!(q-j)!} i^j (n-k)^{q-j} - \sum_{i=0}^{k}\beta_i \frac{\lambda^q}{(q-1)!}\sum_{j=0}^{q-1}\frac{(q-1)!}{j!(q-1-j)!} i^j (n-k)^{q-1-j} \\
&= \sum_{i=0}^{k}\alpha_i\lambda^q \left(\frac{(n-k)^q}{q!} + \sum_{j=0}^{q-1}\frac{i^{j+1}(n-k)^{q-j-1}}{(j+1)!(q-1-j)!}\right) - \sum_{i=0}^{k}\beta_i\lambda^q \sum_{j=0}^{q-1}\frac{i^j (n-k)^{q-j-1}}{j!(q-1-j)!} \\
&= \frac{\lambda^q (n-k)^q}{q!}\sum_{i=0}^{k}\alpha_i + \lambda^q \sum_{j=0}^{q-1}\frac{(n-k)^{q-j-1}}{j!(q-1-j)!}\sum_{i=0}^{k} i^j\left(\frac{i\alpha_i}{j+1} - \beta_i\right).
\end{aligned} \tag{3.9}$$

By (2.3),

$$r_q = 0, \quad q = 0, 1, \ldots, p, \qquad r_{p+1} \neq 0. \tag{3.10}$$

Therefore, $R_{n+k} = O(h^{p+1})$. The proof is complete. □

3.1. An Example

Denote $l = m\{n/m\}$, then we can define the coefficients of (3.2) as follows:

$$\begin{aligned}
\widehat{\alpha}_i^n &= (1+b)^{[(k-l-i)/m]+1}\alpha_i, \\
\widehat{\beta}_i^n &= (1+b)^{[(k-l-i)/m]+1}\beta_i.
\end{aligned} \tag{3.11}$$

Theorem 3.4. *Assume that* (2.3) *holds. Then the improved linear multistep methods formed by* (3.11) *are of order p for* (1.2).

Proof. We only need to verify that the condition in Theorem 3.3 holds. Note that

$$t_{n+i-k} = \frac{n+i-k}{m} = \left[\frac{n}{m}\right] + \frac{m\{n/m\}+i-k}{m} = \left[\frac{n}{m}\right] + \frac{l+i-k}{m}. \tag{3.12}$$

Thus, $\{t_{n+i-k}\} = 0$ if and only if $[(k-i-l)/m] = (k-i-l)/m$, that is, $i = i_1, \ldots, i_c$, $0 \le i_c < k$. Therefore,

$$[t_{n+i-k}] + \left[\frac{k-i-l}{m}\right] = \begin{cases} \left[\dfrac{n}{m}\right], & \text{if } \left[\dfrac{k-i-l}{m}\right] = \dfrac{k-i-l}{m}, \\[2ex] \left[\dfrac{n}{m}\right] - 1 & \text{if } \left[\dfrac{k-i-l}{m}\right] \neq \dfrac{k-i-l}{m}. \end{cases} \tag{3.13}$$

Hence,

$$\begin{aligned}
&\sum_{i=0, i\neq i_1,\ldots,i_c}^{k}\left(\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n\right)e^{at_{n+i-k}}(1+b)^{[t_{n+i-k}]} \\
&\quad + \sum_{j=1}^{c}\left(\widehat{\alpha}_{i_j}^n - ha\widehat{\beta}_{i_j}^n\right)e^{at_{n+i_j-k}}(1+b)^{[t_{n+i_j-k}]-1} \\
&= \sum_{i=0, i\neq i_1,\ldots,i_c}^{k}(\alpha_i - ha\beta_i)e^{at_{n+i-k}}(1+b)^{[t_{n+i-k}]+[(k-i-l)/m]+1} \\
&\quad + \sum_{j=1}^{c}\left(\alpha_{i_j} - ha\beta_{i_j}\right)e^{at_{n+i_j-k}}(1+b)^{[t_{n+i_j-k}]+[(k-i_j-l)/m]} \\
&= (1+b)^{[n/m]}\sum_{i=0}^{k}(\alpha_i - ha\beta_i)e^{at_{n+i-k}}.
\end{aligned} \tag{3.14}$$

Thus, the conditions in Theorem 3.3 are satisfied with $\mu_n = (1+b)^{[n/m]}$ and $\lambda = a$. Thus, the proof is complete. □

Remark 3.5. If $b = 0$ in (3.11), that is, the impulsive differential equations reduce to ODEs, we have $\widehat{\alpha}_i^n = \alpha_i$, $\widehat{\beta}_i^n = \beta_i$, that is, the improved linear multistep methods (3.2) reduce to the classical linear multistep methods.

Remark 3.6. In the improved linear multistep method defined by (3.11), the stepsize $h = 1/m$ can be chosen with arbitrary positive integer m without any restriction.

Remark 3.7. If $m \geq l \geq k$, then $\widehat{\alpha}_i^n = \alpha_i$, $\widehat{\beta}_i^n = \beta_i$. In other words, if all the mesh points are in the same integer interval, then the process (3.2) defined by (3.11) reduces to the classical linear multistep methods (2.1).

4. Stability Analysis

In this section, we will investigate the stability of the improved linear multistep methods (3.1) for (1.2). The following theorem is an extention of Theorem 1.4 in [9], and the proof is obvious.

Theorem 4.1 (see [9]). *The solution $x(t) \equiv 0$ of* (1.2) *is asymptotically stable if and only if $|(1+b)e^a| < 1$.*

The corresponding property of the numerical solution is described as follows.

Definition 4.2. The numerical solution x_n is called asymptotically stable for (1.2) with $|(1+b)e^a| < 1$ if $\lim_{n\to\infty} x_n = 0$ for arbitrary stepsize $h = 1/m$.

Theorem 4.3. *Assume there exist constants ϵ, $C > 0$ and a consequence of functions $\gamma_i = \gamma_i(n,i)$, $i = 0,\ldots,k$ such that $|1/\gamma_i| < C$ for arbitrary n and $0 \leq i \leq k$, and the following equality holds*

$$\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n = (\alpha_i - h\epsilon\beta_i)\gamma_i, \quad i = 0,1,\ldots,k. \tag{4.1}$$

Then, $\lim_{n\to\infty} x_n = 0$ if the corresponding linear multistep methods (2.1) *are A-stable and $\Re\epsilon < 0$.*

Proof. Denote

$$y_{n+i-k} = \gamma_i x_{n+i-k}. \tag{4.2}$$

Then by (4.1), the process (3.2) becomes

$$\sum_{i=0}^{k} \left(\alpha_i - h\epsilon\beta_i\right) y_{n+i-k} = 0. \tag{4.3}$$

Therefore, y_n can be viewed as the numerical solution of the equation $y'(t) = \epsilon y(t)$ calculated by linear multistep methods (2.1).

On the other hand, we know that $\Re\epsilon < 0$ and the methods are A-stable. Therefore, $\lim_{n\to\infty} y_n = 0$. The conclusion is obvious in view of that

$$|x_n| < C\left|y_n\right|. \tag{4.4}$$

□

Corollary 4.4. *The improved linear multistep method* (3.11) *is asymptotically stable for* (1.2) *when the corresponding linear multistep methods* (2.1) *are A-stable, and* $\Re a < 0$ *hold.*

Proof. It is obvious that for (3.11):

$$\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n = \left(\alpha_i - ha\beta_i\right)\gamma_i, \quad i = 0, 1, \ldots, k, \tag{4.5}$$

where $\gamma_i = (1+b)^{[(k-l-i)/m]+1}$. Note that k, l, and i are all bounded when the method and the stepsize are given. Therefore, $|1/\gamma_i|$ are uniformly bounded. Thus the proof is complete. □

Remark 4.5. In fact, the improved linear multistep methods (3.11) cannot preserve the asymptotical stability of all equation (1.2). To illustrate this, we consider the following equation:

$$\begin{gathered} x'(t) = 2x, \quad t > 0,\ t \neq d, \\ \Delta x = \left(e^{-2} - 1.27\right)x, \quad t = d, \\ x(0^+) = 2. \end{gathered} \tag{4.6}$$

Theorem 4.1 implies that $\lim_{t\to\infty} x(t) = 0$. We have drawn the numerical solution calculated by method (3.11) in case of 2-step BDF methods, which is A-stable as we know, on $[0, 500]$ in Figure 1. Figure 1 indicates that the numerical solutions are not asymptotically stable. Hence, we will give another improved linear multistep method in the next section.

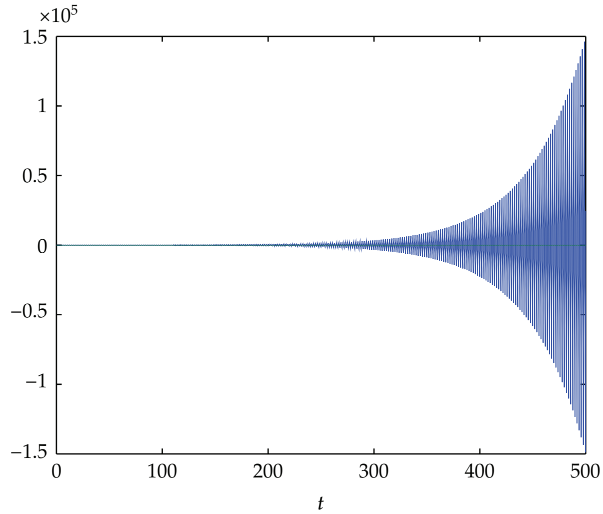

Figure 1: The numerical solution obtained by (3.11) in case of the 2-step BDF method to (4.6).

4.1. Another Improved Linear Multistep Methods

In this section, we give another improved linear multistep methods. We define the coefficients as follows:

$$\widehat{\alpha}_i^n = \begin{cases} (1+b)^{\{t_{n+i-k}\}}\alpha_i, & \{t_{n+i-k}\} \neq 0, \\ (1+b)\alpha_i, & \{t_{n+i-k}\} = 0, \end{cases}$$
$$\widehat{\beta}_i^n = \begin{cases} \dfrac{(a+\ln(1+b))}{a}(1+b)^{\{t_{n+i-k}\}}\beta_i, & \{t_{n+i-k}\} \neq 0, \\ \dfrac{(a+\ln(1+b))}{a}(1+b)\beta_i, & \{t_{n+i-k}\} = 0, \end{cases} \tag{4.7}$$

where we define $\widehat{\beta}_i^n = 0$, when $a = 0$.

Theorem 4.6. *Assume that* (2.3) *holds. Then, the improved linear multistep methods formed by* (4.7) *are of order p for* (1.2).

Proof. It is obvious that

$$\sum_{i=0, i\neq i_1,\ldots,i_c}^{k} \left(\widehat{\alpha}_i^n - ha\widehat{\beta}_i^n\right) e^{at_{n+i-k}} (1+b)^{[t_{n+i-k}]}$$
$$+ \sum_{j=1}^{c} \left(\widehat{\alpha}_{i_j}^n - ha\widehat{\beta}_{i_j}^n\right) e^{at_{n+i_j-k}} (1+b)^{[t_{n+i_j-k}]-1}$$

$$= \sum_{i=0}^{k} (\alpha_i - h(a + \ln(1+b))\beta_i) e^{at_{n+i-k}} (1+b)^{t_{n+i-k}}$$
$$= \sum_{i=0}^{k} (\alpha_i - h(a + \ln(1+b))\beta_i) e^{(a+\ln(1+b))t_{n+i-k}}. \tag{4.8}$$

Thus, the conditions in Theorem 3.3 are satisfied with $\mu_n = 1$, and $\lambda = a + \ln(1+b)$. The proof is compete. □

Theorem 4.7. *Assume that* $\Re(a + \ln(1+b)) < 0$. *Then,* $\lim_{n\to\infty} x_n = 0$ *if the corresponding linear multistep methods* (2.1) *are A-stable, where* x_n *is obtained by the improved linear multistep method* (4.7).

Proof. Define

$$\gamma_i = \begin{cases} (1+b)^{\{t_{n+i-k}\}}, & \{t_{n+i-k}\} \neq 0, \\ (1+b), & \{t_{n+i-k}\} = 0. \end{cases} \tag{4.9}$$

Then (4.1) is satisfied, and $|1/\gamma_i| \le \max\{1, |1/(1+b)|\} = C$. Therefore the conditions of Theorem 4.3 are satisfied, and the conclusion follows. □

4.2. Another Way to View the Improved Linear Multistep Method (4.7)

In fact, the improved linear multistep methods (4.7) can be viewed as the application of the classical linear multistep methods to the modified form of (1.2).

Denote

$$y(t) = \begin{cases} (1+b)^{\{t\}} x(t), & t \ge 0, \quad t \neq d, \\ (1+b) x(d), & t = d. \end{cases} \tag{4.10}$$

Then, it is easy to see that $y(t)$ is continuous for $t \in [0, +\infty)$.

Lemma 4.8. *Assume that* (4.10) *holds. Then* $x(t)$ *is the solution of* (1.2) *if and only if* $y(t)$ *is the solution of*

$$\begin{aligned} y'(t) &= (a + \ln(1+b)) y(t), \\ y(0) &= x_0. \end{aligned} \tag{4.11}$$

Proof. Necessity. In view of Theorem 1.3 and (4.10), we obtain that in the case $t \neq d$:

$$y(t) = (1+b)^{\{t\}} e^{at} (1+b)^{[t]} x_0 = e^{at} (1+b)^t x_0 = e^{(a+\ln(1+b))t} x_0. \tag{4.12}$$

and $y(k) = (1+b)x(k) = e^{(a+\ln(1+b))k} x_0$, which is coincided with the solution of (4.11). The necessity can be proved in the same way, and the proof is complete. □

Table 4: Equation (5.1) in case of mid-point rule for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	1.964613490354896e + 002	8.950836048980485e − 001
20	4.971904772277776e + 001	1.197198445982186e − 001
40	1.248351989479124e + 001	1.553305410469574e − 002
80	3.126205023072544e + 000	2.002751471856357e − 003
160	7.821288588893367e − 001	2.607915957975049e − 004
320	1.955988918562071e − 001	3.491638393382512e − 005
Ratio	3.998636451705015e + 000	7.469032196798133e + 000

Table 5: Equation (5.1) in case of 2-step BDF methods for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	3.675982365982200e + 002	9.516633893408510e − 003
20	9.552174783206283e + 001	2.325825408788229e − 003
40	2.442645505222754e + 001	5.749450957563962e − 004
80	6.181500463695556e + 000	1.429301543146577e − 004
160	1.555180585986818e + 000	3.563220614588580e − 005
320	3.900496418718831e − 001	8.895535536623811e − 006
Ratio	3.987135018310406e + 000	4.005627991612696e + 000

Table 6: Equation (4.13) in case of mid-point rule for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	1.533815775376497e + 003	4.656258034474448e − 006
20	4.273673528511026e + 002	1.126844488719137e − 006
40	1.116157821124107e + 002	2.771258855727155e − 007
80	2.844156657094209e + 001	6.871261282181962e − 008
160	7.173499151609576e + 000	1.710731889481565e − 008
320	1.800995925495954e + 000	4.267991404738325e − 009
Ratio	3.983073503974838e + 000	4.008283352169618e + 000

It follows from (4.10) that the numerical solutions of (1.2) can be approximated by means of y_n as follows:

$$\sum_{i=0}^{k}\alpha_i y_{n+i-k} = h(a+\ln(1+b))\sum_{i=0}^{k}\beta_i y_{n+i-k},$$
$$x_n = \begin{cases}(1+b)^{-\{t_n\}}y_n, & \text{if } \{t_n\}\neq 0,\\ (1+b)^{-1}y_n, & \text{if } \{t_n\}=0.\end{cases} \tag{4.13}$$

It is obvious that the methods (4.7) and (4.13) are the same.

5. Numerical Experiment

In this section, some numerical experiments are given to illustrate the conclusion in the paper.

Table 7: Equation (4.13) in case of 2-step BDF methods for (2.7) and (2.8).

m	Absolute errors for (2.7)	Absolute errors for (2.8)
10	3.034120865054534e + 003	9.093526915804340e − 006
20	8.173504073904187e + 002	2.227110126540310e − 006
40	2.159842038101851e + 002	5.509781502155420e − 007
80	5.579637010861916e + 001	1.370190298999319e − 007
160	1.419913379106947e + 001	3.416407701184454e − 008
320	3.582733454190020e + 000	8.529607353757740e − 009
Ratio	3.963212438944776e + 000	4.005351664492912e + 0 00

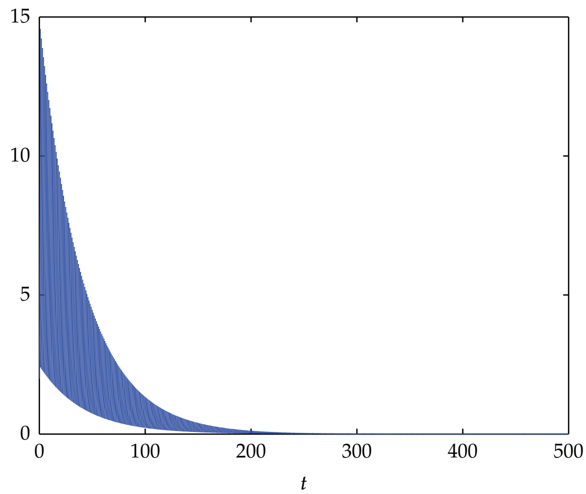

Figure 2: The numerical solution obtained by (4.13) in case of the 2-step BDF method to (4.6).

5.1. Convergence

The improved 2-step linear multistep methods (3.11) takes the form:

$$\begin{aligned} x_{0,0} &= x_0, \\ x_{wm,l} &= -\frac{\alpha_0 - ha\beta_0}{\alpha_2 - ha\beta_2} x_{wm,l-2} - \frac{\alpha_1 - ha\beta_1}{\alpha_2 - ha\beta_2} x_{wm,l-1}, \quad l = 2, \dots, m, \;\; w = 0, 1, \dots, \\ x_{(w+1)m,0} &= (1+b) x_{wm,m}, \\ x_{(w+1)m,1} &= -(1+b) \frac{\alpha_0 - ha\beta_0}{\alpha_2 - ha\beta_2} x_{wm,m-1} - \frac{\alpha_1 - ha\beta_1}{\alpha_2 - ha\beta_2} x_{(w+1)m,0}, \end{aligned} \tag{5.1}$$

where we assume that $x_{0,1}$ has been calculated by a one-step method of order $p \geq 2$. We use the methods (5.1) and (4.13) in case of the mid-point rule and 2-step BDF method. We consider (2.7) and (2.8) and calculate the numerical solutions at $t = 5$ with stepsize $h = 1/m$. We have listed the absolute errors and the ratio of the errors of the case $m = 160$ over the error in the case $m = 320$, from which we can estimate the convergent order. We can see from Tables 4, 5, 6, and 7, that all methods can preserve their original order for ODEs.

5.2. Stability

To illustrate the stability, we consider (4.6). We use method (4.13) in case of the 2-step BDF method. We draw the module of numerical solution on $[0, 500]$ in Figure 2. We can see from the figure that the method can preserve the stability of the analytic solution.

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