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Research Article

Hermite Polynomials and their Applications Associated with Bernoulli and Euler Numbers

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We derive some interesting identities and arithmetic properties of Bernoulli and Euler polynomials from the orthogonality of Hermite polynomials. Let $\mathbf{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Then we show that $\{H_0(x), H_1(x), \ldots, H_n(x)\}$ is a good basis for the space $\mathbf{P}_n$ for our purpose of arithmetical and combinatorial applications.

1. Introduction

As is well known, the Euler polynomials, $E_n(x)$, are defined by the generating function as follows:

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \tag{1.1}$$

(see [1–8]), with the usual convention about replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ is called the nth *Euler number*. From (1.1) and definition of Euler numbers, we note that

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n}\binom{n}{l}E_l x^{n-l} = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}x^l \tag{1.2}$$

with the usual convention about replacing E^n by E_n.

The *Bernoulli numbers* are defined as

$$B_0 = 1, \qquad (B+1)^n - B_n = \delta_{1,n} \tag{1.3}$$

(see [9–14]), where $\delta_{k,n}$ is a Kronecker symbol.

As is well known, *Bernoulli polynomials* are also defined by

$$B_n(x) = (B+x)^n = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l \tag{1.4}$$

with the usual convention about replacing B^n by B_n (see [1, 15–18]).

The *Hermite polynomials* are defined by the generating function as follows:

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{1.5}$$

(see [5, 19]), with the usual convention about replacing $H^n(x)$ by $H_n(x)$.

From (1.5), we can derive the following identities:

$$\begin{aligned} H_n(x) &= \left(\frac{\partial}{\partial t}\right)^n e^{2xt-t^2}\Big|_{t=0} = e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\Big|_{t=0} \\ &= (-1)^n e^{x^2}\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\Big|_{t=0} = (-1)^n e^{x^2}\left(\frac{d^n}{dx^n} e^{-x^2}\right). \end{aligned} \tag{1.6}$$

Let us consider two operators as follows:

$$\begin{aligned} f \longmapsto O_1 f &= -\left(e^{x^2}\frac{d}{dx}e^{-x^2}\right) f = 2xf - \frac{df}{dx}, \\ f \longmapsto O_2 f &= \left(e^{x^2/2}\left(x - \frac{d}{dx}\right)e^{-x^2/2}\right) f = 2xf - \frac{df}{dx}. \end{aligned} \tag{1.7}$$

By (1.7), we get $O_1 = O_2$. In particular, if we take $f = 1$, then we have

$$-e^{x^2}\left(\frac{d}{dx}e^{-x^2}\right) = e^{x^2/2}\left(x - \frac{d}{dx}\right)e^{-x^2/2}. \tag{1.8}$$

We note that

$$(-1)^n e^{x^2}\left(\frac{d^n}{dx^n}e^{-x^2}\right) = \left(-e^{x^2}\frac{d}{dx}e^{-x^2}\right)^n. \tag{1.9}$$

From (1.8), we note that

$$(-1)^n e^{x^2}\left(\frac{d^n e^{-x^2}}{dx^n}\right) = \left(-e^{x^2}\frac{de^{-x^2}}{dx}\right)^n = \left(e^{x^2/2}\left(x-\frac{d}{dx}\right)e^{-x^2/2}\right)^n$$
$$= e^{x^2/2}\left(x-\frac{d}{dx}\right)^n e^{-x^2/2}. \tag{1.10}$$

Thus, by (1.10), we get

$$H_n(x) = e^{x^2/2}\left(x-\frac{d}{dx}\right)^n e^{-x^2/2} \tag{1.11}$$

(see [5, 19–23]). In the special case, $x = 0$, $H_n(0) = H_n$ are called the *Hermite numbers*.

From (1.5), we can derive the following identities:

$$H_n(x) = (H+2x)^n = \sum_{l=0}^{n}\binom{n}{l}H_{n-l}2^l x^l \tag{1.12}$$

(cf. [5, 19]), with the usual convention about replacing H^n by H_n. It is easy to show that

$$\sum_{n=0}^{\infty}H_n\frac{t^n}{n!} = e^{-t^2} = \sum_{l=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}. \tag{1.13}$$

By comparing coefficients on the both sides of (1.13), we get

$$H_{2n} = (-1)^n 2n(2n-1)\cdots(n+1) = \frac{(-1)^n(2n)!}{n!}, \qquad H_{2n-1} = 0, \tag{1.14}$$

where $n \in \mathbb{N}$. From (1.12), we have

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x) \quad (n \in \mathbb{N}). \tag{1.15}$$

Let $\mathbf{P}_n = \{p \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Probably, $\{1, x, x^2, \dots, x^n\}$ is the most natural basis for this space. But $\{H_0(x), H_1(x), H_2(x), \dots, H_n(x)\}$ is also a good basis for the space $\mathbf{P}_n$, for our purpose of arithmetical and combinatorial applications.

For $p(x) \in \mathbf{P}_n$,

$$p(x) = \sum_{k=0}^{n} C_k H_k(x), \tag{1.16}$$

for some uniquely determined $b_l \in \mathbb{Q}$.

The purpose of this paper is to develop methods for computing C_k from the information of $p(x)$. By using these methods, we define some interesting identities.

2. Properties of Hermite Polynomials

From (1.5) and (1.13), we note that

$$\begin{aligned}1 &= \left(\sum_{m=0}^{\infty}\frac{H_m t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{t^{2l}}{l!}\right)\\ &= \left(\sum_{m=0}^{\infty}H_{2m}\frac{t^{2m}}{(2m)!}\right)\left(\sum_{l=0}^{\infty}\frac{(2l)(2l-1)\cdots(l+1)}{(2l)!}t^{2l}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{(2l)(2l-1)\cdots(l+1)}{(2l)!(2n-2l)!}H_{2n-2l}(2n)!\right)\frac{t^{2n}}{(2n)!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}l!\binom{2l}{l}\binom{2n}{2l}H_{2n-2l}\right)\frac{t^{2n}}{(2n)!}.\end{aligned} \tag{2.1}$$

Thus, by (2.1), we obtain the following recurrence formula.

Proposition 2.1. *For $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, one has*

$$\sum_{l=0}^{n}l!\binom{2l}{l}\binom{2n}{2l}H_{2n-2l} = \begin{cases}1, & if\ n = 0\\ 0, & if\ n \neq 0\end{cases}. \tag{2.2}$$

By, (1.5), we get

$$\sum_{n=0}^{\infty}H_n(-x)\frac{t^n}{n!} = e^{2t(-x)-t^2} = e^{2x(-t)-(-t)^2} = \sum_{n=0}^{\infty}H_n(x)(-1)^n\frac{t^n}{n!}. \tag{2.3}$$

From (2.3), we can derive the following reflection symmetric identity of $H_n(x)$:

$$H_n(-x) = (-1)^n H_n(x). \tag{2.4}$$

By (1.5), we easily see that

$$\frac{\partial}{\partial t}\left(e^{2xt-t^2}\right) = (2x-2t)e^{2xt-t^2}. \tag{2.5}$$

Thus, by (1.5) and (2.5), we get

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\right) = (2x-2t)\left(\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\right). \tag{2.6}$$

$$\text{LHS of (2.5)} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!}, \tag{2.7}$$

$$\begin{aligned}\text{RHS of (2.5)} &= \sum_{n=0}^{\infty}\left(2xH_n(x)\frac{t^n}{n!}\right) - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty}\left(2xH_n(x)\frac{t^n}{n!}\right) - \sum_{n=1}^{\infty} 2H_{n-1}(x)\frac{t^n}{(n-1)!} \\ &= \sum_{n=0}^{\infty}(2xH_n(x))\frac{t^n}{n!} - \sum_{n=1}^{\infty} 2nH_{n-1}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Thus, by (2.6) and (2.7), we get

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad (n \in \mathbb{N}). \tag{2.9}$$

From (1.15) and (2.9), we note that

$$H_{n+1}(x) - 2xH_n(x) + H'_n(x) = 0. \tag{2.10}$$

Differentiating on both sides, we have

$$2(n+1)H_n(x) - 2H_n(x) - 2xH'_n(x) + H'_n(x) = 0. \tag{2.11}$$

Thus, we have

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0. \tag{2.12}$$

From (2.12), we note that $H_n(x)$ is a solution of the following second-order linear differential equation:

$$u'' - 2xu' + 2nu = 0. \tag{2.13}$$

From (1.5), we note that

$$\begin{aligned}\sum_{m=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx-t^2} &= \left(\sum_{l=0}^{\infty}\frac{(2x)^l}{l!}t^l\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^{2k}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

Thus, by (2.14), we get

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}$$
$$= \begin{cases} \sum_{l=0}^{n/2} \frac{(-1)^{n/2-l} n! 2^{2l}}{(n/2-l)!(2l)!} x^{2l}, & \text{if } n \equiv 0 \pmod 2, \\ \sum_{l=0}^{(n-1)/2} \frac{(-1)^{(n-1)/2-l} n! 2^{2l+1}}{((n-1)/2-l)!(2l+1)!} x^{2l+1}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2.15}$$

3. Main Results

By (1.6), we easily get

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = (-1)^n \int_{-\infty}^{\infty} \left(\frac{d^n}{dx^n} e^{-x^2} \right) H_m(x) dx. \tag{3.1}$$

From (3.1), we note that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \delta_{m,n}. \tag{3.2}$$

It is easy to show that

$$\int_{-\infty}^{\infty} e^{-x^2} x^l dx = \begin{cases} 0 & \text{if } l \equiv 1 \pmod 2, \\ \frac{l! \sqrt{\pi}}{2^l (l/2)!} & \text{if } l \equiv 0 \pmod 2, \end{cases} \tag{3.3}$$

where $l \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. By (3.3), we get

$$\int_{-\infty}^{\infty} \left(\frac{d^n e^{-x^2}}{dx^n} \right) x^m dx = \begin{cases} 0 & \text{if } n > m \text{ or } n \leq m \text{ with } n - m \equiv 1 \pmod 2, \\ \frac{m! (-1)^n \sqrt{\pi}}{2^{m-n}((m-n)/2)!} & \text{if } n \leq m \text{ with } n - m \equiv 0 \pmod 2. \end{cases} \tag{3.4}$$

From (3.2), we note that $H_0(x), H_1(x), \ldots, H_n(x)$ are orthogonal basis for the space $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ with respect to the inner product

$$\langle p(x), q(x) \rangle = \int_{-\infty}^{\infty} e^{-x^2} p(x) q(x) dx. \tag{3.5}$$

For $p(x) \in \mathbb{P}_n$, the polynomial $p(x)$ is given by

$$p(x) = \sum_{k=0}^{\infty} C_k H_k(x), \tag{3.6}$$

where

$$\begin{aligned} C_k &= \frac{1}{2^k k! \sqrt{\pi}} \langle p(x), H_k(x) \rangle \\ &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) p(x) dx. \end{aligned} \tag{3.7}$$

Let us take $p(x) = x^n \in \mathbb{P}_n$. For $n \equiv 0 \pmod 2$, we compute C_k in (3.6) as follows

$$\begin{aligned} C_k &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) x^n dx \\ &= \begin{cases} \dfrac{(-1)^k}{2^k k! \sqrt{\pi}} \times \dfrac{(-1)^k n! \sqrt{\pi}}{2^{n-k}((n-k)/2)!} & \text{if } k \equiv 0 \pmod 2, \\ 0 & \text{if } k \equiv 1 \pmod 2. \end{cases} \end{aligned} \tag{3.8}$$

Let $n \equiv 1 \pmod 2$. Then we have

$$\begin{aligned} C_k &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) x^n dx \\ &= \begin{cases} \dfrac{n!}{2^n k!((n-k)/2)!} & \text{if } k \equiv 1 \pmod 2, \\ 0 & \text{if } k \equiv 0 \pmod 2. \end{cases} \end{aligned} \tag{3.9}$$

Therefore, by (3.6), (3.8), and (3.9), we obtain the following proposition.

Proposition 3.1. *One has*

$$\begin{aligned} x^{2n} &= \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{1}{(2k)!(n-k)!} H_{2k}(x), \\ x^{2n+1} &= \frac{(2n+1)!}{2^{2n+1}} \sum_{k=0}^{n} \frac{1}{(2k+1)!(n-k)!} H_{2k+1}(x). \end{aligned} \tag{3.10}$$

Let us take $p(x) = B_n(x)$. From (3.4), $P(x)$ can be rewritten by

$$B_n(x) = \sum_{k=0}^{n} C_k H_k(x), \tag{3.11}$$

where

$$C_k = \frac{(-1)^k}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_n(x) dx. \tag{3.12}$$

By integrating by parts, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_n(x) &= (-n)(-(n-1)) \cdots (-(n-k+1)) \int_{-\infty}^{\infty} e^{-x^2} B_{n-k}(x) dx \\
&= (-1)^k \frac{n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-\infty}^{\infty} e^{-x^2} x^l dx \\
&= \frac{(-1)^k n!}{(n-k)!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{(n-k)! B_{n-k-l}}{l!(n-k-l)!} \times \frac{l!\sqrt{\pi}}{2^l (l/2)!} \\
&= (-1)^k n! \sqrt{\pi} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{B_{n-k-l}}{(n-k-l)! 2^l (l/2)!}.
\end{aligned} \tag{3.13}$$

Thus, from (3.11) and (3.13), we have

$$C_k = \frac{n!}{2^k k!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{B_{n-k-l}}{(n-k-l)! 2^l (l/2)!}. \tag{3.14}$$

Therefore, by (3.11) and (3.14), we obtain the following theorem.

Theorem 3.2. *For $n \in \mathbb{Z}_+$, one has*

$$B_n(x) = n! \sum_{k=0}^{n} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{B_{n-k-l}}{2^{k+l} k! (n-k-l)! (l/2)!} H_k(x). \tag{3.15}$$

Remark 3.3. Let us take $p(x) = E_n(x)$. Then, by the same method, we obtain the following identity:

$$E_n(x) = n! \sum_{k=0}^{n} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{E_{n-k-l}}{2^{k+l} k! (n-k-l)! (l/2)!} H_k(x). \tag{3.16}$$

Now, we consider $p(x) = H_n(x)$. From (3.6), we note that $p(x)$ can be rewritten as

$$H_n(x) = \sum_{k=0}^{n} C_k H_k(x), \tag{3.17}$$

where

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) H_n(x) dx. \tag{3.18}$$

By integrating by parts, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) H_n(x) dx &= (-2n) \cdots (-2(n-k+1)) \int_{-\infty}^{\infty} e^{-x^2} H_{n-k}(x) dx \\
&= \frac{(-1)^k 2^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} 2^l H_{n-k-l} \int_{-\infty}^{\infty} e^{-x^2} x^l dx \\
&= \frac{(-1)^k 2^k n!}{(n-k)!} \sum_{\substack{l=0 \\ l \equiv 0 \ (\mathrm{mod}\ 2)}}^{n-k} \frac{2^l (n-k)!}{l!(n-k-l)!} H_{n-k-l} \frac{l! \sqrt{\pi}}{2^l (l/2)!} \\
&= (-1)^k 2^k n! \sqrt{\pi} \sum_{\substack{l=0 \\ l \equiv 0 \ (\mathrm{mod}\ 2)}}^{n-k} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!}.
\end{aligned} \tag{3.19}$$

From (3.17) and (3.19), we note that

$$\begin{aligned}
C_k &= \left(\frac{(-1)^k}{2^k k! \sqrt{\pi}} \right) \times \left((-1)^k 2^k n! \sqrt{\pi} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!} \right) \\
&= \frac{n!}{k!} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!}.
\end{aligned} \tag{3.20}$$

Therefore, by (3.17) and (3.20), we obtain the following theorem.

Theorem 3.4. *For $n \in \mathbb{Z}_+$, one has*

$$H_n(x) = n! \sum_{k=0}^{n} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{H_{n-k-l}}{k!(n-k-l)!(l/2)!} H_k(x). \tag{3.21}$$

From Theorem 3.4, we note that

$$H_n(x) = n! \sum_{k=0}^{n-1} \sum_{\substack{0 \le l \le n-k \\ l \equiv 0 \ (\mathrm{mod}\ 2)}} \frac{H_{n-k-l}}{k!(n-k-l)!(l/2)!} H_k(x) + \frac{n! H_n(x)}{n!}. \tag{3.22}$$

Thus, we have, for $0 \leq k \leq n-k$,

$$\sum_{\substack{0\leq l\leq n-k \\ l\equiv 0 \pmod 2}} \frac{H_{n-k-l}}{(n-k-l)!(l/2)!} = 0. \tag{3.23}$$

Let $l, k \in \mathbb{Z}_+$ with $k \leq l$. Then we easily see that

$$\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_l(x)dx = (-1)^k l! \sqrt{\pi} \sum_{\substack{0\leq j\leq l-k \\ j\equiv 0 \pmod 2}} \frac{B_{l-k-j}}{(l-k-j)! 2^j (j/2)!}, \tag{3.24}$$

$$\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) E_l(x)dx = (-1)^k l! \sqrt{\pi} \sum_{\substack{0\leq j\leq l-k \\ j\equiv 0 \pmod 2}} \frac{E_{l-k-j}}{(l-k-j)! 2^j (j/2)!}. \tag{3.25}$$

Let us consider the following polynomial of degree n in $\mathbb{P}_n$:

$$p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x). \tag{3.26}$$

From (3.6), we note that $p(x)$ can be rewritten as

$$p(x) = \sum_{k=0}^{n} C_k H_k(x), \tag{3.27}$$

where

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) p(x)dx. \tag{3.28}$$

In [15], it is known that

$$\begin{aligned} p(x) &= \sum_{k=0}^{n} B_k(x) B_{n-k}(x) \\ &= \frac{2}{n+2} \sum_{l=0}^{n-2} \binom{n+2}{l} B_{n-l} B_l(x) + (n+1) B_n(x). \end{aligned} \tag{3.29}$$

From (3.23) and (3.29), we have the following:

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \left\{ \frac{2}{n+2} \sum_{l=0}^{n-2} \binom{n+2}{l} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_l(x)dx + (n+1) \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_n(x)dx \right\}, \tag{3.30}$$

By (3.24) and (3.30), we get

$$
\begin{aligned}
C_n &= \left(\frac{(-1)^n}{2^n n!\sqrt{\pi}}\right) \times (n+1)\int_{-\infty}^{\infty}\left(\frac{d^n e^{-x^2}}{dx^n}\right)B_n(x)dx \\
&= \left(\frac{(-1)^n}{2^n n!\sqrt{\pi}}\right) \times \left((n+1)\frac{(-1)^n n!\sqrt{\pi}B_0}{0!2^0 0!}\right) = \frac{n+1}{2^n}, \\
C_{n-1} &= \left(\frac{(-1)^{n-1}}{2^{n-1}(n-1)!\sqrt{\pi}}\right) \times \left((n+1)\int_{-\infty}^{\infty}\left(\frac{d^{n-1} e^{-x^2}}{dx^{n-1}}\right)B_n(x)dx\right) \\
&= \left(\frac{(-1)^{n-1}}{2^{n-1}(n-1)!\sqrt{\pi}}\right) \times \left((n+1)(-1)^{n-1}n!\sqrt{\pi}\sum_{\substack{j=0 \\ j\equiv 0 \ (\mathrm{mod}\ 2)}}^{1}\frac{B_{1-j}}{(1-j)!2^j(j/2)!}\right) \\
&= \left(\frac{(-1)^{n-1}}{2^{n-1}(n-1)!\sqrt{\pi}}\right) \times \left((n+1)(-1)^{n-1}n!\sqrt{\pi}B_1\right) = \frac{-n(n+1)}{2^n}.
\end{aligned}
\tag{3.31}
$$

For $0 \le k \le n-2$, we have

$$
\begin{aligned}
&C_k \\
&= \frac{(-1)^k}{2^k k!\sqrt{\pi}}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\binom{n+2}{l}B_{n-l}\int_{-\infty}^{\infty}\left(\frac{d^k e^{-x^2}}{dx^k}\right)B_l(x)dx + (n+1)\int_{-\infty}^{\infty}\left(\frac{d^k e^{-x^2}}{dx^k}\right)B_n(x)dx\right\} \\
&= \frac{(-1)^k}{2^k k!\sqrt{\pi}}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\binom{n+2}{l}B_{n-l}(-1)^k l!\sqrt{\pi} \times \sum_{\substack{0\le j\le l-k \\ j\equiv 0 \ (\mathrm{mod}\ 2)}}\frac{B_{l-k-j}}{(l-k-j)!2^j(j/2)!}\right. \\
&\qquad\qquad \left. +(n+1)(-1)^k n!\sqrt{\pi}\sum_{\substack{0\le j\le n-k \\ j\equiv 0 \ (\mathrm{mod}\ 2)}}\frac{B_{n-k-j}}{(n-k-j)!2^j(j/2)!}\right\} \\
&= \frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{\substack{0\le j\le l-k \\ j\equiv 0 \ (\mathrm{mod}\ 2)}}\binom{n+2}{l}\frac{B_{n-l}B_{l-k-j}l!}{2^{k+j}k!(l-k-j)!(j/2)!} \\
&\quad + (n+1)!\sum_{\substack{0\le j\le n-k \\ j\equiv 0 \ (\mathrm{mod}\ 2)}}\frac{B_{n-k-j}}{k!(n-k-j)!(j/2)!2^{k+j}}.
\end{aligned}
\tag{3.32}
$$

Therefore, by (3.27) and (3.32), we obtain the following theorem.

Theorem 3.5. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}\sum_{k=0}^{n} B_k(x)B_{n-k}(x) &= \sum_{k=0}^{n-2}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{\substack{0\le j\le n-k\\ j\equiv 0\ (\mathrm{mod}\ 2)}}\binom{n+2}{l}\frac{l!B_{n-l}B_{l-k-j}}{2^{k+j}k!(l-k-j)!(j/2)!}\right.\\ &\left.\quad+(n+1)!\sum_{\substack{0\le j\le n-k\\ j\equiv 0\ (\mathrm{mod}\ 2)}}\frac{B_{n-k-j}}{2^{k+j}k!(n-k-j)!(j/2)!}\right\}H_k(x)\\ &\quad-\frac{n(n+1)}{2^n}H_{n-1}(x)+\frac{n+1}{2^n}H_n(x).\end{aligned} \tag{3.33}$$

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