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Research Article

Bifurcation of Limit Cycles of a Class of Piecewise Linear Differential Systems in $\mathfrak{R}^4$ with Three Zones

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We study the bifurcation of limit cycles from periodic orbits of a four-dimensional system when the perturbation is piecewise linear with two switching boundaries. Our main result shows that when the parameter is sufficiently small at most, six limit cycles can bifurcate from periodic orbits in a class of asymmetric piecewise linear perturbed systems, and, at most, three limit cycles can bifurcate from periodic orbits in another class of asymmetric piecewise linear perturbed systems. Moreover, there are perturbed systems having six limit cycles. The main technique is the averaging method.

1. Introduction and Statement of the Main Result

Piecewise linear systems are used extensively to model many physical phenomena, such as switching circuits in power electronics [1, 2] and impact and dry frictions in mechanical systems [3]. These systems exhibit not only standard bifurcations but also complicated dynamical phenomena not existing in smooth systems. The study and classification of various kinds of bifurcation phenomena for piecewise linear systems have attracted great attentions since the last century, see, for example, [4, 5] and the references therein.

In recent years, many papers studied the bifurcation of limit cycles and the number and distribution of these limit cycles. Most of them studied the planar piecewise linear system, see for example, [6–9] and the references quoted there. There are also some papers which studied bifurcation of limit cycles of 3D piecewise linear systems [10, 11]. For high-dimensional cases, there are a few papers [12–16]. Especially in [12] the authors studied the bifurcation of limit cycles of a class of piecewise linear systems in $\mathfrak{R}^4$. They showed that three is an upper bound for the number of limit cycles that bifurcate from periodic orbits.

In this paper, we study the limit cycles bifurcated from periodic orbits of a linear differential system in $\mathfrak{R}^4$ when the perturbation is piecewise linear with two switching boundaries. We consider two classes of asymmetric perturbation. With the first class of asymmetric perturbation, six is the upper bound for the number of limit cycles bifurcated from periodic orbits, and there are perturbed systems having six limit cycles. With the second class of asymmetric perturbation, three is the upper bound for the number of limit cycles bifurcated from periodic orbits, which generalizes the result of the paper [12].

More precisely, we study the maximum number of limit cycles of the 4-dimensional continuous piecewise linear vector fields with three zones of the form

$$\dot{x} = A_0 x + \varepsilon F(x), \tag{1}$$

for $\varepsilon \neq 0$ sufficiently small real parameter, where

$$A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{2}$$

and $F : \mathfrak{R}^4 \to \mathfrak{R}^4$ is given by

$$F(x) = Ax + \varphi\left(k^T x\right) b, \tag{3}$$

with $A \in M_4(\mathfrak{R})$, $k, b \in \mathfrak{R}^4 \setminus \{0\}$, and $\varphi : \mathfrak{R} \to \mathfrak{R}$ the piecewise linear function if $m_1 < 0 < m_2$,

$$\varphi(x) = \begin{cases} hm_1, & \text{for } x \in (-\infty, m_1), \\ hx, & \text{for } x \in (m_1, m_2), \\ hm_2, & \text{for } x \in (m_2, +\infty); \end{cases} \quad (4)$$

if $m_1 < m_2 < 0$,

$$\varphi(x) = \begin{cases} hm_1 - hm_2, & \text{for } x \in (-\infty, m_1), \\ hx - hm_2, & \text{for } x \in (m_1, m_2), \\ 0, & \text{for } x \in (m_2, +\infty); \end{cases} \quad (5)$$

if $0 < m_1 < m_2$,

$$\varphi(x) = \begin{cases} 0, & \text{for } x \in (-\infty, m_1), \\ hx - hm_1, & \text{for } x \in (m_1, m_2), \\ hm_2 - hm_1, & \text{for } x \in (m_2, +\infty), \end{cases} \quad (6)$$

where $h \in \mathfrak{R} \setminus \{0\}$. The independent variable is denoted by t; vectors of $\mathfrak{R}^4$ are column vectors, and k^T denotes a transposed vector.

For $\varepsilon = 0$, system (1) becomes

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1, \qquad \dot{x}_3 = -x_4, \qquad \dot{x}_4 = x_3. \quad (7)$$

Our main results are the following.

Theorem 1. *If $m_1 m_2 > 0$, six is the upper bound for the number of limit cycles of system (1) which bifurcate from the periodic orbits of system (7) with ε sufficiently small. Moreover, there are systems of form (1) having six limit cycles.*

Theorem 2. *If $m_1 m_2 < 0$, three is the upper bound for the number of limit cycles of system (1) which bifurcate from the periodic orbits of system (7) with ε sufficiently small. Moreover, there are systems of form (1) having three limit cycles.*

It is worth to note that Theorem 2 generalizes the result of paper [12]. The method for computing the number of limit cycles bifurcated from periodic orbits is the averaging method, which is obtained by Buică and Llibre [17]. By means of the result of paper [18], we can study the stability of the limit cycles of Theorem 1; for more details see Remark 10.

Theorems 1 and 2 will be proved in Section 3. In Section 2, we review the results from the averaging theory necessary for proving these two theorems. Further discussions on the number of limit cycles of the perturbed system are present in Section 4. There is a conclusion given in the last section.

2. First-Order Averaging Method

The aim of this section is to review the first-order averaging method which is obtained by Buică and Llibre [17]. The advantage of this method is that the smoothness assumptions for the vector field of the differential system are minimal.

Theorem 3 (see [17]). *Consider the following differential system:*

$$\dot{x}(t) = \varepsilon H(t, \varepsilon) + \varepsilon^2 R(t, x, \varepsilon), \quad (8)$$

where $H : \mathfrak{R} \times D \to \mathfrak{R}^n$, $R : \mathfrak{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathfrak{R}^n$ are continuous functions. T-periodic in the first variable, and D is an open subset of $\mathfrak{R}^n$. We define $h : D \to \mathfrak{R}^n$ as

$$h(z) = \int_0^T H(s, z)\, ds, \quad (9)$$

and assume that

(i) *H and R are locally Lipschitz with respect to x;*

(ii) *for $a \in D$ with $h(a) = 0$, there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(h, V, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists an isolated T-periodic solution $\phi(\cdot, \varepsilon)$ of system (8) such that $\phi(\cdot, \varepsilon) \to a$ as $\varepsilon \to 0$.

We remind here that $d_B(h, V, a)$ denotes the Brouwer degree of the function h with respect to the set V and the point a, as is defined in [19]. The following fact is useful for the proof of Theorems 1 and 2.

Fact 1. Let $h : D \to \mathfrak{R}^n$ be a C^1 function, with $h(a) = 0$, where D is an open subset of $\mathfrak{R}^n$ and $a \in D$. Whenever a is a simple zero of h (i.e., $J(a) \neq 0$), there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$. Then, $d_B(h, V, 0) \in \{-1, 1\}$.

3. Proof of Main Theorems

The proof of Theorems 1 and 2 is based on the first-order averaging method presented in the previous section. In order to apply this method, we will first reduce the four parameters of the vector k in the definition of the function $F(x)$ to one, and then we will change the variables in order to transform the system into the standard form for the averaging method. After that, we will calculate the number of its isolated zeros.

Lemma 4. *By a linear change of variables, system (1) can be transformed into the system*

$$\dot{x} = A_0 x + \varepsilon \overline{A} x + \varepsilon \varphi(x_1) \overline{b}, \quad (10)$$

where $\overline{A} \in M_4(\mathfrak{R})$ is an arbitrary matrix and $\overline{b} = (\overline{b}_1, \overline{b}_2, 0, 0)^T$ or $\overline{b} = e_3$.

Proof. A linear change of variables $x = Py$, with P invertible, transforms system (1) into

$$\dot{y} = P^{-1} A_0 P y + \varepsilon P^{-1} A P y + \varepsilon \varphi\left(k^T P y\right) P^{-1} b, \quad (11)$$

where $b = (b_1, b_2, b_3, b_4)^T$, $k = (k_1, k_2, k_3, k_4)^T$.

We have to find P invertible which satisfies

$$\begin{gathered} P^{-1} A_0 P = A_0, \\ k^T P = e_1^T. \end{gathered} \quad (12)$$

It is easy to obtain that P^{-1} has the following form:

$$P^{-1} = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ -k_2 & k_1 & -k_4 & k_3 \\ p_{31} & p_{32} & p_{33} & p_{34} \\ -p_{32} & p_{31} & -p_{34} & p_{33} \end{pmatrix}. \quad (13)$$

Thus, we have

$$P^{-1} b = \bar{b}, \quad (14)$$

where

$$\bar{b}_1 = \sum_{i=1}^{4} k_i b_i, \qquad \bar{b}_2 = -k_2 b_1 + k_1 b_2 - k_4 b_3 + k_3 b_4,$$
$$\bar{b}_3 = \sum_{i=1}^{4} p_{3i} b_i, \qquad \bar{b}_4 = -p_{32} b_1 + p_{31} b_2 - p_{34} b_3 + p_{33} b_4. \quad (15)$$

If $\bar{b}_1^2 + \bar{b}_2^2 \neq 0$, it is easy to find P^{-1} invertible with p_{31}, p_{32}, p_{33}, p_{34} satisfying

$$\bar{b}_3 = \sum_{i=1}^{4} p_{3i} b_i = 0,$$
$$\bar{b}_4 = -p_{32} b_1 + p_{31} b_2 - p_{34} b_3 + p_{33} b_4 = 0. \quad (16)$$

If $\bar{b}_1^2 + \bar{b}_2^2 = 0$, it is easy to find P^{-1} invertible with p_{31}, p_{32}, p_{33}, p_{34} satisfying

$$\bar{b}_3 = \sum_{i=1}^{4} p_{3i} b_i = 1,$$
$$\bar{b}_4 = -p_{32} b_1 + p_{31} b_2 - p_{34} b_3 + p_{33} b_4 = 0. \quad (17)$$

Changing variables y to x with $x = y$, then we obtain system (10). □

The standard form of the averaging method is obtained by changing variables (x_1, x_2, x_3, x_4) to (r, θ, ρ, s) with

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta,$$
$$x_3 = \rho\cos(\theta + s), \qquad x_4 = \rho\sin(\theta + s). \quad (18)$$

Thus, system (10) is transformed into the following system:

$$\frac{dr}{d\theta} = \varepsilon H_1(\theta, r, \rho, s) + \varepsilon^2 O(1),$$
$$\frac{d\rho}{d\theta} = \varepsilon H_2(\theta, r, \rho, s) + \varepsilon^2 O(1), \quad (19)$$
$$\frac{ds}{d\theta} = \varepsilon H_3(\theta, r, \rho, s) + \varepsilon^2 O(1),$$

where H_1, H_2, and H_3 are given by

$$H_1 = \cos\theta F_1 + \sin\theta F_2,$$
$$H_2 = \cos(\theta + s) F_3 + \sin(\theta + s) F_4,$$
$$H_3 = \frac{1}{r}\sin\theta F_1 - \frac{1}{r}\cos\theta F_2 - \frac{1}{\rho}\sin(\theta + s) F_3 + \frac{1}{\rho}\cos(\theta + s) F_4 \quad (20)$$

and for every $i = 1, 2, \ldots, 4$,

$$F_i = a_{i1} r\cos\theta + a_{i2} r\sin\theta + a_{i3}\rho\cos(\theta + s) + a_{i4}\rho\sin(\theta + s) + \varphi(r\cos\theta)\bar{b}_i, \quad (21)$$

where a_{ij} are elements of the matrix $\overline{A}$ of Lemma 4.

We take ε_0 sufficiently small, m arbitrarily large and

$$D_m = \left\{(r, \rho, s) \mid (r, \rho, s) \in \left(\frac{1}{m}, m\right)^2 \times S\right\}. \quad (22)$$

Then, the vector of system (19) is well defined and continuous on $S \times D_m \times (-\varepsilon_0, \varepsilon_0)$. Moreover, the system is 2π-periodic with respect to variable θ and locally Lipschitz with respect to variables (r, ρ, s). Our next step is to find the corresponding function $h : D \to \mathfrak{R}^3$, $h = (h_1, h_2, h_3)$, where

$$h_i(r, \rho, s) = \int_0^{2\pi} H_i(r, \theta, \rho, s)\, d\theta, \quad (23)$$

for $i = 1, 2, 3$.

In order to calculate the exact expression of h, we denote

$$I_1(r) = \int_0^{2\pi} \varphi(r\cos\theta)\cos\theta d\theta, \quad (24)$$

$$I_2(r) = \int_0^{2\pi} \varphi(r\cos\theta)\sin\theta d\theta, \quad (25)$$

for each $r > 0$, where φ is the piecewise linear function given by (4)–(6). Without loss of generality, we assume that the slope h of φ is positive.

Lemma 5. *The integrals I_1 and I_2 given by* (24)-(25)*, respectively, have the following expressions:*

$$I_2(r) = 0, \quad \forall r > 0,\ m_1, m_2, \quad (26)$$

and

(1) *if* $0 < m_1 < m_2$,

$$I_1(r) = \begin{cases} 0, & if\ \ 0 < r \le m_1, \\ J(r, m_1), & if\ \ m_1 < r < m_2, \\ K(r, m_1, m_2), & if\ \ r \ge m_2; \end{cases} \quad (27)$$

(2) *if* $m_1 < m_2 < 0$,

$$I_1(r) = \begin{cases} 0, & if\ \ 0 < r \le |m_2|, \\ -J(r, m_2), & if\ \ |m_2| < r < |m_1|, \\ K(r, m_1, m_2), & if\ \ r \ge |m_1|; \end{cases} \quad (28)$$

(3) *if* $m_1 < 0 < m_2$ *and* $|m_1| < |m_2|$,

$$I_1(r) = \begin{cases} \pi hr, & \text{if } 0 < r \le |m_1|, \\ \pi hr + J(r, m_1), & \text{if } |m_1| < r < m_2, \\ \pi hr + K(r, m_1, m_2), & \text{if } r \ge m_2; \end{cases} \quad (29)$$

(4) *if* $m_1 < 0 < m_2$ *and* $|m_1| > |m_2|$,

$$I_1(r) = \begin{cases} \pi hr, & \text{if } 0 < r \le m_2, \\ \pi hr - J(r, m_2), & \text{if } m_2 < r < |m_1|, \\ \pi hr + K(r, m_1, m_2), & \text{if } r \ge |m_1|; \end{cases} \quad (30)$$

(5) *if* $m_1 < 0 < m_2$ *and* $|m_1| = |m_2| = m$,

$$I_1(r) = \begin{cases} \pi hr, & \text{if } 0 < r \le m, \\ \pi hr - 2J(r, m), & \text{if } r > m, \end{cases} \quad (31)$$

where

$$J(r, m_i) = hr\left(\arctan\frac{\sqrt{r^2 - m_i^2}}{m_i} - \frac{m_i\sqrt{r^2 - m_i^2}}{r^2}\right), \quad (32)$$

for $i = 1, 2$, *and*

$$K(r, m_1, m_2) = hr\left(\arctan\frac{\sqrt{r^2 - m_1^2}}{m_1} - \arctan\frac{\sqrt{r^2 - m_2^2}}{m_2} - \frac{m_1\sqrt{r^2 - m_1^2}}{r^2} + \frac{m_2\sqrt{r^2 - m_2^2}}{r^2}\right). \quad (33)$$

The proof of this lemma is given in the appendix.

Remark 6. If $m_1 < 0 < m_2$ and $|m_1| = |m_2|$, system (1) can be transformed into the system which is studied in the paper [12].

Lemma 7. *If* $m_1 m_2 > 0$, *one defines* $\overline{m} = \max(|m_1|, |m_2|)$, $\underline{m} = \min(|m_1|, |m_2|)$ *and consider the equation* $I_1(r) = cr$, $r > 0$ *with* I_1 *given by* (24), *and c is a real parameter. Then,*

(1) *if* $c < 0$ *or* $c > h(\arctan(\overline{m}/\underline{m}) - \arctan(\underline{m}/\overline{m}))$, *the equation has no solutions;*

(2) *if* $c = 0$, *then the interval* $(0, \underline{m}]$ *is a continuum of solutions;*

(3) *if* $c = h(\arctan(\overline{m}/\underline{m}) - \arctan(\underline{m}/\overline{m}))$, *there is an unique solution* $r^* = \sqrt{\overline{m}^2 + \underline{m}^2}$;

(4) *if* $c \in (0, h(\arctan(\overline{m}/\underline{m}) - \arctan(\underline{m}/\overline{m})))$, *there are two solutions* $r_1^* < \sqrt{\overline{m}^2 + \underline{m}^2}$ *and* $r_2^* > \sqrt{\overline{m}^2 + \underline{m}^2}$.

Proof. If $0 < m_1 < m_2$, we have $\overline{m} = m_2$ and $\underline{m} = m_1$. It is easy to see that all $r \in (0, \underline{m}]$ are a solution if $c = 0$. If $c \ne 0$ changing the variable $u = \sqrt{r^2 - m_1^2}/m_1$ and defining $g(u) = I_1(r)/hr$, we obtain the equivalent equation

$$g(u) = \begin{cases} \arctan u - \dfrac{u}{1 + u^2}, & \text{if } u \in \left(0, \dfrac{\sqrt{m_2^2 - m_1^2}}{m_1}\right], \\ \arctan u - \dfrac{u}{1 + u^2} - \arctan\dfrac{\sqrt{(1 + u^2)m_1^2 - m_2^2}}{m_2} + \dfrac{m_2\sqrt{(1 + u^2)m_1^2 - m_2^2}}{m_1^2(1 + u^2)}, & \text{if } u \in \left(\dfrac{\sqrt{m_2^2 - m_1^2}}{m_1}, +\infty\right) \end{cases} \quad (34)$$

with simple computation; we find that the function g is strictly monotonically increasing of variable u when $u \in (0, m_2/m_1)$ and strictly monotonically decreasing when $u \in (m_2/m_1, +\infty)$. The function g gets to the maximal value $g_{\max} = \arctan(m_2/m_1) - \arctan(m_1/m_2)$ when $u = m_2/m_1$. Also we have $f(u) \to 0$ as $u \to 0$ and $u \to +\infty$. The proof is similar if $m_1 < m_2 < 0$. □

Lemma 8. *If* $m_1 m_2 < 0$, *one defines* $\underline{m} = \min(|m_1|, |m_2|)$ *and consider the equation* $I_1(r) = cr$, $r > 0$ *with* I_1 *given by* (24), *and c is a real parameter. Then,*

(1) *if* $c < 0$ *or* $c > \pi h$, *the equation has no solutions;*

(2) *if* $c = \pi h$, *then the interval* $(0, \underline{m}]$ *is a continuum of solutions;*

(3) *if* $c \in (0, \pi h)$, *there is an unique solution* r^*.

Proof. We only consider the case when $|m_1| < |m_2|$: the proof is similar when $|m_1| > |m_2|$ and $|m_1| = |m_2|$. It is easy to see that all $r \in (0, |m_1|]$ are a solution if $c = \pi h$. If $c \ne 0$ changing the variable $u = \sqrt{r^2 - m_1^2}/m_1$ and defining $g(u) = I_1(r)/hr$, we obtain the equivalent equation

$$g(u) = \begin{cases} \pi + \arctan u - \dfrac{u}{1 + u^2}, & \text{if } u \in \left[\dfrac{m_2^2 - m_1^2}{m_1}, 0\right), \\ \pi + \arctan u - \dfrac{u}{1 + u^2} - \arctan\dfrac{\sqrt{(1 + u^2)m_1^2 - m_2^2}}{m_2} + \dfrac{m_2\sqrt{(1 + u^2)m_1^2 - m_2^2}}{m_1^2(1 + u^2)}, & \text{if } u \in \left(-\infty, \dfrac{\sqrt{m_2^2 - m_1^2}}{m_1}\right). \end{cases} \quad (35)$$

With simple computation, we find that the function g is strictly monotonically increasing of variable u. It is easy to know $f(u) \to \pi$ as $u \to 0$ and $f(u) \to 0$ as $u \to -\infty$. □

With Lemma 5, we obtain the expressions for the components of function h,

$$h_1(r, \rho, s) = c_1 r + c_2 \rho \cos s + c_3 \rho \sin s + \overline{b}_1 I_1(r),$$

$$h_2(r,\rho,s) = c_7\rho + c_5 r\cos s + c_6 r\sin s + \bar{b}_3\cos s I_1(r),$$
$$h_3(r,\rho,s) = c_4 + (c_3\cos s - c_2\sin s)\frac{\rho}{r} + (c_6\cos s - c_5\sin s)\frac{r}{\rho} - \bar{b}_3\sin s\frac{I_1(r)}{\rho} - \bar{b}_2\frac{I_1(r)}{r}, \quad (36)$$

where c_i are constants that depend linearly on a_{ij}

$$\begin{aligned} c_1 &= (a_{11}+a_{22})\pi, \quad c_2 = (a_{13}+a_{24})\pi, \\ c_3 &= (a_{14}-a_{23})\pi, \quad c_4 = (a_{43}+a_{12}-a_{34}-a_{21})\pi, \\ c_5 &= (a_{31}+a_{42})\pi, \quad c_6 = (a_{41}-a_{32})\pi, \\ c_7 &= (a_{33}+a_{44})\pi. \end{aligned} \quad (37)$$

According to Theorem 3 and Fact 1, for each simple zero (r^*,ρ^*,s^*) of (36) there is an isolated 2π-periodic solution $\phi(\cdot,\varepsilon)$ of system (19) with $|\varepsilon|\neq 0$ sufficiently small such that $\phi(\cdot,\varepsilon)\to(r^*,\rho^*,s^*)$ as $\varepsilon\to 0$. Any isolated 2π-periodic solution of system (19) with $|\varepsilon|\neq 0$ sufficiently small corresponds to a limit cycle of system (10). Thus, the most important task is to calculate the number of the simple zeros of function h. We solve the two first equations of (36), then, we get

$$I_1(r) = \frac{k_2(s)}{d(s)}\rho, \qquad r = \frac{k_1(s)}{d(s)}\rho, \quad (38)$$

where

$$\begin{aligned} d(s) &= (\bar{b}_1 c_5 - \bar{b}_3 c_1)\cos s + \bar{b}_1 c_6\sin s, \\ k_1(s) &= \bar{b}_3\cos s(c_2\cos s + c_3\sin s) + \bar{b}_1 c_7, \\ k_2(s) &= c_1c_7 - c_2c_5\cos^2 s - c_3c_6\sin^2 s - (c_2c_6 + c_3c_5)\sin s\cos s. \end{aligned} \quad (39)$$

Substituting (38) into the third equation, we obtain

$$h_3(r,\rho,s) = \frac{f(s)}{d(s)k_1(s)} = 0, \quad (40)$$

where

$$\begin{aligned} f(s) &= c_4 d(s)k_1(s) + (c_3\cos - c_2\sin s)d^2(s) \\ &\quad + (c_6\cos s - c_5\sin s)k_1^2(s) \\ &\quad - \bar{b}_3\sin s k_1(s)k_2(s) - \bar{b}_2 d(s)k_2(s). \end{aligned} \quad (41)$$

It is necessary to study the zeros of f instead of the zeros of h.

Lemma 9. *The function $f:[0,2\pi)\to\mathfrak{R}$ given by formula (41) can have at most six isolated zeros, and they appear in pairs $\{s^*, s^*+\pi(\text{mod }2\pi)\}$.*

Proof. Substituting $\cos s = x$ and $\sin s = \sqrt{1-x^2}$ in $f(s)=0$ we get

$$D_1x + D_3x^3 + (D_0 + D_2x^2)\sqrt{1-x^2} = 0, \quad (42)$$

where

$$\begin{aligned} D_0 &= \bar{b}_1^2(c_2c_6^2 - c_4c_6c_7 + c_5c_7^2) + \bar{b}_1\bar{b}_2(c_3c_6^2 - c_1c_6c_7), \\ D_1 &= \bar{b}_1^2(2c_2c_5c_6 - c_3c_6^2 - c_4c_5c_7 - c_6c_7^2) + \bar{b}_3^2(c_1c_3c_7 - c_6c_3^2) \\ &\quad + \bar{b}_1\bar{b}_2(2c_2c_5c_6 + c_3c_5^2 - c_1c_5c_7), \\ D_2 &= \bar{b}_1^2(c_2c_5^2 - c_2c_6^2 - 2c_3c_5c_6) \\ &\quad + \bar{b}_3^2(c_1c_2c_7 + c_2c_1^2 - c_1c_3c_4 - 2c_2c_3c_6) \\ &\quad + \bar{b}_1\bar{b}_2(2c_2c_5c_6 + c_3c_5^2 - c_3c_6^2), \\ D_3 &= \bar{b}_1^2(c_3c_6^2 - c_3c_5^2 - 2c_2c_5c_6) \\ &\quad + \bar{b}_3^2(c_3^2c_6 - c_3c_1^2 - c_1c_2c_4 - c_1c_3c_7 - c_6c_2^2) \\ &\quad + \bar{b}_1\bar{b}_2(c_2c_5^2 - 2c_3c_5c_6 - c_2c_6^2). \end{aligned} \quad (43)$$

When we consider the case $\cos s = x$ and $\sin s = -\sqrt{1-x^2}$, $f(s)=0$ becomes

$$D_1x + D_3x^3 - (D_0 + D_2x^2)\sqrt{1-x^2} = 0. \quad (44)$$

It follows that we have to find solutions of (42) or (44) in the interval $[-1,1]$. This is equivalent to

$$D_1x + D_3x^3 - (D_0 + D_2x^2)^2(1-x^2) = 0 \quad (45)$$

which is the polynomial equation

$$(D_3^2 + D_2^2)x^6 + (2D_1D_3 + 2D_0D_2 - D_2^2)x^4 + (D_1^2 + D_0^2 - 2D_0D_2)x^2 - D_0^2 = 0. \quad (46)$$

This equation can have at most six roots in the interval $[-1,1]$. Then, $f(s)=0$ has at most six solutions $s\in[0,2\pi)$. Since $f(s+\pi) = -f(s)$ for all $s\in[0,2\pi)$, it is clear that if s^* is a zero of f then $s^*+\pi(\text{mod }2\pi)$ is also a zero. □

The functions $f(s)$, $d(s)$, $k_1(s)$, and $k_2(s)$ have the properties $f(s+\pi) = -f(s)$, $d(s+\pi) = -d(s)$, $k_1(s+\pi) = k_1(s)$, and $k_2(s+\pi) = k_2(s)$. So, we have

$$\frac{k_1(s)}{d(s)} > 0 \Longrightarrow \frac{k_1(s+\pi)}{d(s+\pi)} < 0. \quad (47)$$

Thus, the equation $h_3 = 0$ at most three zeros that satisfy $k_1(s)/d(s) > 0$. With Lemma 7 for a fixed s^*, we at most find two isolated value of r^* from $I_1(r)/r = k_2(s^*)/d(s^*)$. With Lemma 8 for a fixed s^*, we at most find one isolated value of r^* from $I_1(r)/r = k_2(s^*)/d(s^*)$. For fixed s^* and fixed r^*, $\rho/r = k_1(s^*)/d(s^*)$ gives at most one isolated value for ρ^*. Thus, we conclude that if $m_1m_2 > 0$ the maximum number of limit cycles for system (1) is six, and if $m_1m_2 < 0$ the maximum number of limit cycles for system (1) is three.

Remark 10. Using the main result of [18], the stability of the limit cycles associated with the solution (r^*, ρ^*, s^*) is given by the eigenvalues of the matrix

$$\left.\frac{\partial(h_1, h_2, h_3)}{\partial(r, \rho, s)}\right|_{(r,\rho,s)=(r^*,\rho^*,s^*)}. \tag{48}$$

In order to show that there exist examples with exactly six limit cycles, we consider the following values of the coefficients:

$$\begin{gathered} c_1 = -1, \qquad c_2 = \frac{-4\sqrt{3}}{3}, \qquad c_3 = 0, \\ c_4 = \frac{-4\sqrt{3}}{3}, \qquad c_5 = 0, \quad c_6 = 1, \qquad c_7 = -1, \\ \bar{b}_1 = 1, \qquad \bar{b}_2 = \bar{b}_3 = \bar{b}_4 = 0, \qquad h = 6, \\ m_1 = 1, \qquad m_2 = \sqrt{3}. \end{gathered} \tag{49}$$

More precisely, the system has the following form:

$$\dot{x} = A_0 x + \varepsilon A x + \varepsilon \varphi(x_1)\bar{b}, \tag{50}$$

where

$$\varphi(x_1) = \begin{cases} 0, & \text{for } x_1 \in (-\infty, 1), \\ 6x_1 - 6, & \text{for } x_1 \in (1, \sqrt{3}), \\ 6\sqrt{3} - 6, & \text{for } x_1 \in (\sqrt{3}, +\infty), \end{cases}$$

$$A_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{51}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{14} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & -a_{31} & a_{43} & a_{44} \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $a_{ij} \in \mathfrak{R}$ satisfy

$$\begin{gathered} (a_{11} + a_{22})\pi = -1, \qquad (a_{13} + a_{24})\pi = \frac{-4\sqrt{3}}{3}, \\ (a_{43} + a_{12} - a_{34} - a_{21})\pi = \frac{-4\sqrt{3}}{3}, \\ (a_{41} - a_{32})\pi = 1, \qquad (a_{33} + a_{44})\pi = -1. \end{gathered} \tag{52}$$

It is easy to know $6(\arctan\sqrt{3} - \arctan(\sqrt{3}/3)) = \pi$. Computing the six solutions of $f(s) = 0$, we get $\{\pi/6, \pi/3, \pi/2, 7\pi/6, 4\pi/3, 3\pi/2\}$. The values of $d(s^*)$, $k_1(s^*)$ and $k_2(s^*)$ are given in Table 1.

There are three values of s^* that satisfy $k_1(s^*)/d(s^*) > 0$ and $0 < k_2(s^*)/k_1(s^*) < \pi$. These three solutions are $\{\pi/6, \pi/3, \pi/2\}$.

The six values of solution s^*, r^*, ρ^* and the value of the Jacobian at the solution (r^*, ρ^*, s^*) are given in Table 2.

Table 1: The values of s^*, $d(s^*)$, $k_1(s^*)$, and $k_2(s^*)$.

s^*	$k_1(s^*)$	$d(s^*)$	$k_2(s^*)$
$\pi/6$	1	$1/2$	2
$\pi/3$	1	$\sqrt{3}/2$	2
$\pi/2$	1	1	1
$7\pi/6$	1	$-1/2$	2
$4\pi/3$	1	$-\sqrt{3}/2$	2
$3\pi/2$	1	-1	1

Table 2: The values of solution s^*, r^*, ρ^* and the Jacobian $Jh(r^*, \rho^*, s^*)$.

s^*	r^*	ρ^*	$Jh(r^*, \rho^*, s^*)$
$\pi/6$	1.484	0.742	11.957
$\pi/6$	4.139	2.07	−4.688
$\pi/3$	1.484	1.285	−3.27
$\pi/3$	4.139	3.585	−1.144
$\pi/2$	1.254	1.254	5.732
$\pi/2$	8.672	8.672	−0.964

4. Conclusion

In this paper, we have studied the limit cycles bifurcated from periodic orbits of a linear differential system in $\mathfrak{R}^4$ when the perturbation is piecewise linear with two switching boundaries. We considered two classes of asymmetric perturbation. We have found that the perturbed system could have at most six limit cycles with one class of the asymmetric perturbation and three limit cycles with the other class of asymmetric perturbation, which generalized the result of paper [12].

Appendix

The Proof of Lemma 5

Case 1 $(0 < m_1 < m_2)$. We have $|r\sin\theta| \le m_1$ and $|r\cos\theta| \le m_1$ for all $\theta \in [0, 2\pi)$ if $0 < r \le m_1$. Then, $\varphi(r\cos\theta) = 0$ for every θ. Thus,

$$I_1(r) = I_2(r) = 0. \tag{A.1}$$

We now fix $m_1 < r < m_2$ and consider $\theta_c \in (0, \pi/2)$ which satisfies $r\cos\theta_c = m_1$. Then, we have

$$\begin{aligned} I_1(r) &= \int_0^{\theta_c} (hr\cos\theta - hm_1)\cos\theta d\theta \\ &\quad + \int_{2\pi-\theta_c}^{2\pi} (hr\cos\theta - hm_1)\cos\theta d\theta \\ &= hr\theta_c - hm_1\sin\theta_c, \\ I_2(r) &= \int_0^{\theta_c} (hr\cos\theta - hm_1)\sin\theta d\theta \\ &\quad + \int_{2\pi-\theta_c}^{2\pi} (hr\cos\theta - hm_1)\sin\theta d\theta \\ &= 0. \end{aligned} \tag{A.2}$$

We now fix $r \geq m_2$ and consider $0 < \tilde{\theta}_c < \theta_c < \pi/2$ which satisfies $r\cos\tilde{\theta}_c = m_2$. Then, we have

$$\begin{aligned} I_1(r) &= \int_{-\tilde{\theta}_c}^{\tilde{\theta}_c} (hm_2 - hm_1)\cos\theta d\theta \\ &+ \int_{\tilde{\theta}_c}^{\theta_c} (hr\cos\theta - hm_1)\cos\theta d\theta \\ &+ \int_{2\pi-\theta_c}^{2\pi-\tilde{\theta}_c} (hr\cos\theta - hm_1)\cos\theta d\theta, \\ I_2(r) &= \int_{-\tilde{\theta}_c}^{\tilde{\theta}_c} (hm_2 - hm_1)\sin\theta d\theta \\ &+ \int_{\tilde{\theta}_c}^{\theta_c} (hr\cos\theta - hm_1)\sin\theta d\theta \\ &+ \int_{2\pi-\theta_c}^{2\pi-\tilde{\theta}_c} (hr\cos\theta - hm_1)\sin\theta d\theta. \end{aligned} \tag{A.3}$$

With simple computation, we get

$$\begin{gathered} I_1(r) = hm_2\sin\tilde{\theta}_c - hm_1\sin\theta_c + hr\theta_c - hr\tilde{\theta}_c, \\ I_2(r) = 0, \end{gathered} \tag{A.4}$$

where

$$\sin\theta_c = \frac{\sqrt{r^2 - m_1^2}}{r}, \qquad \sin\tilde{\theta}_c = \frac{\sqrt{r^2 - m_2^2}}{r}, \tag{A.5}$$

$$\theta_c = \arctan\frac{\sqrt{r^2 - m_1^2}}{m_1}, \qquad \tilde{\theta}_c = \arctan\frac{\sqrt{r^2 - m_2^2}}{m_2}. \tag{A.6}$$

Case 2 ($m_1 < m_2 < 0$). We have $|r\sin\theta| \leq |m_2|$ and $|r\cos\theta| \leq |m_2|$ for all $\theta \in [0, 2\pi)$ if $0 < r \leq |m_2|$. Then, $\varphi(r\cos\theta) = 0$ for every θ. Thus,

$$I_1(r) = I_2(r) = 0. \tag{A.7}$$

We fix now $|m_2| < r < |m_1|$ and consider $\theta_c \in (\pi/2, \pi)$ which satisfies $r\cos\theta_c = m_2$. Then, we have

$$\begin{aligned} I_1(r) &= \int_{\theta_c}^{2\pi-\theta_c} (hr\cos\theta - hm_2)\cos\theta d\theta \\ &= \pi hr - hr\theta_c + hm_2\sin\theta_c, \\ I_2(r) &= \int_{\theta_c}^{2\pi-\theta_c} (hr\cos\theta - hm_2)\sin\theta d\theta = 0. \end{aligned} \tag{A.8}$$

We now fix $r \geq |m_1|$ and consider $\tilde{\theta}_c \in (\pi/2, \pi)$ which satisfies $r\cos\tilde{\theta}_c = m_1$. Obviously, $\theta_c < \tilde{\theta}_c$. Then, we have

$$\begin{aligned} I_1(r) &= \int_{\theta_c}^{\tilde{\theta}_c} (hr\cos\theta - hm_2)\cos\theta d\theta \\ &+ \int_{\tilde{\theta}_c}^{2\pi-\tilde{\theta}_c} (hm_1 - hm_2)\cos\theta d\theta \\ &+ \int_{2\pi-\tilde{\theta}_c}^{2\pi-\theta_c} (hr\cos\theta - hm_2)\cos\theta d\theta, \\ I_2(r) &= \int_{\theta_c}^{\tilde{\theta}_c} (hr\cos\theta - hm_2)\sin\theta d\theta \\ &+ \int_{\tilde{\theta}_c}^{2\pi-\tilde{\theta}_c} (hm_1 - hm_2)\sin\theta d\theta \\ &+ \int_{2\pi-\tilde{\theta}_c}^{2\pi-\theta_c} (hr\cos\theta - hm_2)\sin\theta d\theta. \end{aligned} \tag{A.9}$$

With simple computation, we get

$$\begin{gathered} I_1(r) = hm_2\sin\theta_c - hm_1\sin\tilde{\theta}_c - hr\theta_c + hr\tilde{\theta}_c, \\ I_2(r) = 0, \end{gathered} \tag{A.10}$$

where

$$\begin{gathered} \sin\theta_c = \frac{\sqrt{r^2 - m_2^2}}{r}, \qquad \sin\tilde{\theta}_c = \frac{\sqrt{r^2 - m_1^2}}{r}, \\ \theta_c = \pi + \arctan\frac{\sqrt{r^2 - m_2^2}}{m_2}, \qquad \tilde{\theta}_c = \pi + \arctan\frac{\sqrt{r^2 - m_1^2}}{m_1}. \end{gathered} \tag{A.11}$$

Case 3 ($m_1 < 0 < m_2$ and $|m_1| < |m_2|$). We have $|r\sin\theta| \leq |m_1|$ and $|r\cos\theta| \leq |m_1|$ for all $\theta \in [0, 2\pi)$ if $0 < r \leq |m_1|$. Then, $\varphi(r\cos\theta) = hr\cos\theta$ for every θ. Thus,

$$\begin{aligned} I_1(r) &= \int_0^{2\pi} hr\cos^2\theta d\theta = \pi hr, \\ I_2(r) &= \int_0^{2\pi} hr\cos\theta\sin\theta d\theta = 0. \end{aligned} \tag{A.12}$$

We fix now $|m_1| < r < |m_2|$ and consider $\theta_c \in (\pi/2, \pi)$ which satisfies $r\cos\theta_c = m_1$. Then, we have

$$\begin{aligned} I_1(r) &= \int_{-\theta_c}^{\theta_c} hr\cos^2\theta d\theta \\ &+ \int_{\theta_c}^{2\pi-\theta_c} hm_1\cos\theta d\theta = hr\theta_c - hm_1\sin\theta_c, \\ I_2(r) &= \int_{-\theta_c}^{\theta_c} hr\sin\theta\cos\theta d\theta + \int_{\theta_c}^{2\pi-\theta_c} hm_1\sin\theta d\theta = 0. \end{aligned} \tag{A.13}$$

We now fix $r \geq m_2$ and consider $\widetilde{\theta}_c \in (0, \pi/2)$ which satisfies $r\cos\widetilde{\theta}_c = m_2$. Then, we can write

$$I_1(r) = \int_{-\widetilde{\theta}_c}^{\widetilde{\theta}_c} hm_2 \cos\theta d\theta + \int_{\widetilde{\theta}_c}^{\theta_c} hr\cos^2\theta d\theta + \int_{\theta_c}^{2\pi-\theta_c} hm_1 \cos\theta d\theta + \int_{2\pi-\theta_c}^{2\pi-\widetilde{\theta}_c} hr\cos^2\theta d\theta, \quad (A.14)$$

$$I_2(r) = \int_{-\widetilde{\theta}_c}^{\widetilde{\theta}_c} hm_2 \sin\theta d\theta + \int_{\widetilde{\theta}_c}^{\theta_c} hr\cos\theta\sin\theta d\theta + \int_{\theta_c}^{2\pi-\theta_c} hm_1 \sin\theta d\theta + \int_{2\pi-\theta_c}^{2\pi-\widetilde{\theta}_c} hr\cos\theta\sin\theta d\theta. \quad (A.15)$$

With simple computation, we get

$$I_1(r) = hm_2\sin\widetilde{\theta}_c - hm_1\sin\theta_c + hr\theta_c - hr\widetilde{\theta}_c, \quad (A.16)$$

$$I_2(r) = 0, \quad (A.17)$$

where

$$\sin\theta_c = \frac{\sqrt{r^2 - m_1^2}}{r}, \qquad \sin\widetilde{\theta}_c = \frac{\sqrt{r^2 - m_2^2}}{r}, \quad (A.18)$$

$$\theta_c = \pi + \arctan\frac{\sqrt{r^2 - m_1^2}}{m_1}, \qquad \widetilde{\theta}_c = \arctan\frac{\sqrt{r^2 - m_2^2}}{m_2}. \quad (A.19)$$

Case 4 ($m_1 < 0 < m_2$ and $|m_1| > |m_2|$). We have $|r\sin\theta| \leq m_2$ and $|r\cos\theta| \leq m_2$ for all $\theta \in [0, 2\pi)$ if $0 < r \leq m_2$. Then, $\varphi(r\cos\theta) = hr\cos\theta$ for every θ. Thus,

$$I_1(r) = \int_0^{2\pi} hr\cos^2\theta d\theta = \pi hr, \qquad I_2(r) = \int_0^{2\pi} hr\cos\theta\sin\theta d\theta = 0. \quad (A.20)$$

We fix now $m_2 < r < |m_1|$ and consider $\theta_c \in (0, \pi/2)$ which satisfies $r\cos\theta_c = m_2$. Then, we have

$$I_1(r) = \int_{-\theta_c}^{\theta_c} hm_2\cos\theta d\theta + \int_{\theta_c}^{2\pi-\theta_c} hr\cos^2\theta d\theta = \pi hr - hr\theta_c + hm_2\sin\theta_c,$$

$$I_1(r) = \int_{-\theta_c}^{\theta_c} hm_2\sin\theta d\theta + \int_{\theta_c}^{2\pi-\theta_c} hr\cos\theta\sin\theta d\theta = 0. \quad (A.21)$$

We now fix $r \geq |m_1|$ and consider $\widetilde{\theta}_c \in (\pi/2, \pi)$ which satisfies $r\cos\widetilde{\theta}_c = m_1$. Then, we have

$$I_1(r) = \int_{-\theta_c}^{\theta_c} hm_2\cos\theta d\theta + \int_{\theta_c}^{\widetilde{\theta}_c} hr\cos^2\theta d\theta + \int_{\widetilde{\theta}_c}^{2\pi-\widetilde{\theta}_c} hm_1\cos\theta d\theta + \int_{2\pi-\widetilde{\theta}_c}^{2\pi-\theta_c} hr\cos^2\theta d\theta,$$

$$I_2(r) = \int_{-\theta_c}^{\theta_c} hm_2\sin\theta d\theta + \int_{\theta_c}^{\widetilde{\theta}_c} hr\cos\theta\sin\theta d\theta + \int_{\widetilde{\theta}_c}^{2\pi-\widetilde{\theta}_c} hm_1\sin\theta d\theta + \int_{2\pi-\widetilde{\theta}_c}^{2\pi-\theta_c} hr\cos\theta\sin\theta d\theta. \quad (A.22)$$

With simple computation, we get

$$I_1(r) = hm_2\sin\theta_c - hm_1\sin\widetilde{\theta}_c - hr\theta_c + hr\widetilde{\theta}_c, \qquad I_2(r) = 0, \quad (A.23)$$

where

$$\sin\theta_c = \frac{\sqrt{r^2 - m_2^2}}{r}, \qquad \sin\widetilde{\theta}_c = \frac{\sqrt{r^2 - m_1^2}}{r},$$

$$\theta_c = \arctan\frac{\sqrt{r^2 - m_2^2}}{m_2}, \qquad \widetilde{\theta}_c = \pi + \arctan\frac{\sqrt{r^2 - m_1^2}}{m_1}. \quad (A.24)$$

Case 5 ($m_1 < 0 < m_2$ and $|m_1| = |m_2| = m$). We have $|r\sin\theta| \leq m$ and $|r\cos\theta| \leq m$ for all $\theta \in [0, 2\pi)$ if $0 < r \leq m$. Then, $\varphi(r\cos\theta) = hr\cos\theta$ for every θ. Thus,

$$I_1(r) = \int_0^{2\pi} hr\cos^2\theta d\theta = \pi hr, \qquad I_2(r) = \int_0^{2\pi} hr\cos\theta\sin\theta d\theta = 0. \quad (A.25)$$

We fix now $r > m$ and consider $\theta_c \in (0, \pi/2)$ which satisfies $r\cos\theta_c = m$. Then, we have

$$I_1(r) = \int_{-\theta_c}^{\theta_c} hm\cos\theta d\theta + \int_{\theta_c}^{\pi-\theta_c} hr\cos^2\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} hm\cos\theta d\theta + \int_{\pi+\theta_c}^{2\pi-\theta_c} hr\cos^2\theta d\theta,$$

$$I_1(r) = \int_{-\theta_c}^{\theta_c} hm\sin\theta d\theta + \int_{\theta_c}^{\pi-\theta_c} hr\cos\theta\sin\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} hm\sin\theta d\theta + \int_{\pi+\theta_c}^{2\pi-\theta_c} hr\cos\theta\sin\theta d\theta. \quad (A.26)$$

With simple computation, we get

$$I_1(r) = \pi hr + 2hm\sin\theta_c - 2hr\theta_c, \qquad I_2(r) = 0, \quad (A.27)$$

where

$$\sin\theta_c = \frac{\sqrt{r^2 - m^2}}{r}, \qquad \theta_c = \arctan\frac{\sqrt{r^2 - m^2}}{m}. \qquad \text{(A.28)}$$

This completes the proof of the lemma.

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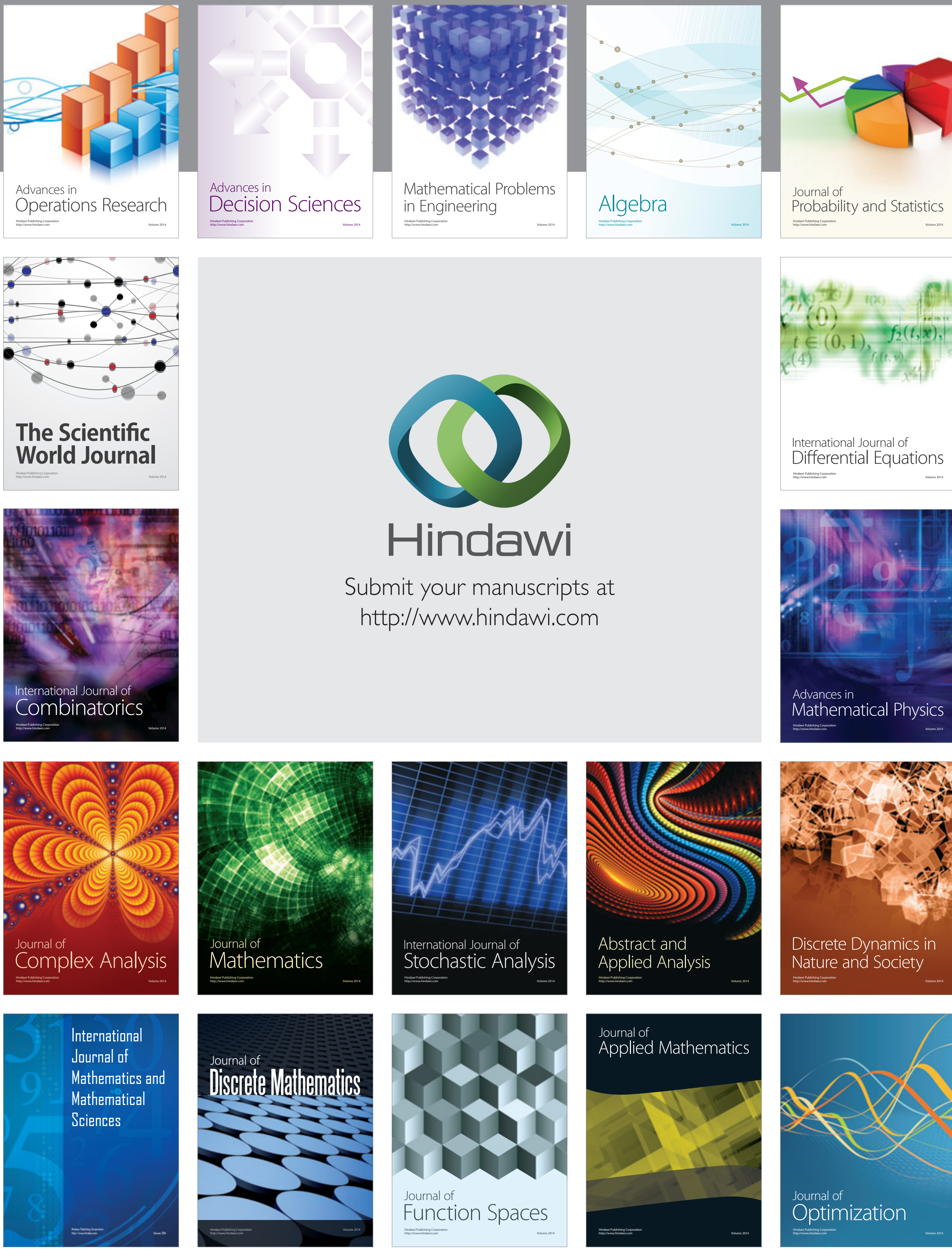
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