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Technological Changes on the Macroeconomic Level—Mathematical Modeling

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We consider two of the technological changes on the macroeconomic level. The first type is due to changes of addresses of mutual deliveries between producers and the second type is due to technological progress.

Keywords: Production factors of current use; Aggregation; Radon transform; Laplace transform

1. MODELS OF TECHNOLOGICAL STRUCTURE

Peculiarities of dynamics of technological structure of economics play a very impotent role in explanation of such phenomena as economic cycles, inflation. Now the V. Leontiev scheme is the main method in description of technological structure and this scheme is widely used in models of inter-industry balance. Primarily in models of inter-industry balance scientists described pure industries in every of which existence of only one technology was allowed. In what follows for description of structural variations in industrial system there was need in considering of pure industries with several technologies, but intensities of technologies use were restricted to production capacities. However, such a generalization turned out to be too narrow for modeling of evolution of technological structure. For example, L.V. Kantorovich and his school considered technologies and capacities differentiated with respect to time of creation. The model of pure industries needs which have, in general, continual set of technologies. Such a model was given in the H.S. Houthakker and the L. Johansen works Houthakker, 1955–1956; Johansen, 1972) and this model was a natural expression of the Leontiev scheme.

1.1. The Houthakker–Johansen Model

We consider a model of pure industry producing homogeneous outputs and using n kinds of production factors of current use (PFCU). Suppose, that in industry there are different technological production processes every of which requires input of n kinds of PFCU in given proportions. Then every technology is given by a vector $x = (x_1, \dots, x_n)$ of coefficients of PFCU input per unit production output. Intensities of technologies use are restricted by available capacities in industry. Suppose that when building up capacities there is selection of technology by which this capacity is functioning. Then at any moment of time, the capacities of industry turn out to be distributed according to technologies. Denote by μ the corresponding non-negative measure in $\mathrm{R}^n_+ = \{x = (x_1, \dots, x_n)x_j \geq 0, \quad j = 1, \dots, n\}$. In order to load capacities completely it is necessary to provide industry with a vector of PFCU $L = (L_1, \dots, L_n)$, where

$$L_i = \int_{R^n_+} x_i \mu(\mathrm{d}x).$$

Let $l = (l_1, \dots, l_n)$ be a vector of PFCU going into disposal of industry. If $l_i < L_i$ then for lack of i-component of PFCU, it is impossible to load all capacities. Denote by $u(x)$ a coefficient of capacity load corresponding to technology x.

We consider a problem of optimal distribution of resources one on purpose of maximization of industry output:

$$\begin{aligned} &\int_{R^n_+} u(x)\mu(\mathrm{d}x) \to \max_{u(x)} \\ &\int_{R^n_+} x u(x)\mu(\mathrm{d}x) \leq 1, \\ &0 \leq u(x) \leq 1. \end{aligned} \tag{1.1}$$

It follows from interpretation of duality theorem which is standard for mathematical economics that optimal mechanisms of resources distribution in problem (1.1) are market mechanisms. The corresponding exact statements are usually called the generalized Neuman–Pirson lemma (see, for example, Shananin, 1984).

The production function $\Psi(l)$ is called function that associates to the vector $l \geq 0$ maximum value of functional in problem (1.1). In such a way defined production function posses the main properties postulated in neo-classical theory.

The question about correspondence between production functions and capacities distribution according to technologies arise in Houthakker (1955–1956) where capacities distributions corresponding to the production functions of Cobb–Douglas type were obtained. In the L. Johansen work (1972) there are another examples and necessary conditions which the production functions corresponding to capacities distribution according to technologies should satisfy. Hildenbrandt (1981) used zonoid theory for characterization of production functions of the same class. However, in such a way a characterization suitable for testing was not obtained.

In Shananin (1984), it is supposed along with production function $\Psi(l)$ to consider profit function $\Pi(p,p_0)$ that is connected with $\Psi(l)$ by Legender transform

$$\begin{aligned} \Pi(p,p_0) &= \sup_{l \geq 0}[p_0\Psi(l) - pl], \\ \Psi(l) &= \frac{1}{p_0}\inf_{p \geq 0}[\Pi(p,p_0) + pl], \end{aligned} \tag{1.2}$$

where $p_0 > 0$ is the price of output; $p = (p_1, \dots, p_n) \geq 0$—prices of PFCU used by industry; $\Pi(p,p_0)$—summary profit of the industry. Profit function and capacities distribution turn out to be connected with integral transform of Radon type:

$$\Pi(p,p_0) = \int_{R^n_+} (p_0 - px)_+ \mu(\mathrm{d}x), \tag{1.3}$$

where

$$(p_0 - px)_+ = \begin{cases} p_0 - px, & \text{if} \quad p_0 - px \geq 0, \\ 0, & \text{if} \quad p_0 - px < 0 \end{cases}$$

From the mathematical point of view the problem is reduced to the inversion problem of Radon transform by incomplete data.

1.2. The Aggregation Procedure

Now we will explain why this problem is interesting for understanding the application frontiers of Leontiev scheme for description of technological structure of economics.

The models of inter-industry balance based on Leontiev scheme find applications for the state regulation of economics. It is very impotent to analyze the main hypothesis of the scheme—existence of capacities distribution according to technologies. It is not obvious that this hypothesis is valid. The fact is that nomenclature of final product produced in industrial countries consists of 10^7-10^9 names, but in macro economic models there are approximately 10^1-10^2 pure industries. Consequently, outputs of pure industries and their production functions are defined with help of aggregation procedures. It is typical for generally accepted aggregation procedures that they are multi-step. That is descriptions obtained as a result of aggregation are subject to aggregation again on the next stage. So natural demand of aggregation procedures is universality principal of description—as a result of aggregation we should obtain the description of the same type that describes the original elements. Test of the main hypothesis is reduced to the question about existence of capacities distribution according to technologies generating aggregated production function.

Now we shall analyze this problem on the example of aggregation of model for inter-industry balance. We consider system of m pure industries, producing final products $X^0 = (X_1^0, \ldots, X_m^0)$. And suppose that demand functions for these final products satisfy integrability conditions and to these functions there corresponds product index $\Psi_0(X^0)$ (positive homogeneous utility function satisfying natural demands) and price index $q_0(q)$ connected with $\Psi_0(X^0)$ by transforms

$$q_0(q) = \inf_{\{X^0 \geq 0 | \Psi_0(X^0)\}} \frac{qX^0}{\Psi_0(X^0)}, \quad \Psi_0(X^0) = \inf_{\{q \geq 0 | q_0(q)\}} \frac{qX^0}{q_0(q)},$$

where $q = (q_1, \ldots, q_m) \geq 0$—a price vector for final product (see Shananin, 1986 and Shananin, Petzov, 1997). We describe every j-industry with help of production function $\Psi_j(X^j, l^j)$ constructed according to model from sections 1.1 or 1.2; $X^j = (X_1^j, \ldots, X_{j-1}^j, X_{j+1}^j, \ldots, X_m^j)$—a vector of costs of output of the rest industries by j-th industry; $l^j = (l_1^j, \ldots, l_n^j)$—a vector of primary resources spent by j-th industry; $s = (s_1, \ldots, s_n) \geq 0$—a price vector for primary resources.

We consider a problem of optimal distribution of primary resources $l = (l_1, \ldots, l_n) \geq 0$ between industries:

$$\Psi_0(X^0) \to \max$$

$$\Psi_j(X^j, l^j) - \sum_{i \neq j} X_j^i - X_j^0 \geq 0,$$

$$(j = 1, \ldots, m),$$

$$\sum_{j=1}^{m} l^j \leq 1, \quad X^0 \geq 0,$$

$$X^1 \geq 0, \ldots, X^m \geq 0, \quad l^1 \geq 0, \ldots, l^m \geq 0. \quad (1.4)$$

Under usual restrictions (productivity and so on) it follows from standard economic interpretation of duality theory that equilibrium market mechanisms (that is market mechanisms under which demand for end products is equal to supply) are optimal mechanisms of resources distribution (see Shananin, 1986).

Function which correspond optimal value of functional in the problem (1.4) with the vector of primary resources $l \geq 0$ is called aggregated production function $\Psi^A(l)$.

In accordance with universality principle it is required to verify whether it is possible to generate according to the model from the section 1.1 aggregated production function $\Psi^A(l)$ from some capacities distribution according to technologies.

To study this question we consider aggregated function of profit $\Pi^A(s,q_0)$. In Shananin (1986) it is proved that summary profit of system of industries is expressed through profit functions of primary industries by formula

$$\Pi^A(s, q_0) = \min_{\{q \geq 0 | q_0(q) \geq q_0\}} \left(\sum_{j=1}^{m} \Pi_j(s, q) \right). \quad (1.5)$$

Besides, functions $\Psi^A(l)$ and $\Pi^A(s,q_0)$ are connected with the Legendre transforms in Eq.

(1.2). So the problem is reduced to the problem of inversion of the profit operator in Eq. (1.3).

1.3. The Inversion of Profit Operator As the Inversion of the Radon Transform by Incomplete Data

We shall consider now the following important questions about transform in Eq. (1.3):

1. Is the measure $\mu(dx)$ uniquely reconstructed through functions $\Pi(p,p_0)$, $(p,p_0) \in R^n_+ \times R^l_+$?
2. What are necessary and sufficient conditions for the given function $\Pi(p,p_0)$ to be the transform in Eq. (1.3) of the nonnegative measure $\mu(dx)$, $x \in R^n_+$?

The following statement give positive answer to the question (1) at least for the case when measure μ with support in R^n_+ has not more than exponential growth at infinity. This means that for some $A > 0$

$$\int e^{-A|x|}\mu(dx) < \infty. \tag{1.6}$$

Theorem. *(Henkin G.M., Shananin A.A. 1990) Let the measures μ_1 and μ_2 with supports in R^n_+ have not more than exponential growth at infinity and satisfy the equality*

$$\int_{R^n_+} (p_0 - px)_+\mu_1(dx) = \int_{R^n_+} (p_0 - px)_+\mu_2(dx) \tag{1.7}$$

for every $p \in \Gamma$ and $p_0 > 0$, where Γ is some open cone in R^n_+. Then $\mu_1 = \mu_2$.

This theorem is rather simple consequence of the classical properties of the Laplace transform. Indeed, it follows from Eqs. (1.6) and (1.7) that

$$\int_{R^n_+} e^{-px}\mu_1(dx) = \int_{R^n_+} e^{-px}\mu_2(dx) \tag{1.8}$$

for every $p \in \Gamma$, $|p| > A$.

Let us consider a finite measures $\mu_j^{(A)} = e^{-A(x_1+\ldots+x_n)}\mu_j$, $j = 1,2$. It implies from Eq. (1.8) that $\Phi_1(p) = \Phi_2(p)$, $p \in \Gamma$, where

$$\Phi_j(p) = \int e^{-px}\mu_j^{(A)}(dx).$$

The functions $\Phi_j(p)$ are analytically extended to the tube domain $R^n_+ + iR^n \subset C^n$ and coincide there. So the Fourier transform $\mu_1^{(A)}$ and $\mu_2^{(A)}$ also coincides. Consequently $\mu_1 = \mu_2$.

Let us consider a class of nonnegative measure with support in R^n_+ exponentially decreasing at infinity, i.e. for some $\varepsilon > 0$

$$\int_{R^n_+} e^{\varepsilon|x|}\mu(dx) < \infty.$$

The effective answer to the question (2) can be given in terms of the Laplace transform

$$\Phi(p) = \int_{x\in R^n_+} e^{-px}\mu(dx) = \int_0^\infty e^{-\tau}\frac{\partial^2\Pi(p,\tau)}{\partial\tau^2}d\tau.$$

Theorem. *(Henkin G.M., Shananin A.A. 1990) A function $\Pi(p,p_0)$, $(p,p_0) \in R^n_+ \times R^l_+$, can be represented in the form (1.3) for a nonnegative exponentially decreasing measure μ with support in R^n_+ iff*

i) *$\Pi(p,p_0)$ is a homogeneous of the first order convex function on $R^n_+ \times R^l_+$;*
ii) *for a fixed $p \in R^n_+$ the measure $((\partial^2\Pi(p,p_0))/\partial p_0^2)$ exponentially decreases as $p_0 \to \infty$ and $\lim_{p_0\to+0}\Pi(p,p_0) = \lim_{p_0\to+0}(\partial\Pi(p,p_0))/\partial p_0) = 0$;*
iii) *the function $\Phi(p) = \int_0^\infty e^{-\tau}((\partial^2\Pi\ (p,\tau))/\partial\tau^2)\,d\tau$ is bounded, completely monotone function on R^n_+.*

The proof of this theorem is based on the following classical result of S. Bernstein ($n = 1$) and V. Gilbert ($n > 1$). A function $\Phi(p), p \in R^n_+$ can be represented as Laplace transform of nonnegative finite measure with support in R^n_+ iff $\Phi(p)$ is completely monotone on R^n_+, i.e. $\Phi(p) \in C(R^n_+) \cap C^\infty(\text{int} R^n_+)$ and for any $p \in \text{int} R^n_+$ and $k = (k_1, \ldots, k_n)$

$$(-1)^{|k|} \frac{\partial^{|k|} \Phi(p)}{\partial p_l^{k_1} \ldots \partial p_n^{k_n}} \geq 0.$$

Let us remark here that the aggregated profit functions of the form (1.5) satisfy automatically to the condition (i) and (ii) of the theorem but not satisfy in general to the condition (iii).

Example. *We touch upon two examples of applications of results giving characterization of a profit functions.*

1. *We consider the production function of the CES type:* $\Psi(l_1, l_2) = (\beta_1 l_1^{-\rho} + \beta_2 l_2^{-\rho})^{-\gamma/\rho}$, *where* $\beta_1 > 0,\ \beta_2 > 0,\ \rho \geq -1,\ \rho \neq 0,\ 0 < \gamma \leq 1$. *Formula (1.2) allows computing the corresponding profit function* $\Pi(p_1,p_2,p_0)$. *It implies from theorem that some efficiency distribution of technologies corresponds to the function* $\Psi(l_1,l_2)$ *iff* $\rho > -1$ *and* $0 < \gamma < 1$.
2. *We consider the profit function*

$$\Pi(p, p_0) = \max_j \Pi_j(p, p_0), \tag{1.9}$$

where functions $\Pi_j(p, p_0)$ *are generated by various capacity distribution of technologies* $\mu_j(dx)$. *Suppose that the following expansion* $R^n_+ = \cup_j \bar{K}_j$ *takes place, where* $K_j = \{p | \Pi(p, p_0) = \Pi_j(p, p_0),\ \Pi(p, p_0) > \Pi_i(p, p_0) \text{ for } \ i \neq j\}$ *is open cone. It follows from uniqueness theorem that for the function* $\Pi(p,p_0)$ *there does not exist capacity distribution over technologies, which generates it. In fact inside cone* K_j *only efficiency distribution over technologies* $\mu_j(dx)$ *may correspond to the function* $\Pi(p,p_0)$. *If structure of prices changes in such a way that vector p comes over from one cone to another then capacity distribution over technologies changes step-wise.*

For example, consider a system of two industries. The output of the first industry in the final product and the expenses for the unit of output production are x_0 units of the second industry product, x_1 units of first primary resource, x_2 units of second primary resource. Suppose the total capacity of the first industry equals k_0. The second industry utilizes the same primary resources as the first one. Let $y^1 = (y^1_1, y^1_2)$ and $y^2 = (y^2_1, y^2_2)$ be two technologies of the second industry and $\mu(dz) = k_1 \delta(z - y^1) + k_2 \delta(z - y^2)$ be its capacity distribution over technologies. Denote by p_0 the price of the first industry product, p_1 and p_2 be the prices of the first and the second primary resources, p_3 be the price of the second industry production. It follows from Eq. (1.5) that the aggregated profit function has the form

$$\begin{aligned}\Pi(p_1, p_2, p_0) = \min_{p_3 \geq 0} \{ & k_0(p_0 - p_3 x_0 - p_1 x_1 - p_2 x_2)_+ \\ & + k_1(p_3 - p_1 y^1_1 - p_2 y^1_2)_+ + k_2(p_3 \\ & - p_1 y^2_1 - p_2 y^2_2)_+ \}\end{aligned}$$

Let us assume that $y^2_1 > y^1_1$, $y^1_2 > y^2_2$, $k_1 + k_2 > x_0 k_0$. Then

$$\begin{aligned}\Pi(p, p_0) = \max \Bigg\{ & \left(k_0 - \frac{k_2}{x_0} \right)_+ (p_0 - p(x + x_0 y^1))_+ \\ & + \min\left(\frac{k_2}{x_0}, k_0 \right) (p_0 - p(x + x_0 y^2))_+, \\ & \min\left(\frac{k_1}{x_0}, k_0 \right) (p_0 - p(x + x_0 y^1))_+ \\ & + \left(k_0 - \frac{k_1}{x_0} \right)_+ (p_0 - p(x + x_0 y^2))_+ \Bigg\}\end{aligned}$$

where $x = (x_1, x_2)$, $p = (p_1, p_2)$. It follows that for $py^2 \leq py^1$

$$\Pi(p, p_0) = \int_{R^2_+} (p_0 - pz)_+ \mu_1(dz),$$

where

$$\mu_1(\mathrm{d}z) = \left(k_0 - \frac{k_2}{x_0}\right)_+ \delta(z - (x + x_0 y^1))_+$$
$$+ \min\left(\frac{k_2}{x_0}, k_0\right) \delta(z - (x + x_0 y^2))$$

and for $py^2 \leq py^1$

$$\Pi(p, p_0) = \int_{\mathrm{R}^2_+} (p_0 - pz)_+ \mu_2(\mathrm{d}z),$$

where

$$\mu_2(\mathrm{d}z) = \min\left(\frac{k_2}{x_0}, k_0\right) \delta(z - (x + x_0 y^1))$$
$$+ + \left(k_0 - \frac{k_1}{x_0}\right)_+ \delta(z - (x + x_0 y^2)).$$

Since $\mu_1(\mathrm{d}z) \neq \mu_2(\mathrm{d}z)$ the passage of the primary resources price structure over the line $py^2 = py^1$ results in changes of aggregated capacity distribution over technologies. In this example due to changes of structure of prices jump of aggregated capacity distribution over technologies takes place because of changes of addresses of mutual deliveries between industries but not due to physical conversion of capacities.

2. MODEL DESCRIBING PROPAGATION OF NEW TECHNOLOGIES

Describing how new technologies are transferred between firms is an old economic problem since economists have realized that in our age technological progress is the major force of economic development. An interesting mathematical model describing propagation new technologies in an industry has been proposed by Henkin and Polterovich, 1988; 1991.

2.1. Henkin–Polterovich's Model

Consider an industry, which contains a large number of firms. According to this model each firm correspond to certain level of efficiency. Efficiency may be defined as profit or added value per unit of capacity. It is assumed that each firm wants to increase its level of efficiency. Let this level be a discrete variable n, which may take any nonnegative integer value $n = 0, 1, \ldots$. Denote by $F_n(t)$ the fraction of the firms, which have efficiency level n or less at time t. Then, for every real nonnegative t sequence $F = \{F_n(t), \quad n = 0, 1, \ldots\}$ is a distribution function and the model describes its evolution in time. It is assumed that this evolution satisfies the following rules:

- The efficiency can only improve in time.
- The firms cannot jump over levels: if a firm has a level n then it may transit to the level $n + 1$ only.
- The speed of transition is a sum of two components: innovation and imitation components.

This goes in accordance with idea developed by famous economist J. Schumpeter who divided the mechanism of technological changes into innovation, i.e. creation of new technologies by a firm itself, and imitation, i.e. adoption of technologies created by other firms. The fraction of firms going over from level n to the level $n + 1$ per unit of time due to imitation is proportion with coefficient β to the share of more efficient firms $F_n(t) - F_{n-1}(t)$, and due to innovation is constant α. Summing up we get the following infinite system of ordinary differential equations:

$$\frac{\mathrm{d}F_n(t)}{\mathrm{d}t} = \alpha(F_{n-1}(t) - F_n(t)) + \beta(1 - F_n(t))(F_{n-1}(t) - F_n(t)) = -(\alpha + \beta(1 - F_n(t))) \times (F_n(t) - F_{n-1}(t)), \quad \alpha, \beta > 0 \tag{2.1}$$

under initial conditions

$$F_0(t) \equiv 0, \;\; 0 < F_n(0) \leq F_{n+1}(0) < 1, \;\; n < N - 1,$$
$$F_n(0) = 1, \;\; n \geq N, \tag{2.2}$$

which contain only finite number N of variables different from one.

Henkin and Polterovich proved crucial theorem about behavior of system Eqs. (2.1) and (2.2). Let us denote

$$B(t)=\prod_{k=1}^{\infty}\left(1+\frac{\beta}{\alpha}(1-F_k(t))\right),$$

$$F_n^*(t,A)=(1+e^{-(n-c(t+\tau))})^{-1},$$

where $c=(\beta/\ln(\alpha+\beta/\alpha))$, $\tau=(\ln A/\beta)$.

Theorem. *(Henkin G.M., Polterovich V.M, 1988). Let F_n, $1\leq n\leq\infty$ be a solution of the problem (2.1) and (2.2). If we define $A=B(0)$ then the following estimation is valid*

$$|F_n(t)-F_n^*(t,A)|\leq\lambda e^{-\gamma t},\quad 1\leq n<\infty,\quad t\geq T$$

where λ, γ, T are constants depending on α, β, $B(0)$, N.

This theorem asserted that all solution to the Cauchy problem (2.1) and (2.2) asymptotically assume a shape of the well known logistic distribution and propagate with speed c. The result agrees with many empirical works. However, some empirical facts cannot be described by the Henkin–Polterovich's model. For example, there exist industries with several technological hierarchies and less effective hierarchies, practically speaking, do not adopt the achievements of more advanced ones. G.M. Henkin and V.M. Polterovich proposed a modification of the Eq. (2.1)

$$\frac{dF_n(t)}{dt}=-\varphi(F_n(t))(F_n(t)-F_{n-1}(t)),$$

where $\varphi(F_n)$ is nonlinear function. For non-monotomic function $\varphi(F_n)$ G.M. Henkin and V.M. Polterovich states that in numerical investigations several technological hierarchies may exist. We consider another modification more close to initial suggestions of J. Schumpeter.

2.2. Modification of Henkin–Polterovich's Model

We have assumed that the imitation process has "local" nature. It means that firms are able to imitate only technologies of the firms from the next higher efficiency level. Assuming that, the imitation component becomes $\beta(F_n-F_{n+1})(F_n-F_{n-1})$ and we come to following system

$$\frac{dF_n(t)}{dt}=-(\alpha+\beta(F_{n+1}(t)-F_n(t))(F_n(t)-F_{n-1}(t))).\tag{2.3}$$

the boundary and initial condition are the same.

Numerical solution (see figures in Appendix A) to the new Cauchy problem (2.3) and (2.2) show that in the long run the distribution curve does not become similar to logistic distribution and depends on initial condition. If one considers densities rather than distribution functions, one can see several maxima of different increasing heights. These maxima move along axis n independently of each other and gradually dissipate in contrast to the presence of only one maximum in the Henkin–Polterovich's model. To explain analytically such a behavior we considered Eq. (2.3) when $\alpha=0$, that is when technologies are propagated due to imitation process only. Taking into account the initial conditions, we have a finite system. All variables from $N+1$ are identically equal to one.

A change of variables

$$\tau=\beta t,\quad c_n(t)=F_{N+1-n}(t)-F_{N-n}(t)$$

leads to system

$$\begin{cases}\dfrac{dc_1}{dt}=c_1c_2,\\ \dfrac{dc_n}{dt}=c_n(c_{n+1}-c_{n-1}),\quad n=2,\dots,N-1,\\ \dfrac{dc_N}{dt}=-c_Nc_{N-1},,\\ c_n(0)=\gamma_n>0,\quad n=1,\dots,N\end{cases}\tag{2.4}$$

known as finite Langmuir's chain. Note that Langmuir's chain is a high-order nonlinear system with quadratic non-linearity. Such systems may exhibit very sophisticated behavior in the long run—there may be odd attractors, limit cycles—

and one hardly succeeds in their complete analytical investigation. Nevertheless, in the case of finite Langmuir's chain it turned out to be possible.

2.3. Investigation of Langumuir's Chain

The stable stationary solutions of Eq. (2.4) have the following structure

$$(y_1, 0, y_2, 0, \ldots, y_k, 0) \quad \text{if} \quad N = 2k,$$

$$(y_1, 0, y_2, 0, \ldots, y_k, 0, y_{k+1}) \quad \text{if} \quad N = 2k + 1$$

The necessary condition for stability is

$$y_1 \geq y_2 \geq \ldots \geq y_k (\geq y_{k+1}) \geq 0.$$

J. Moser proved that if $c_1(t), \ldots, c_n(t)$ the solution of Eq. (2.4) then the eigenvalues of Jacobi matrix

$$L(t) = \begin{pmatrix} 0 & \sqrt{c_1(t)} & & & & \\ \sqrt{c_1(t)} & 0 & \sqrt{c_2(t)} & & & 0 \\ & \sqrt{c_2(t)} & 0 & & & \\ 0 & & & \ddots & 0 & \sqrt{c_N(t)} \\ & & & & \sqrt{c_N(t)} & 0 \end{pmatrix} \tag{2.5}$$

do not depend from t and are different from each other. More over if N is even then eigenvalues of Jacobi matrix in Eq. (2.5) have the structure $-\lambda_1 < -\lambda_2 < \ldots < -\lambda_{N+1/2} < \lambda_{N+1/2} < \ldots \lambda_2 < \lambda_1$ and if N is odd then eigenvalues have the structure $-\lambda_1 < -\lambda_2 < \ldots < -\lambda_{N/2} < 0 < \lambda_{N/2} < \ldots \lambda_2 < \lambda_1$.

Langmuir's chain is a discrete version of Korteweg–de Vries equation. The solution for Eq. (2.4) can be calculated analytically from the following formula proved by J. Moser

$$\cfrac{1}{z - \cfrac{c_1(t)}{z - \cfrac{c_2(t)}{z - \cfrac{\ddots}{\; - \cfrac{c_N(t)}{z}}}}} = \frac{\sum_{n=0}^{N} \frac{m_n e^{\lambda_n^2 t}}{z - \lambda_n}}{\sum_{n=0}^{N} m_n e^{\lambda_n^2 t}} \tag{2.6}$$

Using the information about initial conditions, $c_1(0), \ldots, c_N(0)$ we can calculate from Eq. (2.6) the values of constants $m_0, \ldots, m_N$, $\lambda_0, \ldots \lambda_N$. Then at every moment of time t we can calculate the values $c_1(t), \ldots, c_N(t)$. However, the formula (2.6) can not be used for determination the asymptotic of $c_1(t), \ldots, c_N(t)$ when $t \to \infty$. Nevertheless, the following theorem about the asymptotic behavior of these variable can be proved.

Theorem. *(Tashlitckay Y.M., Shananin A.A.). Solutions to the Cauchy problem (2.4) for the Langmuir's finite chain converges, as $t \to \infty$, to a fixed point, which is determined uniquely by initial data. Moreover, the following relations determine the character of convergence*

$$c_{2k-1}(t) = \lambda_k^2 + O(e^{-\nu t}), c_{2k}(t) = O(e^{(\lambda_{k+1}^2 - \lambda_k^2)t}),$$
$$k = 1, \ldots, n-1,$$

where $\nu = \min\{\lambda_k^2 - \lambda_{k+1}^2 | k = 1, \ldots n-1\} > 0$ *and* $n = [N/2] + 1$ *is a number of different nonzero eigenvalues of Jacobi matrix Eq. (2.5)* $\lambda_1^2 > \lambda_2^2 > \ldots > \lambda_n^2 > 0$ *which is determined by initial data* $c_1(0), \ldots c_N(0)$.

The proof of this theorem is based on the theory of rational approximations and orthogonal polynomials. This theorem explains the results of numerical calculations. For the model (2.3) and (2.2) a several technological hierarchies in the industry should exist when $(1/\nu\beta) \ll t \ll (1/\alpha)$.

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APPENDIX A

Figures A1–A6

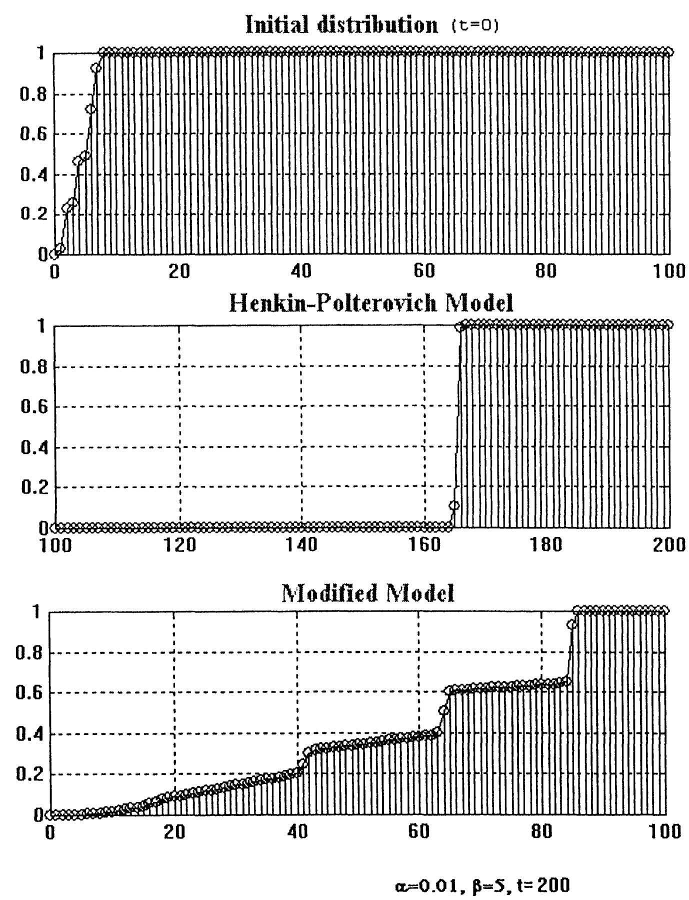

FIGURE A1

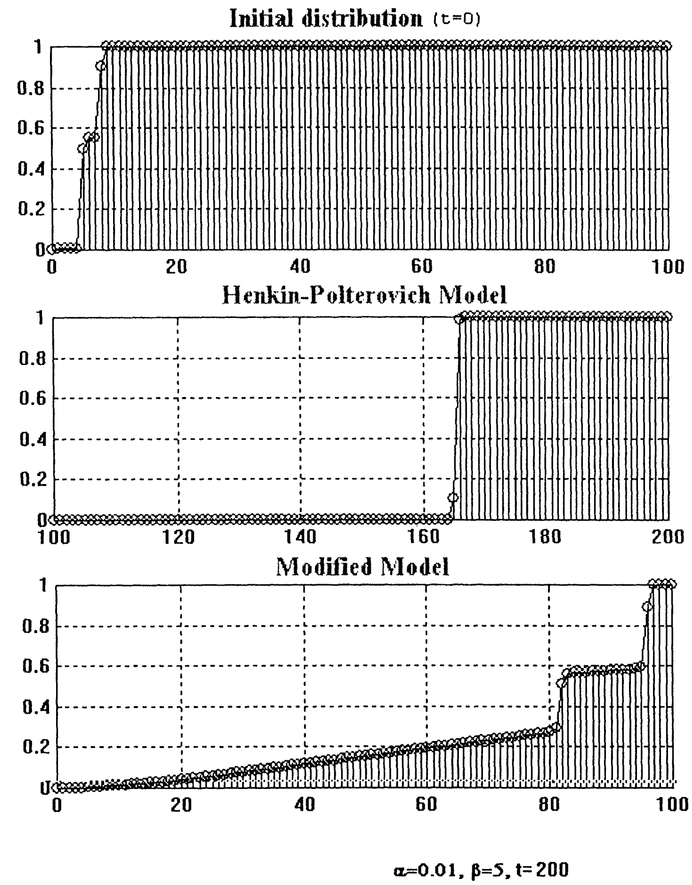
Initial distribution (t=0)
1
0.8
0.6
0.4
0.2
0
0 20 40 60 80 100
Henkin-Polterovich Model
1
0.8
0.6
0.4
0.2
0
100 120 140 160 180 200
Modified Model
1
0.8
0.6
0.4
0.2
0
0 20 40 60 80 100
α=0.01, β=5, t=200

FIGURE A2

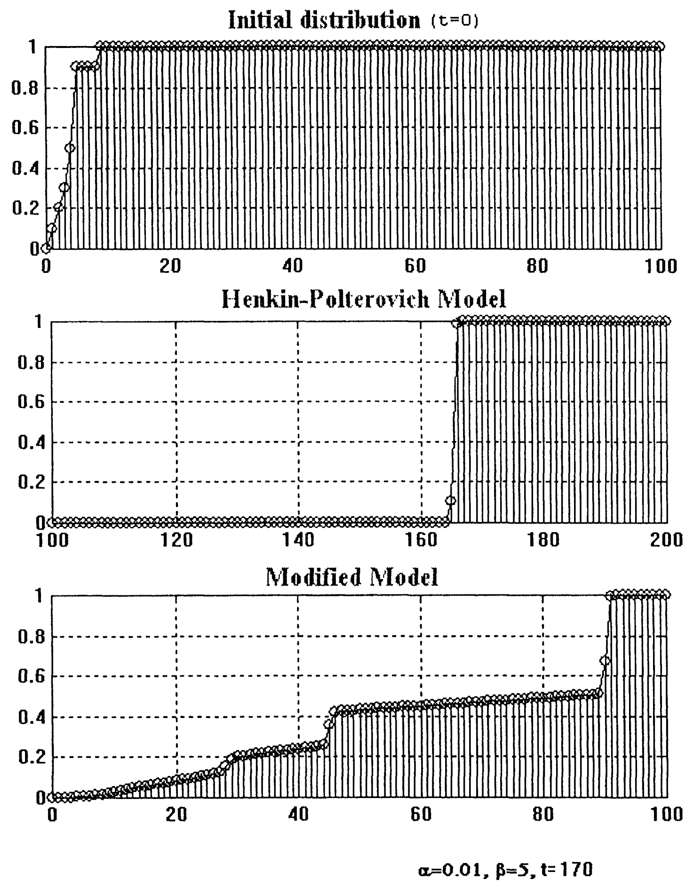
Initial distribution (t=0)
1
0.8
0.6
0.4
0.2
0
0
20
40
60
80
100
Henkin-Polterovich Model
100
120
140
160
180
200
Modified Model
α=0.01, β=5, t=170

FIGURE A3

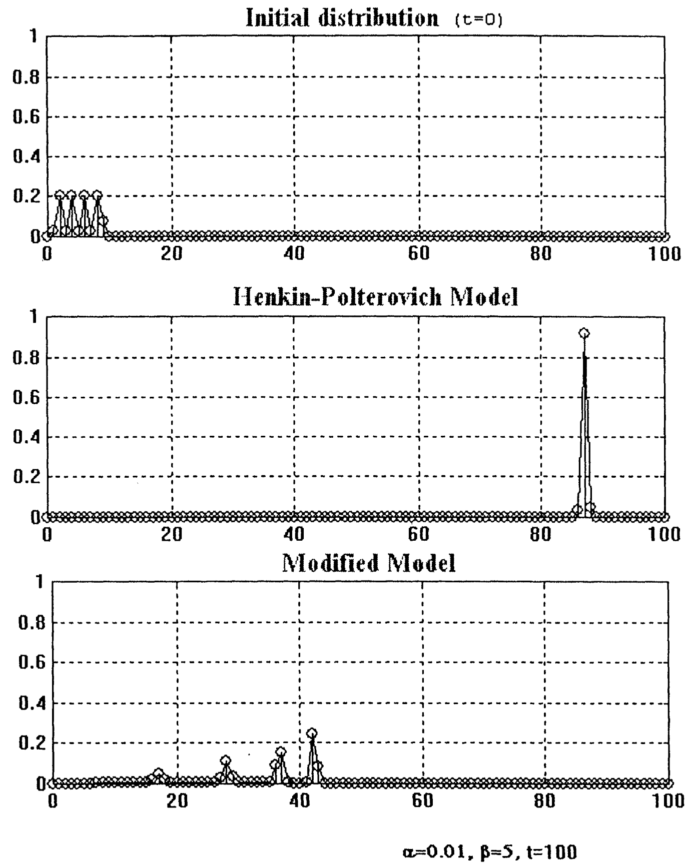
Initial distribution (t=0)
Henkin-Polterovich Model
Modified Model
α=0.01, β=5, t=100

FIGURE A4

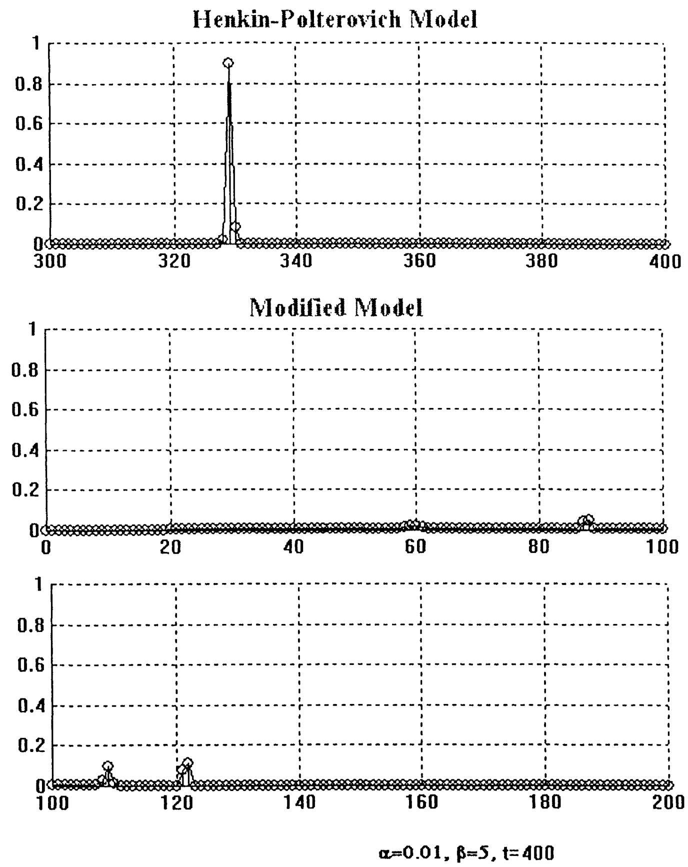
Henkin-Polterovich Model
1
0.8
0.6
0.4
0.2
0
300
320
340
360
380
400
Modified Model
1
0.8
0.6
0.4
0.2
0
0
20
40
60
80
100
1
0.8
0.6
0.4
0.2
0
100
120
140
160
180
200
α=0.01, β=5, t=400

FIGURE A5

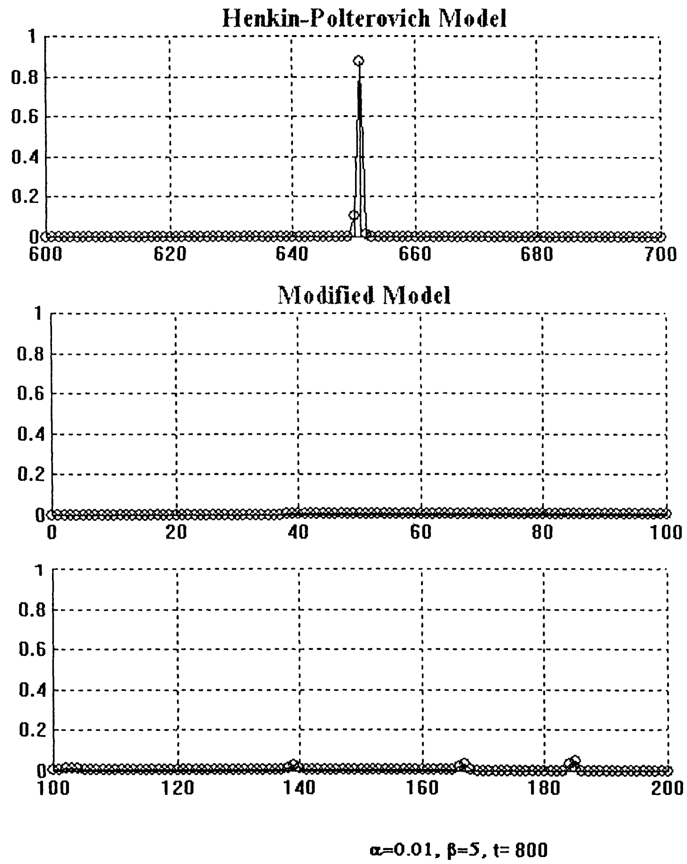
Henkin-Polterovich Model
1
0.8
0.6
0.4
0.2
0
600
620
640
660
680
700
Modified Model
1
0.8
0.6
0.4
0.2
0
0
20
40
60
80
100
1
0.8
0.6
0.4
0.2
0
100
120
140
160
180
200
α=0.01, β=5, t= 800

FIGURE A6