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Review Article

Infinitely Many Solutions for a Robin Boundary Value Problem

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By combining the embedding arguments and the variational methods, we obtain infinitely many solutions for a class of superlinear elliptic problems with the Robin boundary value under weaker conditions.

1. Introduction

In this paper, we consider the following equation:

$$\begin{aligned} -\Delta u &= f(x,u), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + b(x)u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $0 \le b \in L^\infty(\partial\Omega)$. Denote

$$F(x,s) = \int_0^s f(x,t)dt, \qquad \mathcal{F} = f(x,s)s - 2F(x,s), \tag{1.2}$$

and let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_j < \cdots$ be the eigenvalues of $-\Delta$ with the Robin boundary conditions. We assume that the following hold:

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R})$, $\exists q \in (2, 2^*)$ such that

$$|f(x,s)| \leq c\left(1 + |s|^{q-1}\right), \tag{1.3}$$

where $1 \leq s < 2N/(N-2)$, $N \geq 3$. If $N = 1, 2$, let $2^* = \infty$;

(f_2) $f(x,s)s \geq 0$, $\lim_{|s| \to +\infty}(f(x,s)s)/|s|^2 = +\infty$ uniformly for $x \in \Omega$.

(f_3) there exist $\theta \geq 1$, $s \in [0,1]$ s.t.

$$\theta \mathcal{F}(x,t) \geq \mathcal{F}(x,st), \quad (x,t) \in \Omega \times \mathbb{R}; \tag{1.4}$$

(f_4) $f(x,-t) = -f(x,t)$, $(x,t) \in \Omega \times \mathbb{R}$.

Because of (f_2), (1.1) is usually called a superlinear problem. In [1, 2], the author obtained infinitely many solutions of (1.1) with Dirichlet boundary value condition under (f_1), (f_4) and

(AR) $\exists\, \mu > 2$, $R > 0$ such that

$$x \in \Omega, \quad |s| \geq R \Longrightarrow 0 < \mu F(x,s) \leq f(x,s)s. \tag{1.5}$$

Obviously, (f_2) can be deduced form (AR). Under (AR), the (PS) sequence can be deduced bounded. However, it is easy to see that the example [3]

$$f(x,t) = 2t \log(1 + |t|) \tag{1.6}$$

does not satisfy (AR), while it satisfies the aforementioned conditions (take $\theta = 1$ in (f_3)). (f_3) is from [3, 4].

We need the following condition (C), see [3, 5, 6].

Definition 1.1. Assume that X is a Banach space, we say that $J \in C^1(X, \mathbb{R})$ satisfies Cerami condition (C), if for all $c \in \mathbb{R}$:

(i) any bounded sequence $\{u_n\} \subset X$ satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ possesses a convergent subsequence;

(ii) there exist $\sigma, R, \beta > 0$ s.t. for any $u \in J^{-1}([c-\sigma, c+\sigma])$ with $\|u\| \geq R$, $\|J'(u)\|\|u\| \geq \beta$.

In the work in [2, 7], the Fountain theorem was obtained under the condition (PS). Though condition (C) is weaker than (PS), the well-known deformation theorem is still true under condition (C) (see [5]). There is the following Fountain theorem under condition (C).

Assume $X = \overline{\bigoplus_{j=1}^{\infty} X_j}$, where X_j are finite dimensional subspace of X. For each $k \in \mathbb{N}$, let

$$Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j \geq k} X_j}. \tag{1.7}$$

Denote $S_\rho = \{u \in X : \|u\| = \rho\}$.

Proposition 1.2. *Assume that $J \in C^1(X, \mathbb{R})$ satisfies condition (C), and $J(-u) = J(u)$. For each $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(i) $b_k := \inf_{u \in Z_k \cap s_{r_k}} J(u) \to +\infty, k \to \infty$,

(ii) $a_k := \max_{u \in Y_k \cap s_{\rho_k}} J(u) \le 0$.

Then J has a sequence of critical points u_n, such that $J(u_n) \to +\infty$ as $n \to \infty$.

As a particular linking theorem, Fountain theorem is a version of the symmetric Mountain-Pass theorem. Using the aforementioned theorem, the author in [6] proved multiple solutions for the problem (1.1) with Neumann boundary value condition; the author in [3] proved multiple solutions for the problem (1.1) with Dirichlet boundary value condition. In the present paper, we also use the theorem to give infinitely many solutions for problem (1.1). The main results are follows.

Theorem 1.3. *Under assumptions (f_1)–(f_4), problem* (1.1) *has infinitely many solutions.*

Remark 1.4. In the work in [1, 2], they got infinitely many solutions for problem (1.1) with Dirichlet boundary value condition under condition (AR).

Remark 1.5. In the work in [8], they showed the existence of one nontrivial solution for problem (1.1), while we get its infinitely many solutions under weaker conditions than [8].

Remark 1.6. In the work in [9], they also obtained infinitely many solutions for problem (1.1) with Dirichlet boundary value condition under stronger conditions than the aforementioned (f_2) and (f_3) above. Furthermore, function (1.6) does not satisfy all conditions in [9]. Therefore, Theorem 1.3 applied to Dirichlet boundary value problem improves those results in [1, 2, 8, 9].

2. Preliminaries

Let the Sobolev space $X = H^1(\Omega)$. Denote

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right) dx\right)^{1/2} \tag{2.1}$$

to be the norm of u in X, and $|u|_q$ the norm of u in $L^q(\Omega)$. Consider the functional $J : X \to \mathbb{R}$:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} b(x) u^2 dS - \int_\Omega F(x,u) dx. \tag{2.2}$$

Then by (f_1), J is C^1 and

$$\langle J'(u), \phi \rangle = \int_\Omega \nabla u \nabla \phi dx + \int_{\partial\Omega} b(x) u \phi dS - \int_\Omega f(x,u) \phi dx, \quad u, \phi \in X. \tag{2.3}$$

The critical point of J is just the weak solution of problem (1.1).

Since we do not assume condition (AR), we have to prove that the functional J satisfies condition (C) instead of condition (PS).

Lemma 2.1. *Under (f_1)–(f_3), J satisfies condition (C).*

Proof. For all $c \in \mathbb{R}$, we assume that $\{u_n\} \subset X$ is bounded and

$$J(u_n) \longrightarrow c, \qquad J'(u_n) \longrightarrow 0, \qquad n \longrightarrow \infty. \tag{2.4}$$

Going, if necessary, to a subsequence, we can assume that $u_n \rightharpoonup u$ in X, then

$$\begin{aligned}\|u_n - u\|^2 &= \int_\Omega \left(|\nabla(u_n-u)|^2 + (u_n-u)^2\right)dx \\ &= \left\langle J'(u_n) - J'(u), u_n - u\right\rangle + \int_{\partial\Omega} - b(x)(u_n-u)^2 dS \\ &\quad + \int_\Omega \left[(u_n-u)^2 + \left(f(x,u_n) - f(x,u)\right)(u_n-u)\right]dx.\end{aligned} \tag{2.5}$$

that is,

$$\begin{aligned}\|u_n - u\|^2 + \int_{\partial\Omega} b(x)(u_n-u)^2 dS &= \int_\Omega \left(|\nabla(u_n-u)|^2 + (u_n-u)^2\right)dx \\ &= \left\langle J'(u_n) - J'(u), u_n - u\right\rangle \\ &\quad + \int_\Omega \left[(u_n-u)^2 + \left(f(x,u_n) - f(x,u)\right)(u_n-u)\right]dx.\end{aligned} \tag{2.6}$$

Since the Sobolev imbedding $W^{1,2}(\Omega) \hookrightarrow L^\gamma(\Omega)$ $(1 \le \gamma < 2^*)$ is compact, we have the right-hand side of (2.6) converges to 0. While $\int_{\partial\Omega} b(x)(u_n-u)^2 dS \ge 0$, we have $\|u_n - u\|^2 \to 0$. It follows that $u_n \to u$ in X and $J'(u) = 0$, that is, condition (i) of Definition 1.1 holds.

Next, we prove condition (ii) of Definition 1.1, if not, there exist $c \in \mathbb{R}$ and $\{u_n\} \subset X$ satisfying, as $n \to \infty$

$$J(u_n) \longrightarrow c, \qquad \|u_n\| \longrightarrow \infty, \qquad \left\|J'(u_n)\right\| \|u_n\| \longrightarrow 0, \tag{2.7}$$

then we have

$$\lim_{n\to\infty} \int_\Omega \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right)dx = \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle\right) = c. \tag{2.8}$$

Denote $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$, that is, $\{v_n\}$ is bounded in X, thus for some $v \in X$, we get

$$\begin{aligned} v_n &\rightharpoonup v, \quad \text{in } X, \\ v_n &\longrightarrow v, \quad \text{in } L^2(\Omega), \\ v_n &\longrightarrow v, \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.9}$$

If $v = 0$, define a sequence $\{t_n\} \subset \mathbb{R}$ as in [4]

$$J(t_n u_n) = \max_{t\in[0,1]} J(tu_n). \tag{2.10}$$

If for some $n \in \mathbb{N}$, there is a number of t_n satisfying (2.10), we choose one of them. For all $m > 0$, let $\overline{v}_n = 2\sqrt{m} v_n$, it follows by $v_n(x) \rightarrow v(x) = 0$ a.e. $x \in \Omega$ that

$$\lim_{n\to\infty}\int_\Omega F(x,\overline{v}_n)dx = \lim_{n\to\infty}\int_\Omega F\left(x,2\sqrt{m}v_n\right)dx = 0. \tag{2.11}$$

Then for n large enough, by (2.9), (2.11), and $\int_{\partial\Omega} b(x)v_n^2 \geq 0$, we have

$$\begin{aligned} J(t_n u_n) \geq J(\overline{v}_n) &= \frac{1}{2}\int_\Omega |\nabla \overline{v}_n|^2 dx + \frac{1}{2}\int_{\partial\Omega} b(x)\overline{v}_n^2 dS - \int_\Omega F(x,\overline{v}_n)dx \\ &= 2m\int_\Omega |\nabla v_n|^2 dx + 2m\int_{\partial\Omega} b(x)v_n^2 dS - \int_\Omega F(x,\overline{v}_n)dx \\ &= 2m\|v_n\|^2 - \int_\Omega F(x,\overline{v}_n)dx + 2m\int_{\partial\Omega} b(x)v_n^2 dS - 2m\int_\Omega v_n^2 dx \\ &\geq 2m - \int_\Omega F(x,\overline{v}_n)dx \geq m. \end{aligned} \tag{2.12}$$

That is, $\lim_{n\to\infty} J(t_n u_n) = +\infty$. Since $J(0) = 0$ and $J(u_n) \rightarrow c$, then $0 < t_n < 1$. Thus

$$\begin{aligned} &\int_\Omega |\nabla t_n u_n|^2 dx + \int_{\partial\Omega} b(x)(t_n u_n)^2 dS - \int_\Omega f(x,t_n u_n)t_n u_n dx \\ &\quad = \left\langle J'(t_n u_n), t_n u_n \right\rangle = t_n \frac{d}{dt}\bigg|_{t=t_n} J(tu_n) = 0. \end{aligned} \tag{2.13}$$

We see that

$$\begin{aligned} &\frac{1}{2}\int_\Omega f(x,t_n u_n)t_n u_n dx - \int_\Omega F(x,t_n u_n)dx \\ &\quad = \frac{1}{2}\int_\Omega |\nabla t_n u_n|^2 dx + \frac{1}{2}\int_{\partial\Omega} b(x)(t_n u_n)^2 dS - \int_\Omega F(x,t_n u_n)dx = J(t_n u_n). \end{aligned} \tag{2.14}$$

From the aforementioned, we infer that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\quad = \frac{1}{2}\int_\Omega \widetilde{F}(x,u_n)dx \geq \frac{1}{2\theta}\int_\Omega \widetilde{F}(x,t_n u_n)dx \\
&\quad = \frac{1}{\theta}\int_\Omega \left[\frac{1}{2} f(x,t_n u_n)t_n u_n - F(x,t_n u_n)\right]dx \longrightarrow +\infty, \quad n \longrightarrow \infty,
\end{aligned} \tag{2.15}$$

which contradicts (2.8).

If $v \not\equiv 0$, by (2.7)

$$\int_\Omega |\nabla u_n|^2 dx + \int_{\partial\Omega} b(x)u_n^2 dS - \int_\Omega f(x,u_n)\ u_n dx = \langle J'(u_n,u_n)\rangle = o(1). \tag{2.16}$$

That is,

$$\begin{aligned}
\|u_n\|^2 - \int_\Omega f(x,u_n)u_n dx - \int_\Omega u_n^2 dx + \int_{\partial\Omega} b(x)u_n^2 dS = o(1) \\
1 - o(1) = \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^2}dx + \int_\Omega \frac{u_n^2}{\|u_n\|^2}dx - \int_{\partial\Omega}\frac{b(x)u_n^2}{\|u_n\|^2}dS.
\end{aligned} \tag{2.17}$$

Since $\lim_{n\to\infty}\int_\Omega (u_n^2/\|u_n\|^2)dx = \lim_{n\to\infty}\int_\Omega v_n^2 = |v|_2^2$ exists, and by $v_n \rightharpoonup v$ in X (the weakly convergent sequence is bounded), we get

$$\int_{\partial\Omega}\frac{b(x)u_n^2}{\|u_n\|^2}dS = \int_{\partial\Omega} b(x)v_n^2 dS \leq C\|b\|_{L^\infty(\partial\Omega)}\|v_n\|^2 < \infty, \tag{2.18}$$

where C is the constant of Sobolev Trace imbedding from $H^1(\Omega) \to L^2(\partial\Omega)$, see [10]. We have

$$1 - o(1) \geq \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^2}dx - \widetilde{C} = \left(\int_{v\neq 0} + \int_{v=0}\right)\frac{f(x,u_n)u_n}{|u_n|^2}|v_n|^2 dx - \widetilde{C}. \tag{2.19}$$

For $x \in \Omega' := \{x \in \Omega : v(x) \neq 0\}$, we get $|u_n(x)| \to +\infty$. Then by (f_2)

$$\frac{f(x,u_n(x))u_n(x)}{|u_n(x)|^2}|v_n(x)|^2 \longrightarrow +\infty, \quad n \longrightarrow \infty. \tag{2.20}$$

By using Fatou lemma, since the Lebesgue measure $|\Omega'|\ > 0$,

$$\int_{v\neq 0}\frac{f(x,u_n)u_n}{|u_n|^2}|v_n|^2 dx \longrightarrow +\infty, \quad n \longrightarrow \infty. \tag{2.21}$$

On the other hand, by (f_2), there exists $\gamma > -\infty$, such that $(f(x,s)s)/|s|^2 \geq \gamma$ for $(x,s) \in \Omega \times \mathbb{R}$. Moreover,

$$\int_{v=0} \|v_n\|^2 dx \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.22}$$

Now, there is $\Lambda > -\infty$ s.t.

$$\int_{v=0} \frac{f(x,u_n)u_n}{|u_n|^2} |v_n|^2 dx \geq \gamma \int_{v=0} \|v_n\|^2 dx \geq \Lambda > -\infty. \tag{2.23}$$

Together with (2.19) and (2.21), (2.23), it is a contradiction.

This proves that J satisfies condition (C). □

3. Proof of Theorem 1.3

We will apply the Fountain theorem of Proposition 1.2 to the functional in (2.2). Let

$$X_j = \ker(-\Delta - \lambda_i), \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j \geq k} X_j}, \tag{3.1}$$

then $X = \overline{\bigoplus_{j=1}^{\infty} X_j}$. It shows that $J \in C^1(X, \mathbb{R})$ by (f_1) and satisfies condition (C) by Lemma 2.1.

(i) After integrating, we obtain from (f_1) that there exist $c_1 > 0$ such that

$$|F(x,u)| \leq c_1 \left(1 + |u|^q\right). \tag{3.2}$$

Let us define $\beta_k = \sup_{u \in Z_k \cap S_1} |u|_q$. By [2, Lemma 3.8], we get $\beta_k \to 0$ as $k \to \infty$. Since $|u|_2 \leq C(\Omega)|u|_q$, let $c = c_1 + (1/2)C(\Omega)$, and $r_k = (cq\beta_k^q)^{1/2-q}$, then by (3.2), for $u \in Z_k$ with $\|u\| = r_k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} b(x) u^2 dS - \int_{\Omega} F(x,u) dx \\
&\geq \frac{1}{2}\|u\|^2 - c_1 |u|_q^q - c_1|\Omega| + \frac{1}{2}\int_{\partial\Omega} b(x) u^2 dS - \frac{1}{2}\int_{\Omega} u^2 dx \\
&\geq \frac{1}{2}\|u\|^2 - c_1 |u|_q^q - c_1|\Omega| - \frac{1}{2}|u|_2^2 \\
&\geq \frac{1}{2}\|u\|^2 - c |u|_q^q - c_1|\Omega| \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)\left(cq\beta_k^q\right)^{2/(2-q)} - c_1|\Omega|.
\end{aligned} \tag{3.3}$$

Notice that $\beta_k \to 0$ and $q > 2$, we infer that

$$b_k = \inf_{u \in Z_k \cap s_{r_k}} J(u) \longrightarrow +\infty, \quad k \longrightarrow \infty. \tag{3.4}$$

(ii) While

$$\lambda_1 = \inf_{u \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \alpha(x) u^2 dS}{\int_\Omega u^2 dx} > 0, \tag{3.5}$$

we can deduce that $\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \alpha(x) u^2 dS$ is the equivalent norm of $\|u\|^2$ in X. Since $\dim Y_k < +\infty$ and all norms are equivalent in the finite-dimensional space, there exists $C_k > 0$, for all $u \in Y_k$, we get

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} b(x) u^2 dS = \frac{1}{2}\|u\|^2 \le C_k |u|_2^2. \tag{3.6}$$

Next by (f_2), there is $R_k > 0$ such that $F(x,s) \ge 2C_k |s|^2$ for $|s| \ge R_k$. Take $M_k := \max\{0, \inf_{|s|\le R_k} F(x,s)\}$, then for all $(x,s) \in \Omega \times \mathbb{R}$, we obtain

$$F(x,s) \ge 2C_k |s|^2 - M_k. \tag{3.7}$$

It follows from (3.6), (3.7), for all $u \in Y_k$ that

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} b(x) u^2 dS - \int_\Omega F(x,u) dx \\ &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u) dx \\ &\le -C_k |u|_2^2 + M_k |\Omega| \\ &\le -\frac{1}{2}\|u\|^2 + M_k |\Omega|. \end{aligned} \tag{3.8}$$

Therefore, we get that for ρ_k large enough $(\rho_k > r_k)$,

$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} J(u) \le 0. \tag{3.9}$$

By Fountain theorem of Proposition 1.2, J has a sequence of critical points $u_n \in X$, such that $J(u_n) \to +\infty$ as $n \to \infty$, that is, (1.1) has infinitely many solutions.

Remark 3.1. By Theorem 1.3, the following equation:

$$
\begin{aligned}
& -\Delta u = 2u \log(1+|u|), \quad \text{in } \Omega, \\
& \frac{\partial u}{\partial n} + b(x)u = 0, \quad \text{on } \partial\Omega,
\end{aligned}
\tag{3.10}
$$

has infinitely many solutions, while the results cannot be obtained by [1, 2, 8, 9]

Remark 3.2. In the next paper, we wish to consider the sign-changing solutions for problem (1.1).

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