PERIODIC SOLUTIONS FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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We study the existence of a periodic solution for some partial functional differential equations. We assume that the linear part is nondensely defined and satisfies the Hille-Yosida condition. In the nonhomogeneous linear case, we prove the existence of a periodic solution under the existence of a bounded solution. In the nonlinear case, using a fixed-point theorem concerning set-valued maps, we establish the existence of a periodic solution.

1. Introduction

Consider the partial functional differential equation

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + L(t,x_t) + G(t,x_t), \quad \text{for } t \geq 0,\\ x_0 &= \varphi \in C = C([-r,0];E),\end{aligned} \tag{1.1}$$

where $A : D(A) \subset E \to E$ is a nondensely defined linear operator on a Banach space E. Throughout this paper, we suppose that

(H_1) A is a Hille-Yosida operator: there exist $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that

$$(\omega_0, \infty) \subset \rho(A), \qquad ||R(\lambda,A)^n|| \leq \frac{M_0}{(\lambda-\omega_0)^n}, \quad \text{for } n \in \mathbb{N},\ \lambda > \omega_0, \tag{1.2}$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda,A) = (\lambda - A)^{-1}$.

C is the space of continuous functions from $[-r,0]$ into E endowed with the uniform norm topology, and for every $t \geq 0$, the history function $x_t \in C$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \text{for } \theta \in [-r,0]. \tag{1.3}$$

$L : \mathbb{R} \times C \to E$ is continuous, linear with respect to the second argument and ω-periodic in t; $G : \mathbb{R} \times C \to E$ is continuous and ω-periodic in t.

When the operator A generates a strongly continuous semigroup on E, (1.1) has been treated extensively by several authors; for more details, we refer to [14]. Recently in [1, 8],

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the existence, the regularity of solutions, and the local stability have been treated when A is nondensely defined and satisfies the Hille-Yosida condition. In this work, we will deal with the existence of periodic solutions of (1.1) when A satisfies the Hille-Yosida condition. The problem of finding periodic solutions is an important subject in the qualitative study of functional differential equations. The famous Massera's theorem on two-dimensional periodic ordinary differential equations [11] explains the relationship between the boundedness of solutions and periodic solutions. In [15], using Browder's fixed-point theorem, it has been proved that if the solutions of an n-dimensional periodic ordinary differential equation are either uniformly bounded or uniformly ultimately bounded, then the system has a periodic solution. In [5], the existence of a periodic solution has been established under the existence of a bounded solution for some inhomogeneous, linear functional differential equation in infinite dimensional space. In [10], using Horn's fixed-point theorem, the existence of periodic solutions for functional differential equation with finite delay was established. Recently in [12], several criteria were obtained to ensure the existence and uniqueness of a periodic solution for some inhomogeneous linear partial functional differential equations with infinite delay. In [4], we developed some results dealing with the existence of a periodic solution for (1.1) when A generates a strongly continuous semigroup on E. In [7], it was established that the existence of bounded and ultimate bounded solutions of (1.1) implies the existence of periodic solutions. The approach that was used was based on Horn's fixed-point theorem. In this paper, we generalize the results obtained in [4, 5, 11] for (1.1), where the operator A is not necessarily densely defined but satisfies the Hille-Yosida condition. In Section 2, we prove the existence of periodic solutions in the nonhomogeneous linear case under the assumption that a bounded solution on $\mathbb{R}^+$ exists. In Section 3, we study the nonlinear case; our approach makes use of a fixed-point theorem for set-valued maps to obtain sufficient conditions, ensuring the existence of a periodic solution for (1.1). Section 4 is devoted to an example.

2. Inhomogeneous linear case

Definition 2.1 [1, 8]. A continuous function $x : [-r,b] \to E$ $(b > 0)$ is called an integral solution of (1.1) if

(i) $\int_0^t x(s)ds \in D(A)$, for $t \in [0,b]$,

(ii) $x(t) = \varphi(0) + A\int_0^t x(s)ds + \int_0^t L(s,x_s)ds + \int_0^t G(s,x_s)ds$, for $t \in [0,b]$,

(iii) $x_0 = \varphi$.

It follows from the closedness of A that if x is an integral solution of (1.1), then $x(t) \in \overline{D(A)}$, for $t \geq 0$. The following result dealing with the existence and the uniqueness of the integral solution was established.

Theorem 2.2 [1, 8]. *Assume that (H_1) holds and G is Lipschitz with respect to the second argument. Then for all $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, (1.1) has a unique integral solution on $\mathbb{R}^+$. Moreover, the integral solution depends continuously on the initial data.*

Let A_0 be the part of A in $\overline{D(A)}$ given by

$$A_0 = A \quad \text{on } D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}. \tag{2.1}$$

Then, from [2], A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. Moreover, from [13], if the integral solution of (1.1) exists, then it is given by this variation of constant formula

$$x(t) = \begin{cases} T_0(t)\varphi(0) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B_\lambda\big(L(s,x_s) + G(s,x_s)\big)ds, & t \geq 0, \\ \varphi(t), \quad t \in [-r,0], \end{cases} \tag{2.2}$$

where $B_\lambda = \lambda(\lambda - A)^{-1}$.

Consider the equation

$$\begin{aligned} \frac{d}{dt}x(t) = Ax(t) + L(t,x_t) + f(t), \quad \text{for } t \geq 0, \\ x_0 = \varphi \in C = C([-r,0];E), \end{aligned} \tag{2.3}$$

where f is continuous and ω-periodic in t, and suppose the hypothesis stated below.

(H_2) The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$, meaning that for $t > 0$, the operator $T_0(t)$ is compact on $\overline{D(A)}$.

Theorem 2.3. *Assume that (H_1) and (H_2) hold. Then the following are equivalent:*

(i) *there exists a $\varphi \in C$ such that (2.3) has a bounded integral solution defined on $\mathbb{R}^+$,*

(ii) *equation (2.3) has an ω-periodic solution.*

Let u be the bounded integral solution of (2.3) on $\mathbb{R}^+$, then the following two lemmas are needed in the proof of Theorem 2.3.

Lemma 2.4. *$\{u(t) : t \geq 0\}$ is relatively compact in E and u is uniformly continuous. Consequently, $\{u_t : t \geq 0\}$ is relatively compact in C.*

Proof of Lemma 2.4. For simplicity, we equate $F(t,\varphi) = L(t,\varphi) + f(t)$, and let $\varepsilon > 0$ and $t > \varepsilon$. Then,

$$u(t) = T_0(t)u(0) + \lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-s)B_\lambda F(s,u_s)ds + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)ds. \tag{2.4}$$

It follows that

$$\begin{aligned} u(t) &= T_0(\varepsilon)\left[T_0(t-\varepsilon)u(0) + \lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda F(s,u_s)ds\right] \\ &\quad + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)ds, \\ u(t) &= T_0(\varepsilon)u(t-\varepsilon) + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)ds. \end{aligned} \tag{2.5}$$

The compactness property of the semigroup $(T_0(t))_{t\geq 0}$ and the boundedness of the solution u show that $\{T_0(\varepsilon)u(t-\varepsilon) : t > \varepsilon\}$ is relatively compact in E. Using the boundedness of B_λ and F, there exists a positive constant a such that

$$\left\|\lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda F(s,u_s)ds\right\| \leq a\varepsilon. \tag{2.6}$$

Hence, $\{u(t) : t \geq 0\}$ is relatively compact in E.

To show the uniform continuity of u, let $t > \tau > 0$. Then,

$$\begin{aligned} u(t) - u(\tau) = {} & (T_0(t) - T_0(\tau))u(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda F(s,u_s)\,ds \\ & - \lim_{\lambda\to\infty}\int_0^\tau T_0(\tau-s)B_\lambda F(s,u_s)\,ds. \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} u(t) - u(\tau) = {} & (T_0(t-\tau) - I)T_0(\tau)u(0) + (T_0(t-\tau) - I)\lim_{\lambda\to\infty}\int_0^\tau T_0(\tau-s)B_\lambda F(s,u_s)\,ds \\ & + \lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds, \end{aligned} \tag{2.8}$$

we have

$$u(t) - u(\tau) = (T_0(t-\tau) - I)u(\tau) + \lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds. \tag{2.9}$$

Now the range of u is relatively compact, so

$$\lim_{h\to 0}(T_0(h) - I)\xi = 0, \quad \text{uniformly in } \xi \in \overline{\{u(t) : t \geq 0\}}. \tag{2.10}$$

Consequently,

$$\lim_{\substack{t-\tau\to 0\\ t>\tau}} \|(T_0(t-\tau) - I)u(\tau)\| = 0. \tag{2.11}$$

On the other hand, we have

$$\lim_{\substack{t-\tau\to 0\\ t>\tau}} \left\| \lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda F(s,u_s)\,ds \right\| = 0. \tag{2.12}$$

Therefore,

$$\lim_{\substack{t-\tau\to 0\\ t>\tau}} \|u(t) - u(\tau)\| = 0. \tag{2.13}$$

Using a similar argument, one can also show that

$$\lim_{\substack{t-\tau\to 0\\ t<\tau}} \|u(t) - u(\tau)\| = 0. \tag{2.14}$$

From the uniform continuity of u, we determine that $\{u_t : t \geq 0\}$ is an equicontinuous family of functions on $[-r,0]$; moreover, the range of u is relatively compact. Hence, by Arzèla-Ascoli theorem, we determine that $\{u_t : t \geq 0\}$ is relatively compact in C. □

Lemma 2.5 [9]. *Let X be a Banach space, let $\Phi : X \to X$ be a continuous linear operator, let $y \in X$ be given, and let $\Theta : X \to X$ be given by $\Theta x = \Phi x + y$. Suppose that there exists $x_0 \in X$ such that $\{\Theta^n x_0 : n \in \mathbb{N}\}$ is relatively compact. Then Θ has a fixed point.*

Proof of Theorem 2.3. As usual, define the Poincaré map $P(\varphi) = x_\omega(\cdot,\varphi,f)$ on the phase space $C_0 = \{\varphi \in C : \varphi(0) \in \overline{D(A)}\}$, where $x(\cdot,\varphi,f)$ is the integral solution of (2.3). Because of the uniqueness property, it is enough to show that P has a fixed point to get an ω-periodic solution of (2.3). Also, the uniqueness property of the solution with respect to φ allows the Poincaré map P to be decomposed as

$$P(\varphi) = x_\omega(\cdot,\varphi,0) + x_\omega(\cdot,0,f), \tag{2.15}$$

where $x_\omega(\cdot,\varphi,0)$ is the integral solution of (2.3) with $f = 0$, and $x_\omega(\cdot,0,f)$ is the integral solution of (2.3) with $\varphi = 0$. Let u be the bounded solution of (2.3) on $[0,+\infty)$ and $u_0 = \varphi$. Then, by Lemma 2.4,

$$\{P^n\varphi : n \in \mathbb{N}\} = \{u_{n\omega} : n \in \mathbb{N}\} \tag{2.16}$$

is relatively compact in C_0, and the mapping P has a fixed point in C_0 using Lemma 2.5. Hence, (2.3) has an ω-periodic solution. □

3. Nonlinear case

Consider the nonlinear equation

$$\frac{d}{dt}x(t) = Ax(t) + L(t,x_t) + G(t,x_t), \quad \text{for } t \geq 0, \tag{3.1}$$

and assume the hypothesis stated below.

(H_3) G takes every bounded set into a bounded set.

Let B_ω be the space of all continuous ω-periodic functions from $\mathbb{R}^+$ into E, endowed with the uniform norm topology.

Theorem 3.1. *Assume that (H_1), (H_2), and (H_3) hold. Further, assume that there exists a positive ρ such that for any $y \in S_\rho = \{v \in B_\omega : \|v\| \leq \rho\}$, the equation*

$$\frac{d}{dt}x(t) = Ax(t) + L(t,x_t) + G(t,y_t), \quad \text{for } t \in \mathbb{R}^+, \tag{3.2}$$

has an ω-periodic integral solution in S_ρ. Then, (3.1) has an integral ω-periodic solution on $\mathbb{R}^+$.

For the proof, we need the following definition and theorem.

Definition 3.2 (see [16, Definition 9.3]). Let $\mathcal{G} : M \to 2^M$ be a multivalued map, where M is a subset of a Banach space and 2^M is the power set of M.

(i) For $D \subset M$, the inverse image $\mathcal{G}^{-1}(D)$ is the set of all $x \in M$ such that $\mathcal{G}(x) \cap D \neq \varnothing$.

(ii) The map $\mathcal{G}$ is called upper semicontinuous if $\mathcal{G}^{-1}(D)$ is closed for all closed set D in M.

Theorem 3.3 (see [16, Corollary 9.8]). *Let $\mathcal{G}: M \to 2^M$ be a multivalued map, where M is a nonempty convex set in the Banach space X such that*

(i) *the set $\mathcal{G}(x)$ is nonempty, closed, and convex for all $x \in M$,*

(ii) *the set $\mathcal{G}(M)$ is relatively compact,*

(iii) *the map $\mathcal{G}: M \to 2^M$ is upper semicontinuous.*

Then $\mathcal{G}$ has a fixed point in the sense that there exists $x \in M$ such that $x \in \mathcal{G}(x)$.

Proof of Theorem 3.1. Define the set-valued mapping $\mathcal{G}: S_\rho \to 2^{S_\rho}$, for $y \in S_\rho$, by

$$\mathcal{G}(y) = \left\{x \in S_\rho : x(t) = T_0(t)x(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds,\ t \geq 0\right\}. \tag{3.3}$$

We will show that the mapping $\mathcal{G}$ satisfies the conditions of Theorem 3.3.

(i) Let $y \in S_\rho$, $x_1, x_2 \in \mathcal{G}(y)$, and $\lambda \in [0,1]$. Then, $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{G}(y)$, which implies that $\mathcal{G}(y)$ is convex. From the continuity of L and G, we obtain that $\mathcal{G}(y)$ is a closed set.

(ii) Let $x \in \mathcal{G}(S_\rho)$, then there exists $y \in S_\rho$ such that

$$x(t) = T_0(t)x(0) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds, \quad t \geq 0. \tag{3.4}$$

We first show that $\{x(t) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E. Let $t > 0$ and $\varepsilon > 0$ such that $t > \varepsilon$. Then,

$$\begin{aligned} x(t) = {} & T_0(t)x(0) + T_0(\varepsilon)\lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds \\ & + \lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds. \end{aligned} \tag{3.5}$$

From the boundedness of L, G and (H_2), we deduce that

$$\left\{T(\varepsilon)\lim_{\lambda\to\infty}\int_0^{t-\varepsilon} T_0(t-\varepsilon-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds : x \in \mathcal{G}(S_\rho)\right\} \tag{3.6}$$

is relatively compact in E. On the other hand, for some positive constant b, we have

$$\left\|\lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)\right\|ds \leq b\varepsilon, \quad \forall x \in \mathcal{G}(S_\rho). \tag{3.7}$$

Hence, $\{x(t) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E, for every $t > 0$, and by periodicity, we also have that $\{x(0) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E. For the equicontinuity, one has, for $t > \tau > 0$,

$$\begin{aligned} \|x(t) - x(\tau)\| \leq {} & \|T_0(t) - T_0(\tau)\|\rho + \left\|\lim_{\lambda\to\infty}\int_\tau^t T_0(t-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds\right\| \\ & + \left\|\big(T_0(t-\tau) - I\big)\lim_{\lambda\to\infty}\int_0^\tau T_0(\tau-s)B_\lambda\big(L(s,x_s) + G(s,y_s)\big)ds\right\|. \end{aligned} \tag{3.8}$$

The semigroup $(T_0(t))_{t\geq 0}$ is compact, so $(T_0(t))_{t\geq 0}$ is continuous in the uniform topology whenever $t > 0$. Hence,

$$\lim_{t\to\tau} ||T_0(t) - T_0(\tau)|| = 0. \tag{3.9}$$

By (H_3), we deduce that for some positive constant c,

$$\int_\tau^t ||T_0(t-s)B_\lambda(L(s,x_s) + G(s,y_s))||ds \leq c(t-\tau), \quad \text{uniformly for } x, y \in S_\rho. \tag{3.10}$$

Since $\{x(t) : x \in \mathcal{G}(S_\rho)\}$ is relatively compact in E for every $t \geq 0$, $\{x(t) - T(t)x(0) : x \in \mathcal{G}(S_\rho)\}$ is also relatively compact in E. Moreover, there exists a compact set K in E such that

$$\lim_{\lambda\to\infty} \int_0^\tau T_0(\tau - s)B_\lambda(L(s,x_s) + G(s,y_s))ds \in K, \quad \forall x \in \mathcal{G}(S_\rho). \tag{3.11}$$

Consequently,

$$\begin{gathered} \lim_{h\to 0} (T_0(h) - I)\xi = 0, \quad \text{uniformly in } \xi \in K, \\ \lim_{\substack{t\to\tau \\ t>\tau}} \sup_{x\in\mathcal{G}(S_\rho)} ||x(t) - x(\tau)|| = 0. \end{gathered} \tag{3.12}$$

Similarly, one can also prove that

$$\lim_{\substack{t\to\tau \\ t<\tau}} \sup_{x\in\mathcal{G}(S_\rho)} ||x(t) - x(\tau)|| = 0. \tag{3.13}$$

Therefore, $\mathcal{G}(S_\rho)$ is a family of uniformly bounded and equicontinuous ω-periodic functions. By the Arzèla-Ascoli theorem, we deduce that $\mathcal{G}(S_\rho)$ is relatively compact in B_ω.

(iii) To prove that $\mathcal{G}$ is upper semicontinuous, it is enough to show that $\mathcal{G}$ is closed. Let $(y^n)_{n\geq 0}$ and $(z^n)_{n\geq 0}$ be sequences, respectively, in S_ρ and $\mathcal{G}(S_\rho)$ such that

$$y^n \longrightarrow y, \quad z^n \longrightarrow z \quad \text{as } n \longrightarrow \infty,\ z^n \in \mathcal{G}(y^n),\ \forall n \geq 0. \tag{3.14}$$

Then,

$$z^n(t) = T_0(t)z^n(0) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B_\lambda(L(s,z_s^n) + G(s,y_s^n))ds, \quad t \geq 0. \tag{3.15}$$

Letting n go to infinity and by a continuity argument, we obtain

$$z(t) = T_0(t)z(0) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B_\lambda(L(s,z_s) + G(s,y_s))ds, \quad t \geq 0. \tag{3.16}$$

Hence, $z \in \mathcal{G}(y)$, which implies that $\mathcal{G}$ is closed. Now let D be a closed set in S_ρ and take a sequence $(x_n)_n \subset \mathcal{G}^{-1}(D)$ such that $x_n \to x$ as $n \to \infty$. Since $x_n \in \mathcal{G}^{-1}(D)$, it follows that there exists $y_n \in D$ such that $y_n \in \mathcal{G}(x_n)$. Moreover, $\mathcal{G}(S_\rho)$ is compact; thus, there exists a subsequence $(y_n')_n$ of $(y_n)_n$ such that $y_n' \to y$ as $n \to \infty$. Therefore, $\mathcal{G}$ is closed and it follows that $y \in \mathcal{G}(x)$ and $y \in \mathcal{G}^{-1}(D)$. Consequently, $\mathcal{G}$ is upper semicontinuous. All the assumptions of Theorem 3.3 hold. Hence, there exists $x \in S_\rho$ such that $x \in \mathcal{G}(x)$. Finally, x is an ω-periodic solution of (3.1) on $\mathbb{R}^+$. □

To prove that (3.2) has an ω-periodic solution in S_ρ, it suffices, by Theorem 2.3, to show that it has a solution which is bounded by ρ.

Corollary 3.4. *Assume that* (H_1), (H_2), *and* (H_3) *hold. If there exists a positive* ρ *such that for any* $y \in S_\rho = \{v \in B_\omega : \|v\| \leq \rho\}$, *the nonhomogeneous linear equation* (3.2) *has an integral solution that is bounded by* ρ. *Then,* (3.1) *has an integral* ω*-periodic solution on* $\mathbb{R}^+$.

Proof. Let u be a bounded solution of (3.2) such that $u_0 = \varphi$. Following the proof of [9, Theorem 2.5], the map P has a fixed point which belongs to $\overline{\text{co}}\{P^n\varphi : n \geq 0\}$, where $\overline{\text{co}}$ denotes the closure of the convex hull. Let ψ be the fixed point of P and $x(\cdot,\psi,f)$ the associated integral solution; by virtue of the continuous dependence on the initial data, the solution $x(\cdot,\psi,f)$ is also bounded by ρ. □

4. Application

To apply the previous results, we consider the partial differential equation with delay:

$$\begin{gathered}
\frac{\partial}{\partial t}w(t,x) = \frac{\partial^2}{\partial x^2}w(t,x) + b_1(t)w(t-r,x) + b_2(t)h(w(t-r,x)) + g(t,x), \quad t \geq 0,\ x \in [0,\pi], \\
w(t,0) = w(t,\pi) = 0, \quad t \geq 0, \\
w(\theta,x) = \phi(\theta,x), \quad \theta \in [-r,0],\ x \in [0,\pi],
\end{gathered} \tag{4.1}$$

where $b_1, b_2 : \mathbb{R}^+ \to \mathbb{R}$ are continuous and ω-periodic, $h : \mathbb{R} \to \mathbb{R}$ is continuous such that

$$|h(x)| \leq k|x|, \quad x \in \mathbb{R}, \tag{4.2}$$

$g : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$ is continuous and ω-periodic in t, and $\phi : [-r,0] \times [0,\pi] \to \mathbb{R}$ is continuous. Let $Y = C([0,\pi];\mathbb{R})$ and Δ the Laplacian operator on $[0,\pi]$ with domain

$$D(\Delta) = \{z \in C([0,\pi];\mathbb{R}) : \Delta z \in C([0,\pi];\mathbb{R}),\ z(0) = z(\pi) = 0\}. \tag{4.3}$$

Then, by [6], Δ satisfies the Hille-Yosida condition in Y; more precisely, one has

$$(0,+\infty) \subset \rho(\Delta), \qquad ||R(\lambda,\Delta)|| \leq \frac{1}{\lambda}, \quad \text{for } \lambda > 0. \tag{4.4}$$

Moreover,

$$\overline{D(\Delta)} = \{z \in C([0,\pi];\mathbb{R}) : z(0) = z(\pi) = 0\} = C_0([0,\pi];\mathbb{R}). \tag{4.5}$$

Let Δ_0 be the part of Δ in $\overline{D(\Delta)}$ given by

$$D(\Delta_0) = \{z \in C_0([0,\pi];\mathbb{R}) : \Delta z \in C_0([0,\pi];\mathbb{R})\}, \qquad \Delta_0 z = \Delta z. \tag{4.6}$$

Then, by [3], Δ_0 generates a compact semigroup $(T_0(t))_{t\geq 0}$ on $C_0([0,\pi];\mathbb{R})$ such that

$$||T_0(t)|| \leq e^{-t}, \quad t \geq 0. \tag{4.7}$$

Let $L, G : \mathbb{R} \times C([-r,0];Y) \to Y$ be defined, for $t \in \mathbb{R}^+$, $\varphi \in C([-r,0];Y)$, and $x \in [0,\pi]$, by

$$\begin{aligned} (L(t,\varphi))(x) &= b_1(t)\varphi(-r)(x), \\ (G(t,\varphi))(x) &= b_2(t)h(\varphi(-r)(x)) + g(t,x). \end{aligned} \tag{4.8}$$

Then, (4.1) takes the abstract form

$$\frac{d}{dt}x(t) = \Delta x(t) + L(t,x_t) + G(t,x_t), \quad \text{for } t \geq 0. \tag{4.9}$$

Hence, (H_1), (H_2), and (H_3) are satisfied, and we have the following proposition.

Proposition 4.1. *Assume that there exists $d \in (0,1)$ such that*

$$|b_1(t)| + |b_2(t)|k \leq 1 - d, \quad \textit{for } t \in [0,\omega]. \tag{4.10}$$

Then, (4.9) has an ω-periodic solution.

Proof. Let $m = \max_{t\in[0,\omega],x\in[0,\pi]} |g(t,x)|$ and $\rho = 1 + m/d$. We claim that if y is a continuous ω-periodic function such that $\|y\| \leq \rho$, then for all φ with $\|\varphi\| < \rho$, the solution x of

$$\begin{gathered} \frac{d}{dt}x(t) = \Delta x(t) + L(t,x_t) + G(t,y_t), \quad \text{for } t \geq 0, \\ x_0 = \varphi \in C([-r,0];Y), \end{gathered} \tag{4.11}$$

satisfies $\|x(t)\| \leq \rho$, for all $t \geq 0$. Proceeding by contradiction, suppose that there exists t_1 such that $\|x(t_1)\| > \rho$ and let

$$t_0 = \inf\{t > 0 : \|x(t)\| > \rho\}. \tag{4.12}$$

By continuity, we get $\|x(t_0)\| = \rho$ and there exists $\delta > 0$ such that $\|x(t)\| > \rho$, for $t \in (t_0, t_0 + \delta)$. By using the variation of constant formula (2.2),

$$x(t_0) = T_0(t_0)\varphi(0) + \lim_{\lambda\to\infty}\int_0^{t_0} T_0(t_0 - s)B_\lambda(L(s,x_s) + G(s,y_s))ds, \quad t \geq 0. \tag{4.13}$$

By (4.8), we get that

$$\|x(t_0)\| \leq e^{-t_0}\rho + ((|b_1| + |b_2|k)\rho + m)(1 - e^{-t_0}), \tag{4.14}$$

and by condition (4.10), we obtain

$$\|x(t_0)\| \leq \rho + (m - \rho d)(1 - e^{-t_0}) \tag{4.15}$$

or $\|x(t_0)\| \leq \rho - d(1 - e^{-t_0})$, which gives that $\|x(t_0)\| < \rho$. This contradicts the definition of t_0. Consequently, $\|x(t)\| \leq \rho$ for all $t \geq 0$, and by Corollary 3.4, (4.9) has an ω-periodic solution in S_ρ. □

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