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Research Article

Algebraic Polynomials with Random Coefficients with Binomial and Geometric Progressions

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The expected number of real zeros of an algebraic polynomial $a_o+a_1x+a_2x^2+\cdots+a_nx^n$ with random coefficient a_j, $j=0,1,2,\ldots,n$ is known. The distribution of the coefficients is often assumed to be identical albeit allowed to have different classes of distributions. For the nonidentical case, there has been much interest where the variance of the jth coefficient is $\operatorname{var}(a_j)=\binom{n}{j}$. It is shown that this class of polynomials has significantly more zeros than the classical algebraic polynomials with identical coefficients. However, in the case of nonidentically distributed coefficients it is analytically necessary to assume that the means of coefficients are zero. In this work we study a case when the moments of the coefficients have both binomial and geometric progression elements. That is we assume $E(a_j)=\binom{n}{j}\mu^{j+1}$ and $\operatorname{var}(a_j)=\binom{n}{j}\sigma^{2j}$. We show how the above expected number of real zeros is dependent on values of σ^2 and μ in various cases.

1. Origin of Polynomials

Let $(\Omega,\Pr,\mathcal{A})$ be a fixed probability space, and for $\omega\in\Omega$ let $\{a_j(\omega)\}_{j=0}^{n}$ be a sequence of independent identically distributed random variables defined on Ω. There has been considerable work on obtaining the expected number of real zeros of algebraic

$$P_n(x,\omega)\equiv P_n(x)=\sum_{j=0}^{n}a_j(\omega)x^j, \tag{1.1}$$

and trigonometric $\sum_{j=0}^{n}a_j(\omega)\cos j\theta$ polynomials with random coefficients $a_j(\omega)$s. The study of the random algebraic polynomials was initiated by Kac [1], and the recent works include [2, 3]. It is shown that under general assumptions for the distribution of coefficients the expected number of real zeros is asymptotic to $(2/\pi)\log n$ as $n\to\infty$. For the case of random trigonometric polynomials, Dunnage [4] obtained the first result which was later generalized by Wilkins Jr. [5, 6] and recently studied in [7, 8]. It is shown that, again for a wide class of distributions for the coefficients, there are significantly more real zeros in the case of trigonometric polynomial compared with the algebraic case. The asymptotic value for the expected number of zeros for the latter case is $2n/\sqrt{3}$. Besides the comprehensive book of Bharucha-Reid and Sambandham [9], the earlier results of general topics on random polynomials are reviewed in [10].

Motivated by the interesting work of Edelman and Kostlan [11], who, among others, considered polynomials of the form $\sum_{j=0}^{n} a_j(\omega)\binom{n}{j}^{1/2} x^j$, [2, 12] obtained many characteristics, like the number of real zeros or the number of maxima of these types of polynomials. This is interesting as they showed that for this case of nonidentically distributed coefficients the expected number of real zeros is $O(\sqrt{n})$, which is significantly more than the classical algebraic case but less than that of trigonometric polynomials. Also in this direction of nonidentical coefficients, a case in which the mean of coefficients $a_j(\omega)$ increases with j is studied in [3, 13]. Now it would be interesting to study a random polynomial formed by combining the above two distribution laws. It is natural to ask, for instance, what would be the behavior of $P_n(x)$ in (1.1) if for constants μ and σ the mean and variance of coefficients are $E(a_j(\omega)) = \binom{n}{j}\mu^{j+1}$ and $\operatorname{var}(a_j(\omega)) = \binom{n}{j}\sigma^{2j}$.

With the latter assumption of the distribution of the coefficients, we first show that if $\mu = 0$, the expected number of real zeros of $P_n(x)$ denoted by $EN_{n,P}(0,\infty) \equiv EN_n(0,\infty)$ is independent of σ. The case of nonzero μ is studied in Theorem 1.2. The analysis for the general case is complicated, and we only give the result for a case that $\mu = \sigma^2$. We prove the following theorem.

Theorem 1.1. *For $\mu = 0$ and $\sigma^2 > 0$, the expected number of real zeros of $P_n(x)$ is independent of σ^2. That is*

$$EN_n(-\infty,0) = EN_n(0,\infty) = \frac{\sqrt{n}}{2}. \tag{1.2}$$

The analysis for the case of $\mu \neq 0$ would be complicated. Without loss of much generality and certainly interest, we restrict ourselves to the case of $\mu = \sigma^2$. We prove the following theorem.

Theorem 1.2. *The expected number of real zeros of $P_n(x)$ for different values of μ satisfies*

$$EN_n(0,\infty)\begin{cases} = O(1), & \text{if } \mu = \sigma > 1, \\ \sim \left(\dfrac{\sqrt{n}}{2}\right)\left\{1 - \arctan\left(\dfrac{2\sqrt{\mu}}{1-\mu}\right)\right\}, & \text{if } 0 < \mu = \sigma < 1. \end{cases} \tag{1.3}$$

For x negative and for every $\mu = \sigma^2$,

$$EN_n(-\infty,0) \sim \frac{\sqrt{n}}{2}. \tag{1.4}$$

2. Moments

In order to obtain the expected number of real zeros we use a generalization of the well-known Kac-Rice formula initiated in [1, 14, 15]. To this end, we need the following moments of $P_n(x)$ and its dervative $P_n'(x)$. First, we assume the general assumptions on the means and the variances of coefficients as stated above. That is, $E(a_j) = \binom{n}{j}\mu^{j+1}$ and $\operatorname{var}(a_j) = \binom{n}{j}\sigma^{2j}$. Since these coefficients are independent, it is easy to show

$$\alpha = E(P_n(x)) = \mu\sum_{j=0}^{n}\binom{n}{j}(x\mu)^j = \mu(1+\mu x)^n, \tag{2.1}$$

$$\beta = E(P_n'(x)) = \mu^2\sum_{j=0}^{n} j\binom{n}{j}(\mu x)^{j-1} = n\mu^2(1+\mu x)^{n-1}, \tag{2.2}$$

$$A^2 = \operatorname{var}(P_n(x)) = \sum_{j=0}^{n}\binom{n}{j}(\sigma x)^{2j} = (1+\sigma^2x^2)^n, \tag{2.3}$$

$$B^2 = \operatorname{var}(P_n'(x)) = \sigma^2\sum_{j=0}^{n} j^2\binom{n}{j}(\sigma x)^2 = n\sigma^2(1+\sigma^2x^2)^{n-2}(1+n\sigma^2x^2), \tag{2.4}$$

and finally

$$C = \operatorname{cov}(P_n(x), P_n'(x)) = \sigma^2 x\sum_{j=0}^{n} j\binom{n}{j}(\sigma x)^{2j-2} = n\sigma^2 x(1+\sigma^2x^2)^{n-1}. \tag{2.5}$$

Then from (2.3)–(2.5) we can obtain

$$\Delta^2 = A^2B^2 - C^2 = n\sigma^2(1+\sigma^2x^2)^{2n-2}. \tag{2.6}$$

With the above notations, we can now write the Kac-Rice for the expected number of real zeros of $P_n(x)$ in the interval (a,b) as, see also [10, page 43],

$$EN_n(a,b) = I_1(a,b) + I_2(a,b), \tag{2.7}$$

where

$$I_1(a,b) = \int_a^b \frac{\Delta}{\pi A^2}\exp\left(-\frac{\alpha^2B^2+\beta^2A^2-2\alpha\beta C}{2\Delta^2}\right)dx, \tag{2.8}$$

$$I_2(a,b) = \int_a^b \frac{\sqrt{2}|\beta A^2 - C\alpha|}{\pi A^3}\exp\left(-\frac{\alpha^2}{2A^2}\right)\operatorname{erf}\left(\frac{|\beta A^2 - C\alpha|}{\sqrt{2}A\Delta}\right)dx, \tag{2.9}$$

where as usual $\operatorname{erf}(x) = \int_0^x \exp(-t^2)dt$. Now we can progress and evaluate further the following terms required in the Kac-Rice formulae (2.7)–(2.9). From (2.1)–(2.5) we can derive

$$\begin{aligned}\alpha^2B^2 &+ \beta^2A^2 - 2\alpha\beta C\\ &= n\mu^2(1+\mu x)^{2n-2}(1+\sigma^2x^2)^{n-2}(n\sigma^4x^2+\sigma^2+2\mu x\sigma^2+\mu^2x^2\sigma^2+n\mu^2-2n\mu\sigma^2x).\end{aligned} \tag{2.10}$$

This together with (2.6) yields

$$\frac{\alpha^2 B^2 + \beta^2 A^2 - 2\alpha\beta C}{2\Delta^2} = \frac{\mu^2(1+\mu x)^{2n-2}\{n(\sigma^2 x - \mu)^2 + \sigma^2(1+\mu x)^2\}}{2\sigma^2(1+\sigma^2 x^2)^n}. \tag{2.11}$$

3. Proof of Theorems

First in the case of $\mu = 0$ from (2.7) and (2.3)–(2.6) by letting $y = \sigma x$ we can show

$$\begin{aligned} EN_n(0,\infty) &= \frac{\sqrt{n}}{\pi}\int_0^\infty \frac{\sigma}{1+\sigma^2 x^2}\,dx \\ &= \frac{\sqrt{n}}{\pi}\int_0^\infty \frac{dy}{1+y^2} = \frac{\sqrt{n}}{2}. \end{aligned} \tag{3.1}$$

This proves Theorem 1.1. Now we proceed with the more general case of $\mu \neq 0$. As explained above, in order to simplify the analysis we let $\mu = \sigma^2$. This yields (2.11) to

$$\frac{\alpha^2 B^2 + \beta^2 A^2 - 2\alpha\beta C}{2\Delta^2} = f_n(x,\mu) g_n(x,\mu), \tag{3.2}$$

where

$$g_n(x,\mu) = n\mu^2(x-1)^2 + \mu(1+\mu x)^2, \tag{3.3}$$

and for all sufficiently large n,

$$f_n(x,\mu) = \frac{(1+\mu x)^{2n-2}}{(1+\mu x^2)^n} \sim \frac{(1+\mu x)^{2n}}{(1+\mu x^2)^n} = \left(\frac{1+\mu^2 x^2 + 2\mu x}{1+\mu x^2}\right)^n. \tag{3.4}$$

Now we assume $x > 0$. Then if we let $\mu > 1$, since

$$\frac{1+\mu^2 x^2 + 2\mu x}{1+\mu x^2} > 1, \tag{3.5}$$

we can see that $f_n(x,\mu) \to \infty$ as $n \to \infty$. Therefore the exponential term that appears in (2.8) tends to zero exponentially fast. Hence the only contribution to $EN_n(0,\infty)$ is from $I_2(0,\infty)$. In the following, we show that the latter is $O(1)$. To this end, we note that since from the definition for all x, $\operatorname{erf}(x) \leq \sqrt{\pi}/2$, then

$$I_2(0,\infty) < \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\beta A^2 - C\alpha}{A^3}\exp\left(-\frac{\alpha^2}{2A^2}\right)dx. \tag{3.6}$$

Now we let $u = \alpha/A$, and since $(d/dx)(\alpha/A) = (\beta A^2 - \alpha C)/A^3$ from (3.6) we obtain

$$I_2(0,\infty) < \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{u^2}{2}\right)du \leq \frac{1}{2}. \tag{3.7}$$

This completes the first part of Theorem 1.2. On the other hand, if $\mu < 1$, the behavior of $f_n(x,\mu)$ will depend on x. That is for $0 < x < 2/(1-\mu)$, $f_n(x,\mu) \to \infty$ as $n \to \infty$ and for $x > 2/(1-\mu)$, $f_n(x,\mu) \to 0$ as $n \to \infty$. Therefore the only contribution to $EN_n(0,\infty)$ from I_1 is in the interval $(2/(1-\mu),\infty)$ as $I_1(0,2/(1-\mu))$ will tend to zero exponentially fast. Also for $v = \sqrt{\mu}x$,

$$\begin{aligned} I_1\left(\frac{2}{1-\mu},\infty\right) &\sim \frac{\sqrt{n}}{\pi}\int_{2/(1-\mu)}^\infty \frac{\sqrt{\mu}}{1+\mu x^2}dx = \frac{\sqrt{n}}{\pi}\int_{2/(1-\mu)}^\infty \frac{dv}{1+v^2} \\ &\sim \left(\frac{\sqrt{n}}{2}\right)\left\{1 - \arctan\left(\frac{2\sqrt{\mu}}{1-\mu}\right)\right\}. \end{aligned} \tag{3.8}$$

The above argument for $I_2(0,\infty)$ in (3.7) remains valid, and therefore we have proof of the first part of Theorem 1.2.

For $x < 0$ without loss of generality, we only consider the case of $\mu > 0$ (since $\mu = \sigma^2$). For this case $g_n(x,\mu)$ remains positive. However, for $x^2 > \epsilon/\mu$, where for $a = 1 - \log\log n^{10}/\log n$ we let $\epsilon = n^{-a}$, we have, (see also [10, page 31]),

$$(1+\mu x^2)^n > (1+\epsilon)^n \{(1+n^{-a})^{n^a}\}^{n^{1-a}} = \exp(n^{1-a}) \sim n^{10}. \tag{3.9}$$

Hence

$$f_n(x,\mu) < (1+\mu x^2)^{-n} < n^{-10}, \tag{3.10}$$

which tends to zero very fast as $n \to \infty$. Therefore the exponential term in I_1 tends to be one, and hence

$$I_1\left(\sqrt{\frac{\epsilon}{\mu}},\infty\right) \sim \sqrt{\frac{n}{\pi}}\int_{\sqrt{\epsilon/\mu}}^\infty \frac{dx}{1+x^2} \sim \frac{\sqrt{n}}{2}. \tag{3.11}$$

Also in the interval $(0,\sqrt{\epsilon/\mu})$,

$$I_1\left(0,\sqrt{\frac{\epsilon}{\mu}}\right) < \int_0^{\sqrt{\epsilon/\mu}} \frac{\Delta}{\pi A^2}\,dx < \frac{\sqrt{n\mu}}{\pi}\int_0^{\sqrt{\epsilon/\mu}} \frac{dx}{1+\mu x^2} \sim \frac{\sqrt{n}}{2\pi}\arctan\epsilon, \tag{3.12}$$

which is small. This completes the proof of Theorem 1.2.

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