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ON THE EXISTENCE OF SOLUTIONS FOR VOLTERRA INTEGRAL INCLUSIONS IN BANACH SPACES[1]

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ABSTRACT

In this paper we examine a class of nonlinear integral inclusions defined in a separable Banach space. For this class of inclusions of Volterra type we establish two existence results, one for inclusions with a convex-valued orientor field and the other for inclusions with nonconvex-valued orientor field. We present conditions guaranteeing that the multivalued map that represents the right-hand side of the inclusion is α-condensing using for the proof of our results a known fixed point theorem for α-condensing maps.

Key words: Volterra integral inclusions, Aumann selection theorem, radial retraction, α-condensing map.

AMS (MOS) subject classifications: 35R15, 34G20, 34A60.

1. INTRODUCTION-PRELIMINARIES

In this paper we examine a class of nonlinear integral inclusions defined in a separable Banach space and we establish two existence results. One for inclusions with a convex-valued orientor field and the other for inclusions with a nonconvex valued orientor field. Our work extends existence results of Ragimkhanov [11] and Lyapin [7] and the infinite dimensional results of Chuong [3] and Papageorgiou [10], where the hypotheses on the orientor field $F(t,x)$ are too restrictive (see theorem 3.1 of Chuong and theorems 3.1-3.3 of Papageorgiou).

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this work we will be using the following notations:

$$P_{f(c)} = \{A \subseteq X\colon \text{ nonempty, closed (convex)}\}$$

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and

$$P_{(w)k(c)}(X) = \{A \subseteq X: \text{nonempty, } (w) \text{ compact, (convex)}\}.$$

A multifunction $F: \Omega \rightarrow P_f(X)$ is said to be measurable (see Wagner [13]), if for every $x \in X$, $\omega \rightarrow d(x, F(\omega)) = inf\{\| x - z \| : z \in F(\omega)\}$ is measurable. When there is a σ-field measure $\mu(\cdot)$ on (Ω, Σ) and Σ is μ-complete, then the above definition of measurability is equivalent to saying that $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability).

By S_F we will denote the set of measurable selectors of $F(\cdot)$ while by S_F^p $(1 \leq p \leq \infty)$ the set of measurable selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(X)$, i.e. $S_F^p = \{f \in L^p(X): f(\omega) \in F(\omega) \mu\text{-a.e.}\}$. This set may be empty. It is nonempty if and only if $\omega \rightarrow inf\{\| z \| : z \in F(\omega)\} \in L_+^p$.

In particular this is the case if $\omega \rightarrow | F(\omega) | = sup\{\| z \| : z \in F(\omega)\} \in L_+^p$ in which case we say that $F(\cdot)$ is L^p-integrably bounded.

If Y, Z are Hausdorff topological spaces and $G: Y \rightarrow 2^Z \setminus \{\emptyset\}$ then we say that $G(\cdot)$ is lower semicontinuous (*l.s.c.*), if for all $U \subset Z$ open, the set $G^-(U) = \{y \in Y: G(y) \cap U \neq \emptyset\}$ is open in Y.

If furthermore Y, Z are metric spaces, then the above definition is equivalent to saying that for all $y_n \rightarrow y$ we have $G(y) \subseteq \underline{lim} G(y_n) = \{z \in Z: z = lim z_n, z_n \in G(y_n)\}$.

Also the multifunction $F: Y \rightarrow 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous (*u.s.c.*) if and only if for every $W \subseteq Z$ open, the set $F^+(W) = \{y \in Y: F(y) \subset W\}$ is open in Y.

Finally we say that a multifunction $G: Y \rightarrow 2^Z \setminus \{\emptyset\}$ is closed if and only if the set $GrG = \{(y, z) \in z \in G(y)\}$ is closed in $Y \times Z$.

2. EXISTENCE THEOREMS

Let $T = [0, b]$, $b > 0$ and let X be a separable Banach space. The integral inclusion of Volterra type which we will be studying is the following:

$$x(t) \in p(t) + \int_0^t K(t,s) F(s, x(s)) ds, t \in T \qquad (*)$$

where $p(\cdot) \in C(T,X)$.

By a solution of $(*)$ we understand a function $x(\cdot) \in C(T,X)$ such that

$$x(t) \in p(t) + \int_0^t K(t,s)f(s)ds,\ t \in T, \text{ with } f \in S^1_{F(\cdot,x(\cdot))}.$$

First we prove an existence result for the case where the orientor field $F(t,x)$ is convex valued. For that purpose we will need the following hypotheses on the data of $(*)$.

H(F): $F: T \times X \to P_{wkc}(X)$ is a multifunction such that:

(1) $t \to F(t,x)$ is measurable,

(2) $x \to F(t,x)$ is *u.s.c.* from X into X_w,

(3) $|F(t,x)| \leq a(t) + b(t)\|x\|$ *a.e.* with $a(\cdot), b(\cdot) \in L^1_+$,

(4) for any $\epsilon > 0$ and $V \subseteq X$ bounded, there exists $I_\epsilon \subseteq T$ open such that $\mu(I_\epsilon) \leq \epsilon$ and $\alpha(F(J \times V)) \leq sup_{t \in J}\eta(t)\alpha(V)$ for any $J \subseteq T \backslash I_\epsilon$ closed and with $\eta(\cdot) \in L^1_+$.

Remark: We can replace the sublinear growth condition $H(F)(3)$ by a hypothesis of the form "for every $B \subseteq X$ bounded there exists $a_B(\cdot) \in L^1_+$ *s.t.* $sup_{x \in B}|F(t,x)| \leq a_B(t)$". In this case though the existence result is only local.

H(K): $K: T \times T \to \mathcal{L}(X)$ is continuous (we can have K defined only on Δ and set $K(t,s) = K(t,t),\ t \leq s$).

Now we are ready for our first result:

Theorem 1: *<u>If</u> hypotheses $H(F)$ and $H(K)$ hold and $M\|\eta\|_1 \leq 1$ where*

$$\|K(t,s)\|_{\mathcal{L}} \leq M,$$

<u>then</u> $(*)$ admits a solution.

Proof: First we will establish an a priori bound for the solutions of $(*)$. So let $x(\cdot) \in C(T,X)$ be such a solution. We have:

$$\|x(t)\| \leq \|p\|_\infty + \int_0^t M|F(s,x(s))|ds$$

for all $t \in T$ and with $\|K(t,s)\|_{\mathcal{L}} \leq M$ for all $(t,s) \in \Delta$ (see hypothesis $H(K)$).

Using hypothesis $H(F)(3)$, we get:

$$\| x(t) \| \leq \| p \|_{\infty} + \int_0^t (Ma(s) + Mb(s) \| x(s) \|)ds, \quad t \in T.$$

Invoking Gronwall's inequality, we get $M_1 > 0$ *s.t.*

$$\| x(t) \| \leq M_1$$

for all $t \in T$ and all solutions $x(\cdot) \in C(T, X)$ of $(*)$.

Let $\widehat{F}(t,x) = F(t, p_{M_1}(x))$, with $p_{M_1}(\cdot)$ being the M_1-radial retraction. We will consider the integral inclusion $(*)$ with the orientor field $F(t,x)$ replaced by $\widehat{F}(t,x)$. Note that because of hypothesis $H(F)(1)$ the multifunction $t \rightarrow \widehat{F}(t,x)$ is also measurable. Also recalling that $p_{M_1}(\cdot)$ is Lipschitz continuous and using hypothesis $H(F)(2)$, we get from theorem 7.3.11 (*ii*), p. 87 of Klein-Thompson [5], that $x \rightarrow \widehat{F}(t,x)$ is *u.s.c.* from X into X_w. Furthermore $|\widehat{F}(t,x)| \leq a(t) + b(t)M_1 = \phi(t)$ *a.e.* with $\phi(\cdot) \in L^1_+$.

Finally in hypothesis $H(F)(4)$ we have

$$\alpha(\widehat{F}(J \times V)) = \alpha(F(J \times p_{M_1}(V)) \leq sup_{t \in J} \eta(t) \alpha(p_{M_1}(V)).$$

But note that $p_{M_1}(V) \subseteq \overline{conv}[\{0\} \cup V] \Rightarrow \alpha(p_{M_1}(V)) = \alpha[\{0\} \cup V] \leq \alpha(V)$. So we have $\alpha(\widehat{F}(J \times V)) \leq sup_{t \in J} \eta(t) \alpha(V)$ and so we have checked that $\widehat{F}(t,x)$ satisfies hypothesis $H(F)(4)$.

Set

$$H = \{y \in C(T,X): y(t) = p(t) + \int_0^t K(t,s)g(s)ds, \quad t \in T, \ \| g(t) \| \leq \phi(t) \ a.e.\}$$

Next let $R: H \rightarrow 2^H$ be defined by

$$R(x) = \{y \in C(T,X): y(t) = p(t) + \int_0^t K(t,s)f(s)ds, t \in T, f \in S^1_{\widehat{F}(\cdot, x(\cdot))}\}.$$

First we will show that $R(\cdot)$ has nonempty values. Let $\{s_n\}_{n \geq 1}$ be simple functions such that $s_n(t) \xrightarrow{s} x(t)$ *a.e.* in X.

Then for each $n \geq 1$, $t \rightarrow \widehat{F}(t, s_n(t))$ is measurable (since $t \rightarrow \widehat{F}(t,x)$ is measurable). So by Aumann's selection theorem (see Wagner [13], theorem 5.10), we get $f_n: T \rightarrow X$ measurable such that $f_n(t) \in \widehat{F}(t, s_n(t))$. Clearly $f_n \in L^1(X)$. Note that because $\widehat{F}(t, \cdot)$ is *u.s.c.* from X into X_w,

$U(t) = \overline{\bigcup_{n \geq 1} F(t, s_n(t))}^w \in P_{wk}(X)$ (see Klein-Thompson [5], theorem 7.4.2, p. 90) and $t \to U(t)$ is measurable. Hence $t \to \overline{conv}U(t) \equiv U_c(t)$ is an integrably bounded, $P_{wkc}(X)$-valued multifunction (Krein-Smulian theorem). So from Papageorgiou [8] (see proposition 3.1), we get that $S^1_{U_c}$ is w-compact in $L^1(X)$.

But observe that $\{f_n\}_{n \geq 1} \subseteq S^1_{U_c}$. So by passing to a subsequence, if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. Then from [9] (see theorem 3.1), we get that

$$f(t) \in \overline{conv}w\text{-}\overline{lim}\{f_n(t)\}_{n \geq 1} \subseteq \overline{conv}w\text{-}\overline{lim}\widehat{F}(t, s_n(t))$$
$$\subseteq \widehat{F}(t, x(t)) \text{ a.e.}$$

the last inclusion following from the upper semicontinuity of $\widehat{F}(t, \cdot)$ from X into X_w, the fact that $s_n(t) \xrightarrow{s} x(t)$ *a.e.* in X and the fact that $\widehat{F}(\cdot, \cdot)$ is $P_{wkc}(X)$-valued. So $S^1_{\widehat{F}(\cdot, x(\cdot))} \neq \emptyset \Rightarrow R(x) \neq \emptyset$ for all $x \in C(T, X)$. Also since $S^1_{\widehat{F}(\cdot, x(\cdot))} \in P_{wkc}(L^1(X))$ (see proposition 3.1 of [8]), we can easily check that $R(\cdot)$ has closed, convex values in $2^{C(T,X)} \backslash \{\emptyset\}$.

Next we will show that $R(\cdot)$ has a closed graph. To this end let $[x_n, y_n] \in GrR$ and assume that $[x_n, y_n] \to [x, y]$ in $C(T, X) \times C(T, X)$. Then by definition for every $n \geq 1$ we have

$$y_n(t) = p(t) + \int_0^t K(t, s) f_n(s) ds, \text{ for } t \in T \text{ and with } f_n \in S^1_{\widehat{F}(\cdot, x_n(\cdot))}.$$

Note that by the Krein-Smulian theorem (see for example Diestel-Uhl [4], theorem II, p. 51), we have that $\overline{conv} \bigcup_{n \geq 1} \widehat{F}(t, x_n(t)) \in P_{wkc}(X)$ for all $t \in T$. So from proposition 3.1 of [8] and by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^1(X)$. Then as above using theorem 3.1 of [9] and the properties of $\widehat{F}(t, x)$, we get

$$f(t) \in \overline{conv}w\text{-}\overline{lim}\{f_n(t)\}_{n \geq 1} \subseteq \overline{conv}w\text{-}\overline{lim}\widehat{F}(t, x_n(t)) \subseteq \widehat{F}(t, x(t)), \text{ a.e.}$$

Also $\int_0^t K(t, s) f_n(s) ds \xrightarrow{w} \int_0^t K(t, s) f(s) ds$ in X. Hence in the limit as $n \to \infty$ we get:

$$y(t) = p(t) + \int_0^t K(t, s) f(s) ds, \; t \in T$$

with $f \in S^1_{\widehat{F}(\cdot, x(\cdot))}$. Therefore $[x,y] \in GrR \Rightarrow R(\cdot)$ has a closed graph.

Next by Lusin's theorem, given $\epsilon > 0$ there exists $I^1_\epsilon \subseteq T$ open such that $\lambda(I^1_\epsilon) < \epsilon/2$, $\eta \mid_{T \setminus I^1_\epsilon} \in C$ and $\| \phi \chi_{I_\epsilon} \|_1 \leq \epsilon/2M$. Also from hypothesis $H(F)(4)$ (which as we have already checked earlier, is also valid for the orientor field $\widehat{F}(t,x)$), given V a nonempty subset of H we can find $I^2_\epsilon \subseteq T$ open with $\lambda(I^2_\epsilon) < \epsilon/2$ and

$$\alpha(F(J \times \widehat{V})) \leq sup_{s \in J} \eta(s) \alpha(\widehat{V}) \text{ and } \| \phi \chi_{I^2_\epsilon} \|_1 \leq \epsilon/2M$$

where $J \subseteq L = T \setminus I_\epsilon$ closed, with $I_\epsilon = I^1_\epsilon \cup I^2_\epsilon$ and $\widehat{V} = \{x(t): x \in V, t \in T\}$.

Note that because of hypothesis $H(K)$ and since by the choice of $L \eta \mid_L$ continuous, the map $(s,w) \rightarrow \| K(t,s) \|_{\mathfrak{L}} \eta(w)$ is continuous, hence uniformly continuous on $([0,t] \cap L) \times L$. Thus given $\delta > 0$ we can find $\theta > 0$ *s.t.*

$$\left| \, \| K(t,s) \|_{\mathfrak{L}} \eta(w) \alpha(\widehat{V}) - \| K(t,\tau) \|_{\mathfrak{L}} \eta(z) \alpha(\widehat{V}) \, \right| \leq \delta \tag{1}$$

for all $s, \tau \in [0,t] \cap L$ with $|s - \tau| \leq \theta$ and all $w, z \in L$ with $|w - z| \leq \theta$.

Let $0 = t_0 < t_1 < \ldots < t_n = b$ be a subdivision of T into $(n+1)$-parts such that $t_i - t_{i-1} \leq \theta$ and let $L_i = [t_{i-1}, t_i] \setminus I_\epsilon$ $i = 1, 2, \ldots n$.

Also let $v_i \in L_i$ and $s_i \in L_i$ $i = 1, 2, \ldots, n$ be such that

$$\| K(t, v_i) \|_{\mathfrak{L}} = sup_{s \in L_i} \| K(t,s) \|_{\mathfrak{L}}$$

and $\eta(s_i) = sup_{s \in L_i} \eta(s)$. Their existence is guaranteed by hypothesis $H(K)$ and since $\eta \mid_L$ is continuous. Then we have:

$$\alpha[\widehat{F}(L_i \times \widehat{V})] \leq \eta(s_i) \alpha(\widehat{V}).$$

Also from the "Mean Value Theorem" for Bochner integrals (see Diestel-Uhl, [4], corollary 8, p. 48), we have:

$$\{ \int_{L_i} K(t,s) \widehat{F}(s, x(s)) ds : x \in V \} \subseteq \mu(L_i) \overline{conv}[K(t,s) \widehat{F}(s,y) : s \in S_i, y \in \widehat{V}].$$

So we have

$$\{ \int_L K(t,s) \widehat{F}(s, x(s)) ds : x \in V \} \subseteq \sum_{i=1}^n \mu(L_i) \overline{conv}[K(t,s) \widehat{F}(s,y) : s \in S_i, \ \ y \in \widehat{V}].$$

Using the subadditivity of the $\alpha(\cdot)$ measure of non-compactness, we get

$$\alpha\{\int_L K(t,s)\widehat{F}(s,x(s))ds: x \in V\} \leq \sum_{i=1}^{n} \mu(L_i) \| K(t,v_i) \|_{\mathcal{L}} \alpha[\widehat{F}(L_i,\widehat{V})]$$

$$\leq \sum_{i=1}^{n} \mu(L_i) \| K(t,v_i) \|_{\mathcal{L}} \eta(s_i)\alpha(\widehat{V}).$$

From (1) above we get

$$\alpha\{\int_L K(t,s)\widehat{F}(s,x(s))ds: x \in V\} \leq \int_L \| K(t,s) \|_{\mathcal{L}} \eta(s)\alpha(\widehat{V})d\tau + \delta\mu(L).$$

Also recall from the initial choice of the sets I_ϵ^1 and I_ϵ^2 that

$$\int_{I_\epsilon} \| K(t,s) \|_{\mathcal{L}} \phi(s)ds \leq \epsilon.$$

So finally we have:

$$\alpha(R(V)(t)) \leq \int_L \| K(t,s) \|_{\mathcal{L}} \eta(s)\alpha(\widehat{V})ds + \delta\mu(S) + \epsilon$$

$$\leq \int_0^b M\eta(s)\alpha(\widehat{V})ds + \delta\mu(S) + \epsilon.$$

Since $\epsilon, \delta > 0$ were arbitrary, we get

$$\alpha(R(V)(t)) \leq \int_0^b M\eta(s)\alpha(\widehat{V})ds = \alpha(\widehat{V})M \| \eta \|_1$$

Since H is bounded and equicontinuous, from Ambrosetti's theorem (see theorem 1.4.2 p. 20 of Lakshmikantham-Leela [6]) we have that

$$\alpha(\widehat{V}) \leq \widehat{\alpha}(V)$$

and $sup_{t \in T}\alpha(R(V)(t)) = \widehat{\alpha}(R(V))$. Thus we get

$$\widehat{\alpha}(R(V)) \leq M \| \eta \|_1 \widehat{\alpha}(V).$$

Since by hypothesis $M \| \eta \|_1 < 1$, we get that $R(\cdot)$ is $\widehat{\alpha}(\cdot)$-condensing. Apply theorem 4.1 of Tarafdar-Vyborny [12], to get $x \in R(x)$. Then $x \in C(T,X)$ solves $(*)$ with the orientor field $\widehat{F}(t,x)$. Using the definition of $\widehat{F}(t,x)$ and same estimation as in the beginning of the proof, we get that

$$\| x(t) \| \leq M_1 \Rightarrow \widehat{F}(t,x(t)) = F(t,x(t)) \Rightarrow x(\cdot) \in C(T,X) \text{ solves } (*).$$

<u>*Q.E.D.*</u>

We can have a variant of Theorem 1, where the orientor field is not convex-valued. For this we will need the following hypothesis.

H(F)': $F: T \times X \rightarrow P_k(X)$ is a multifunction such that

(1) $(t,x) \rightarrow F(t,x)$ is graph measurable,

(2) $x \rightarrow F(t,x)$ is *l.s.c.*,

and the hypotheses $H(F)(3)$ and (4) also hold.

Theorem 2: *If hypotheses $H(F)'$ and $H(K)$ hold and $M \|\eta\|_1 < 1$, then $(*)$ admits a solution.*

Proof: As in the proof of Theorem 1, we can show that for every solution $x(\cdot) \in C(T,X)$ of $(*)$, we have $\|x\|_{C(T,X)} \leq M_1$. Then define $\widehat{F}(t,x) = F(t, p_{M_1}(x))$. This has the same measurability and continuity properties as $F(t,x)$ satisfies $H(F)(4)$, (see the proof of Theorem 1) and $|\widehat{F}(t,x)| \leq \phi(t)$ *a.e.* with $\phi(\cdot) \in L^1_+$.

Let $\Gamma: C(T,X) \rightarrow P_f(L^1(X))$ be defined by

$$\Gamma(x) = S^1_{\widehat{F}(\cdot, x(\cdot))}.$$

Then from Papageorgiou [9] (see theorem 4.1) we get that $\Gamma(\cdot)$ is *l.s.c.* Apply theorem 3 of Bressan-Colombo [2] to get a continuous map $\gamma: C(T,X) \rightarrow L^1(X)$ such that $\gamma(x) \in \Gamma(x)$ for all $x \in C(T,X)$.

As in the proof of Theorem 1, let

$$H = \{y \in C(T,X): y(t) = p(t) + \int_0^t K(t,s)g(s)ds, t \in T, \|g(t)\| \leq \phi(t) \text{ a.e.}\}.$$

This is bounded and equicontinuous. Let $R: H \rightarrow H$ be defined by

$$R(x)(t) = p(t) + \int_0^t K(t,s)\gamma(x)(s)ds.$$

Since $\gamma(\cdot)$ is continuous, we can easily check that $R(\cdot)$ is continuous too. From the proof of Theorem 1, we know that it is $\widehat{\alpha}$-condensing. So there exists $x = R(x)$. This is the desired solution of $(*)$.

Q.E.D.

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