Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 673275, 8 pages
doi:10.1155/2008/673275

Research Article

On the Plane Geometry with Generalized Absolute Value Metric

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Received 7 March 2008; Revised 5 June 2008; Accepted 1 July 2008

Recommended by Mehrdad Massoudi

Metric spaces are among the most important widely studied topics in mathematics. In recent years, Mathematicians began to investigate using other metrics different from Euclidean metric. These metrics also find their place computer age in addition to their importance in geometry. In this paper, we consider the plane geometry with the generalized absolute value metric and define trigonometric functions and norm and then give a plane tiling example for engineers underlying Schwarz's inequality in this plane.

1. Introduction

In the late 1800's, Minkowski [1] published a whole family of metrics providing new insight into the study of plane geometry. In recent years, mathematicians and engineers begin to investigate non-Euclidean metrics used in real-life applications. For example, Eisenberg and Khabbaz [2] discussed the applications of taxicab metric in geometry and network theory. Similarly, Burman et al. [3] utilized the concept of λ-geometry and discussed its applications in the problem of global routing of multiterminal nets.

Our goal is to further develop the plane geometry with the *generalized absolute value metric* d_g suggested by Kaya et al. [4], and to see how it can lead to future researches in related areas. In $\mathbb{R}^2$, the d_g-distances between $X=(x_1,y_1)$ and $Y=(x_2,y_2)$are defined by

$$\begin{aligned} d_g(X,Y) &= d_g\big((x_1,y_1),(x_2,y_2)\big) \\ &= a\,\max\big\{\lvert x_1-x_2\rvert,\lvert y_1-y_2\rvert\big\}+b\,\min\big\{\lvert x_1-x_2\rvert,\lvert y_1-y_2\rvert\big\} \end{aligned} \tag{1.1}$$

for all $a,b\in\mathbb{R}$, $a\ge b\ge 0$, $a\neq 0$. According to the definition of d_g-distances, the shortest way between two points X and Y is the union of a vertical or a horizontal line segment and a line segment with the slope $\pm 2ab/(a^2-b^2)$.

We will show that the d_g-distance defined above is a metric. First, it is positive definite and symmetric, and it can be shown that d_g satisfies the triangle inequality as follows. For $X = (x_1, y_1)$, $Y = (x_2, y_2)$, and $Z = (x_3, y_3)$, since

$$\begin{aligned} \left|x_1 - x_2\right| &= \left|x_1 - x_3 + x_3 - x_2\right| \le \left|x_1 - x_3\right| + \left|x_3 - x_2\right|, \\ \left|y_1 - y_2\right| &= \left|y_1 - y_3 + y_3 - y_2\right| \le \left|y_1 - y_3\right| + \left|y_3 - y_2\right|, \end{aligned} \tag{1.2}$$

one obtains

$$\begin{aligned} d_g(X, Y) &= a \max\left\{\left|x_1 - x_2\right|, \left|y_1 - y_2\right|\right\} + b \min\left\{\left|x_1 - x_2\right|, \left|y_1 - y_2\right|\right\} \\ &= a \max\left\{\left|x_1 - x_3 + x_3 - x_2\right|, \left|y_1 - y_3 + y_3 - y_2\right|\right\} \\ &\quad + b \min\left\{\left|x_1 - x_3 + x_3 - x_2\right|, \left|y_1 - y_3 + y_3 - y_2\right|\right\} \\ &\le a \max\left\{\left|x_1 - x_3\right| + \left|x_3 - x_2\right|, \left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right\} \\ &\quad + b \min\left\{\left|x_1 - x_3\right| + \left|x_3 - x_2\right|, \left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right\} = k. \end{aligned} \tag{1.3}$$

Let us examine the following four cases:

(i) $|x_1 - x_3| \ge |y_1 - y_3|$ and $|x_3 - x_2| \ge |y_3 - y_2|$;

(ii) $|x_1 - x_3| \ge |y_1 - y_3|$ and $|x_3 - x_2| \le |y_3 - y_2|$;

(iii) $|x_1 - x_3| \le |y_1 - y_3|$ and $|x_3 - x_2| \le |y_3 - y_2|$;

(iv) $|x_1 - x_3| \le |y_1 - y_3|$ and $|x_3 - x_2| \ge |y_3 - y_2|$.

We will only prove the first two cases since the remaining cases can be proved similarly.

If $|x_1 - x_3| \ge |y_1 - y_3|$ and $|x_3 - x_2| \ge |y_3 - y_2|$, then

$$\begin{aligned} d_g(X, Y) \le k &= a\left(\left|x_1 - x_3\right| + \left|x_3 - x_2\right|\right) + b\left(\left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right) \\ &= a\left|x_1 - x_3\right| + b\left|y_1 - y_3\right| + a\left|x_3 - x_2\right| + b\left|y_3 - y_2\right| \\ &= d_g(X, Z) + d_g(Z, Y). \end{aligned} \tag{1.4}$$

If $|x_1 - x_3| \ge |y_1 - y_3|$ and $|x_3 - x_2| \le |y_3 - y_2|$, then there are two possible situations.

(i) Let $|x_1 - x_3| + |x_3 - x_2| \ge |y_1 - y_3| + |y_3 - y_2|$. Then

$$d_g(X, Y) \le k = a\left(\left|x_1 - x_3\right| + \left|x_3 - x_2\right|\right) + b\left(\left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right). \tag{1.5}$$

Using $|x_1 - x_3| \ge |y_1 - y_3|$ and $|x_3 - x_2| \le |y_3 - y_2|$,

$$\begin{aligned} d_g(X, Z) &= a\left|x_1 - x_3\right| + b\left|y_1 - y_3\right|, \\ d_g(Z, Y) &= a\left|y_3 - y_2\right| + b\left|x_3 - x_2\right|. \end{aligned} \tag{1.6}$$

Also, $d_g(X,Y) \leq d_g(X,Z) + d_g(Y,Z)$ if and only if

$$\begin{aligned} & d_g(X,Y) \leq k = a\left(\left|x_1 - x_3\right| + \left|x_3 - x_2\right|\right) + b\left(\left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right) \\ & k \leq a\left(\left|x_1 - x_3\right| + \left|y_3 - y_2\right|\right) + b\left(\left|y_1 - y_3\right| + \left|x_3 - x_2\right|\right) \\ & \quad \Longleftrightarrow a\left(\left|x_3 - x_2\right| - \left|y_3 - y_2\right|\right) + b\left(\left|y_3 - y_2\right| - \left|x_3 - x_2\right|\right) \leq 0 \\ & \quad \Longleftrightarrow (a-b)\left(\left|x_3 - x_2\right| - \left|y_3 - y_2\right|\right) \leq 0 \\ & \quad \Longleftrightarrow \left|x_3 - x_2\right| \leq \left|y_3 - y_2\right|. \end{aligned} \tag{1.7}$$

(ii) Let $|x_1 - x_3| + |x_3 - x_2| \leq |y_1 - y_3| + |y_3 - y_2|$. Then

$$d_g(X,Y) \leq k = a\left(\left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right) + b\left(\left|x_1 - x_3\right| + \left|x_3 - x_2\right|\right). \tag{1.8}$$

Using $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$,

$$\begin{aligned} & d_g(X,Z) = a\left|x_1 - x_3\right| + b\left|y_1 - y_3\right|, \\ & d_g(Z,Y) = a\left|y_3 - y_2\right| + b\left|x_3 - x_2\right|, \\ & d_g(X,Y) \leq d_g(X,Z) + d_g(Y,Z) \Longleftrightarrow \\ & d_g(X,Y) \leq k = a\left(\left|y_1 - y_3\right| + \left|y_3 - y_2\right|\right) + b\left(\left|x_1 - x_3\right| + \left|x_3 - x_2\right|\right) \\ & k \leq a\left(\left|x_1 - x_3\right| + \left|y_3 - y_2\right|\right) + b\left(\left|y_1 - y_3\right| + \left|x_3 - x_2\right|\right) \\ & \quad \Longleftrightarrow a\left(\left|y_1 - y_3\right| - \left|x_1 - x_3\right|\right) + b\left(\left|x_1 - x_3\right| - \left|y_1 - y_3\right|\right) \leq 0 \\ & \quad \Longleftrightarrow (a-b)\left(\left|y_1 - y_3\right| - \left|x_1 - x_3\right|\right) \leq 0 \\ & \quad \Longleftrightarrow \left|y_1 - y_3\right| \leq \left|x_1 - x_3\right|. \end{aligned} \tag{1.9}$$

Note that the generalized absolute value metric d_g is a generalization of the taxicab metric d_T, maximum metric $d_{\max}$, and Chinese checker metric d_C in the following manner: $d_g = d_T$ if $a = b = 1$; $d_g = d_{\max}$ if $a = 1$ and $b = 0$; $d_g = d_C$ if $a = 1$ and $b = \sqrt{2} - 1$, respectively. Furthermore, the unit circle in $(\mathbb{R}^2, d_g)$ is the set of points (x,y) in the plane which satisfies

$$a \max\{|x|,|y|\} + b \min\{|x|,|y|\} = 1, \quad a \geq b \geq 0, a \neq 0. \tag{1.10}$$

It is well known that Schwarz's inequality is a fundamental inequality of the Euclidean plane, which states that the magnitude of the inner product of two vectors cannot exceed the product of the lengths of these two vectors in the plane. Our main goal is to show the validity of Schwarz's inequality in $(\mathbb{R}^2, d_g)$. To achieve this goal, our presentation is organized as follows. After the definitions of trigonometric functions in $(\mathbb{R}^2, d_g)$ are given, we define the norm, prove Schwarz's inequality, and give an area formula of a triangle in this plane (Section 3). Finally, we give a plane tiling example underlying Schwarz's inequality in Section 4.

2. The trigonometric functions

We know that if $X = (x,y)$ is a point on the Euclidean unit circle, then $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle with the positive x-axis as the initial side and the radial line passing through point (x,y) as the terminal side. Now, we can define the trigonometric functions cosine and sine of the angle θ in $(\mathbb{R}^2, d_g)$ in the same way as the Euclidean cosine and sine functions. Let (x, y) be a point on the unit circle in $(\mathbb{R}^2, d_g)$. Since the unit circle is an octagon in $(\mathbb{R}^2, d_g)$ (see [4]), trigonometric functions can be defined according to eight quadrants. If we show the cosine and sine functions by $\cos_g\theta$ and $\sin_g\theta$, they can be given as follows with the method in [5]:

$$\cos_g\theta = \begin{cases} \dfrac{\cos\theta}{a|\cos\theta| + b|\sin\theta|} & \text{if } |\tan\theta| \le 1, \\ \dfrac{\cos\theta}{b|\cos\theta| + a|\sin\theta|} & \text{if } |\tan\theta| \ge 1, \end{cases}$$
$$\sin_g\theta = \begin{cases} \dfrac{\sin\theta}{a|\cos\theta| + b|\sin\theta|} & \text{if } |\tan\theta| \le 1, \\ \dfrac{\sin\theta}{b|\cos\theta| + a|\sin\theta|} & \text{if } |\tan\theta| \ge 1. \end{cases} \tag{2.1}$$

One can easily see that if θ' is the related angle in first or second quadrant of given angle θ in quadrant i, $(\pi/4)(i-1) \le \theta \le (\pi/4)i$, $i = 1, \ldots, 8$. The relationship between $\cos_g\theta$ and $\cos_g\theta'$ or $\sin_g\theta$ and $\sin_g\theta'$ is the same as the Euclidean one.

We note that there is the nonuniform increment in arc length when the angle is incremented by a fixed amount whereas on the Euclidean unit circle. So, we want to define trigonometric functions any angle θ between any two lines by using reference angle α of θ [6].

Definition 2.1. Let θ be an angle with the reference angle α which is the angle between θ and the positive direction of the x-axis in d_g-unit circle. Then the cosine and sine functions of angle θ with the reference anglesα, $g\cos\theta$, and $g\sin\theta$ are defined by

$$\begin{aligned} g\cos\theta &:= \cos_g(\alpha+\theta)\cdot\cos_g\alpha + \sin_g(\alpha+\theta)\cdot\sin_g\alpha, \\ g\sin\theta &:= \sin_g(\alpha+\theta)\cdot\cos_g\alpha - \cos_g(\alpha+\theta)\cdot\sin_g\alpha. \end{aligned} \tag{2.2}$$

One can easily see that if $\alpha = 0$, then $g\cos\theta = \cos_g\theta$ and $g\sin\theta = \sin_g\theta$.

2.1. Variation of d_g-lengths under rotations

It is well known that all translations and rotations of the plane preserve the Euclidean distance. It is obvious that d_g-distances are invariant under all translations. Now, we want to investigate the change of d_g-lengths of a line segment after rotations as in [7].

Theorem 2.2. *Let any two points be A and B in $(\mathbb{R}^2, d_g)$, and let the line segment AB be not parallel to the x-axis and the angle α between the line segment AB and the positive direction of x-axis. If $A'B'$ is the image of AB under the rotations with the angle θ, then*

$$d_g(A', B') = \sqrt{\frac{\cos_g^2\alpha + \sin_g^2\alpha}{\cos_g^2(\alpha+\theta) + \sin_g^2(\alpha+\theta)}}\cdot d_g(A, B). \tag{2.3}$$

Proof. We can translate the line segment AB to the line segment OX such that

$$d_g(A,B) = d_g(O,X) = k \tag{2.4}$$

since all the translations preserve the d_g-distance. $X = (k \cdot \cos_g\alpha, k \cdot \sin_g\alpha)$ and under the rotation θ, OX transforms to OX', $X' = (k \cdot \cos_g(\alpha+\theta), k \cdot \sin_g(\alpha+\theta))$. If we calculate $d_g(O,X') = k'$ by using $d_E(O,X) = d_E(O,X')$, we obtain

$$k' = k \cdot \sqrt{\frac{\cos_g^2\alpha + \sin_g^2\alpha}{\cos_g^2(\alpha+\theta) + \sin_g^2(\alpha+\theta)}}. \tag{2.5}$$

□

Corollary 2.3. *If the segment AB is parallel to the x-axis, then*

$$d_g(A',B') = \frac{d_g(A,B)}{\sqrt{\cos_g^2\theta + \sin_g^2\theta}}. \tag{2.6}$$

Proof. In this situation, since $\alpha = 0$, the proof is obvious. □

3. Norm in $(\mathbb{R}^2, d_g)$

In this section, we will define the norm in $(\mathbb{R}^2, d_g)$ by using the standard inner product in $\mathbb{R}^2$ as in [8] .

Definition 3.1. Let $A = (x,y)$ be any vector in $(\mathbb{R}^2, d_g)$. Then the norm of A is defined by

$$\|A\|_g = \sqrt{\frac{\langle A,A\rangle}{\cos_g^2\alpha + \sin_g^2\alpha}}, \tag{3.1}$$

where α is the reference angle of A.

Corollary 3.2. $\|A\|_g = d_g(O,A)$.

Proof. If $A = (x,y) = (d_g(O,A)\cos_g\alpha,\ d_g(O,A)\sin_g\alpha)$ is used,

$$d_E(O,A) = \sqrt{\cos_g^2\alpha + \sin_g^2\alpha} \cdot d_g(O,A). \tag{3.2}$$

If we use $d_E(O,A) = \sqrt{\langle A,A\rangle}$ and Definition 3.1,

$$\|A\|_g = d_g(O,A) \tag{3.3}$$

is obtained. □

As in Euclidean geometry, it is easily seen that the norm function $\|\cdot\|_g$ satisfies the following properties. So, the following proposition is given without proof.

Proposition 3.3. *Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$ be any three vectors in $(\mathbb{R}^2, d_g)$ and $r \in \mathbb{R}$. Then*

(i) $\|A\|_g \geq 0$ *and* $(\|A\|_g = 0 \Leftrightarrow A = 0)$,

(ii) $\|rA\|_g = |r| \cdot \|A\|_g$,

(iii) $\|A + B\|_g \leq \|A\|_g + \|B\|_g$,

(iv) $\|A - B\|_g \geq \|A\|_g - \|B\|_g$,

(v) $\|A - B\|_g \leq \|A - C\|_g + \|C - B\|_g$.

It is important that Schwarz's inequality is valid for a metric. For example, matched filters are found in nearly every communication device, such as cell phones and television broadcasts. The detector utilizes Cauchy-Schwarz's inequality to compare functions. When the inner product of the functions is "large," then there are significant similarities between the two signals. This detector will help us better select which tile is appropriate for a small section of the final image. If a quantized block of the target image is mostly one color on one side but another color on the opposite side, then the matched filter detector will find a tile with similar attributes.

The following proposition shows that Schwarz's inequality is valid for some restricted cases of d_g.

Proposition 3.4 (Schwarz's inequality). *If $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are any two vectors in $(\mathbb{R}^2, d_g)$ and $b \geq 1$, then*

$$\left|\langle A, B\rangle\right| \leq \|A\|_g \cdot \|B\|_g. \tag{3.4}$$

Proof. (i) If $A = 0$, then $\langle A, B\rangle = 0$ and $|\langle A, B\rangle| = 0$. Furthermore, $\|A\|_g = 0$ and $\|A\|_g \cdot \|B\|_g = 0$.

(ii) Let $A \neq 0$, then we know that

$$\left|\langle A, B\rangle\right| = \left|x_1x_2 + y_1y_2\right| \leq \left|x_1x_2\right| + \left|y_1y_2\right|, \tag{3.5}$$

$$\begin{aligned} \|A\|_g \cdot \|B\|_g &= \left(a \max\left\{\left|x_1\right|, \left|y_1\right|\right\} + b \min\left\{\left|x_1\right|, \left|y_1\right|\right\}\right) \cdot \left(a \max\left\{\left|x_2\right|, \left|y_2\right|\right\} + b \min\left\{\left|x_2\right|, \left|y_2\right|\right\}\right) \\ &= a^2\left|x_1x_2\right| + ab\left(\left|x_1y_2\right| + \left|y_1x_2\right|\right) + b^2\left|y_1y_2\right|. \end{aligned} \tag{3.6}$$

In this case, there are four subcases.

(1) $|x_1| \geq |y_1|$ and $|x_2| \geq |y_2|$.

(2) $|x_1| \geq |y_1|$ and $|x_2| \leq |y_2|$.

(3) $|x_1| \leq |y_1|$ and $|x_2| \geq |y_2|$.

(4) $|x_1| \leq |y_1|$ and $|x_2| \leq |y_2|$.

Here, subcase (1) is proved; the others can be similarly proved.

If the right-hand sides of (3.4) and (3.5) are observed,

$$ab\left|x_1y_2\right| + ab\left|y_1x_2\right| + a^2\left|x_1x_2\right| + b^2\left|y_1y_2\right| \geq \left|x_1x_2 + y_1y_2\right| \tag{3.7}$$

is valid since $b \geq 1$, $|x_1| \geq |y_1|$, and $|x_2| \geq |y_2|$. This completes the proof. □

Now, we will give the area of the triangle $\overset{\Delta}{ABC}$.

Theorem 3.5. *Let $\overset{\Delta}{ABC}$ be any triangle in $(\mathbb{R}^2, d_g)$ and let θ be the angle between AC and BC. Then the area $\mathcal{A}$ of the triangle $\overset{\Delta}{ABC}$ is*

$$\mathcal{A} = \frac{1}{2} \cdot \|AC\|_g \cdot \|BC\|_g \cdot g\sin\theta. \tag{3.8}$$

Proof. One can take that C is the origin since all the translations preserve d_g-distance. One can easily obtain the following relation between $g\sin\theta$ and $\sin\theta$:

$$g\sin\theta = \frac{\|OA\| \cdot \|OB\|}{\|OA\|_g \cdot \|OB\|_g} \cdot \sin\theta. \tag{3.9}$$

From (3.9) and $2\mathcal{A} = \|OA\| \cdot \|OB\| \cdot \sin\theta$ for $0 \leq \theta \leq \pi$,

$$\mathcal{A} = \frac{1}{2} \cdot \|AC\|_g \cdot \|BC\|_g \cdot g\sin\theta. \tag{3.10}$$

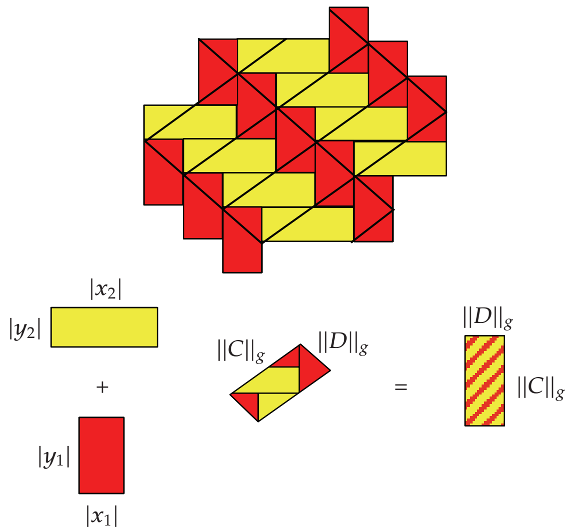
□

4. Application

In Latin, tessella was a small cubical piece of clay, stone, or glass used to make mosaics as mentioned in [9, 10]. The word "tessella" means "small square" (from "tessera," square, derived from the Greek word for "four"). It also corresponds to the everyday term "tiling" which refers to the applications of tessellation, often made of glazed clay.

Certain shapes of tiles, most obviously rectangles, can be replicated to cover a surface with no gaps. These shapes are said to tessellate (from the Latin tessella, "tile"). Over the centuries, many ceramic engineers and artists have employed plane tiling in their work. They used tiles for floors and walls because they are durable, waterproof, and beautiful. If you are mathematically inclined, no doubt you see some mathematics in the tiling. There are some examples where plane tiling on floors, walls, and in paintings underlies proofs of some well-known (and some not so well-known) theorems. However, tilings provide "picture proofs" for many theorems. They also illustrate inequalities such as the arithmetic mean-geometric mean inequality. Squares and rectangles are some of the most popular tiling used by engineers and artists to tile the plane. The design of geometric shapes, which fit together to cover a surface without gaps or overlapping, has a long history. Now, we will give a plane tiling example satisfying Schwarz's inequality in $(\mathbb{R}^2, d_g)$ as an application.

Suppose that the road is tiled with bricks of two different sizes as illustrated in Figure 1. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two vectors in $(\mathbb{R}^2, d_g)$ and the side lengths of two bricks are $|x_1|$, $|x_2|$ and $|y_1|$, $|y_2|$, respectively; let d_g satisfy the condition $b \geq 1$ such



$$|x_1x_2 + y_1y_2| \leq |x_1||x_2| + |y_1||y_2| = \|D\|_g \cdot \|C\|_g \cdot g\sin\theta \leq \|D\|_g \cdot \|C\|_g$$

Figure 1

that $C = (x_2, y_1)$ and $D = (x_1, y_2)$. Then the following tiling forms part of a proof of some special cases of Schwarz's inequality in two dimensions.

Acknowledgment

The authors would like to thank Professor Rüstem Kaya for his kind help and valuable suggestions.

References

[1] H. Minkowski, *Gesammelte Abhandlungen*, Chelsea, New York, NY, USA, 1967.
[2] B. Eisenberg and S. Khabbaz, "Optimal locations," *The College Mathematics Journal*, vol. 23, no. 4, pp. 282–289, 1992.
[3] S. Burman, H. Chen, and N. Sherwani, "Improved Global Routing using λ-geometry," in *Proceedings of the 29th Annual Allerton Conference on Communication, Control, and Computing*, pp. 1083–1092, Urbana, Ill, USA, October 1991.
[4] R. Kaya, Ö. Gelişken, S. Ekmekçi, and A. Bayar, "Isometries of the plane with generalized Absolute Value Metric," submitted to *Rocky Mountains*.
[5] A. Bayar and S. Ekmekçi, "On the Chinese-checkers sine and cosine functions," *International Journal of Mathematics and Analysis*, vol. 1, no. 3, pp. 249–259, 2006.
[6] K. Thompson and T. Dray, "Taxicab angles and trigonometry," *The Pi Mu Epsilon Journal*, vol. 11, pp. 87–97, 2000.
[7] M. Özcan, S. Ekmekçi, and A. Bayar, "A note on the variation of the taxicab lengths under rotations," *The Pi Mu Epsilon Journal*, vol. 11, pp. 381–384, 2002.
[8] Z. Akça, A. Bayar, and S. Ekmekçi, "The norm in CC-plane geometry," *The Pi Mu Epsilon Journal*, vol. 12, no. 6, pp. 321–324, 2007.
[9] G. Branko and G. C. Skephard, *Tillings and Patterns*, W. H. Freeman, New York, NY, USA, 1987.
[10] H. S. M. Coxeter, *Regular Polytopes*, Dover, New York, NY, USA, 3rd edition, 1973.