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On the Structure of Nash Equilibrium Sets in Partially Convex Games

Matthias Asche
Humboldt-Universität, Math.-Naturwiss. Fakultät II,
Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany.
e-mail: asche@mathematik.hu-berlin.de

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The paper describes the geometrical structure of Nash equilibrium sets in partially convex games without constraints. A condition characterizing a distinct class of Nash equilibrium sets is given. A complete description of such sets in two dimensions as well as some pictures illustrating the appearing difficulties are presented.

1. Introduction

In non-cooperative game theory the concept of a Nash equilibrium point is the most important one ([9], [10]). Such a point has the property that every players choice of his strategy is the best response to these of the other players. This concept is a generalization of the equilibrium concept developed by Cournot in 1838 ([5]). Many applications of it can be found in economic theory ([3]). Based on this concept a staggering number of solution concepts were developed (see [7]). An introduction to the theoretical aspects the interested reader can find in [11], [6] or [8].

In this paper we investigate the set of all Nash equilibrium points for partially convex games without constraints. Our purpose is a geometrical characterization of such sets to get a survey of the possible structures. At first we want to give a condition allowing us to characterize a distinct class of Nash equilibrium sets in any dimension. Furthermore we try to give an equivalent description of Nash sets in the 2-dimensional space. Some pictures will show the difficulties which can occur.

2. Preliminaries

We consider the n-dimensional euclidean vector space $\mathbb{R}^n$. With n_i we denote the dimension of the strategy space of player i, and with k players we have $n_1 + ... + n_k = n$. So $x \in \mathbb{R}^n$ can be written as $x = (x_1, ..., x_k)$ with $x_i \in \mathbb{R}^{n_i}$, and x_{-i} denotes $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_k) \in \mathbb{R}^{n-i}$ such that $x = (x_i, x_{-i})$. Furthermore we define for a set

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$G \subset \mathbb{R}^n$

$$G_{-i} = \left\{ x_{-i} \in \mathbb{R}^{n-i} \middle| \exists\, x_i \in \mathbb{R}^{n_i} \,:\, (x_i, x_{-i}) \in G \right\}$$
$$G(x_{-i}) = \left\{ y \in G \middle| y_{-i} = x_{-i} \right\}$$

A partially convex (or convex) game in our sense is a k-tuple $(f_1, ..., f_k)$ of continuous functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are convex for any fixed $x_{-i} \in \mathbb{R}^{n-i}$, $(i = 1, ..., k)$. Such a tuple is called a game in strategic form. Furthermore the set $\phi(x_{-i}) = \operatorname{argmin} f_i(\cdot, x_{-i})$ has to be nonempty and compact for all $x \in \mathbb{R}^n$ and $i = 1, ..., k$. A point x^* with $x_i^* \in \phi(x_{-i}^*)$ for i=1,...,k is called a Nash equilibrium point.

3. Nash sets in $\mathbb{R}^n$

3.1. A first characterization

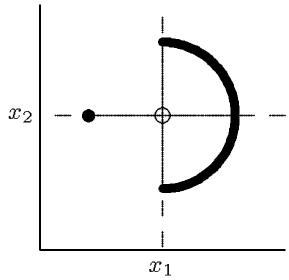


Figure 1

At the beginning we want to consider a simple two-dimensional set G consisting of a half circle and a single point (see Figure 1). We assume this set to be the set of Nash equilibrium points of a convex game. Then the payoff function h_2 of player 2 is convex on every vertical line, especially $\arg\min h_2(x_1, \cdot)$ is convex for arbitraryly fixed x_1. Since the intersection of such a vertical line and G belongs to $\arg\min h_2(x_1, \cdot)$ we can conclude the following: If a vertical line through the point $(x_1, 0)$ intersects the set G then the convex hull of the whole intersection belongs to the set minimizing $h_2(x_1, \cdot)$, i.e. $\operatorname{conv} G(x_1) \subseteq \{x_1\} \times \arg\min h_2(x_1, \cdot)$. With the same argument for any horizontal line we can derive a simple condition for Nash sets in two dimensions: Our set G has to fulfil $\operatorname{conv} G(x_1) \cap \operatorname{conv} G(x_2) \subseteq G$ for all $x \in \mathbb{R}^2$. It is easy to see that the set considered above cannot be a Nash set of a convex game for this reason.
An extension of the derived condition for arbitrary convex games could have the following form:

$$\text{If } G \subset \mathbb{R}^n \text{ and } n = n_1 + ... + n_k \text{ with } n_i \in \mathbb{N}, \text{then}$$
$$\bigcap_{i=1}^{k} \operatorname{conv} G(x_{-i}) \subseteq G \text{ for all } x \in \mathbb{R}^n. \qquad \textbf{(B1)}$$

Lemma 3.1. *Assume $n \in \mathbb{N}$ and $n = n_1 + ... + n_k$ with $n_i \in \mathbb{N}$ $(i = 1, ..., k)$.*
Let $(f_1, ..., f_k)$ be a convex game with Nash set G.
Then the condition (B1) *is satisfied.*

Proof. For the proof we fix $x \in \mathbb{R}^n$, $x^* \in \bigcap_{i=1}^{k} \operatorname{conv} G(x_{-i})$ and $i \in \{1, ..., k\}$. Since x^* belongs to $\operatorname{conv} G(x_{-i})$ it is a convex combination of vectors $x^1, ..., x^{n_i+1} \in G(x_{-i})$. The definition of $G(x_{-i})$ ensures $G(x_{-i}) \subset G$, hence $x^j \in G$ for all j. Thus we have $x_i^j \in \arg\min f_i(\cdot, x_{-i})$ for $j = 1, ..., n_i + 1$ and finally $x_i^* \in \arg\min f_i(\cdot, x_{-i})$, because $x_{-i} = x_{-i}^j = x_{-i}^*$ and $\arg\min f_i(\cdot, x_{-i})$ is convex. Since we chose an arbitrary i the proof is complete. □

The next question we deal with is the search for a class of sets for which our condition (B1) is a sufficient one. To do this we also need a possibility to construct a convex game based on a given set.

3.2. Construction of convex games

Let $G \subset \mathbb{R}^n$ be an arbitrary nonempty, closed subset of $\mathbb{R}^n$, and $n = n_1 + n_2$ with $n_1, n_2 \in \mathbb{N}$.
The distance function is a simple function with the minimal set G. If the set G is not convex the distance function does not satisfy the desired convexity properties. So the distance function has to be changed.
We define $F(x) := d(x, G)$. Then we can get a function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ from changing the epigraph of F. We take $\operatorname{epi} h(\cdot, x_2) = \operatorname{cl} \operatorname{conv} \operatorname{epi} F(\cdot, x_2)$ for all fixed $x_2 \in \mathbb{R}^{n_2}$. The values of h can explicitly be given, too (comp. [12, p. 157]).

Lemma 3.2.
(i) $h(\cdot, x_2)$ *is convex* $\forall\, x_2 \in \mathbb{R}^{n_2}$
(ii) h *is continuous in* $\mathbb{R}^n$
(iii) $h(x) \leq F(x)\ \forall\, x \in \mathbb{R}^n$

The proof shall not be given here, it is easy to verify the assertions.
The closure in the definition of h is not very useful. But it is possible to show that in all cases being interesting for us the use of the closure can be avoided. For this we recall the following definition: $G(x_2)$ is said to be locally bounded at x_2^* if for all bounded sets Ω including x_2^* $\|x_1\| \leq L < \infty$ for all $x_2 \in \Omega$ for all $x_1 \in G(x_2)$ holds. If this holds for all $x_2^* \in \mathbb{R}^{n_2}$ then $G(x_2)$ is locally bounded.

Lemma 3.3. *If* $G(x_2)$ *is locally bounded, then* $\operatorname{conv} \operatorname{epi} F(\cdot, x_2)$ *is closed* $\forall x_2 \in \mathbb{R}^{n_2}$.

Proof. We fix $x_2^* \in \mathbb{R}^{n_2}$. At first we assume, there is a sequence $\{x_1^l\}_{l=1}^{\infty} \subset \mathbb{R}^{n_1}$ such that $\|x_1^l\| \longrightarrow \infty$ and $F(x_1^l, x_2^*) \leq c$ for a fixed $c < \infty$ and all $l \in \mathbb{N}$. Then we have vectors $(\bar{x}_1^l, \bar{x}_2^l) \in G$ with $d((x_1^l, x_2^*), (\bar{x}_1^l, \bar{x}_2^l)) \leq c$ for all l. So we get the boundedness of $\{\bar{x}_2^l\}_{l=1}^{\infty}$, and with the properties of $G(x_2)$ the sequence $\{\bar{x}_1^l\}$ has to be bounded, too. Therefore we conclude $F(x_1, x_2^*) \longrightarrow \infty$ for $\|x_1\| \longrightarrow \infty$.
Now we consider a sequence $\{z_l\} \subset \operatorname{conv} \operatorname{epi} F(\cdot, x_2^*)$ converging to z, where $z_l = (x_l, y_l)$ for all $l \in \mathbb{N}$ and $z = (x, y)$. Then every vector z is a convex combination of vectors z_l^i, $i = 1, ..., n_1 + 1$ [12, p. 157], i.e. $z_l = \sum_{i=1}^{n_1+1} \lambda_l^i z_l^i$ with $\lambda_l^i \geq 0$, $\sum_{i=1}^{n_1+1} \lambda_l^i = 1$ and $z_l^i = (x_l^i, y_l^i)$,

where $y_l^i \geq F(x_l^i, x_2^*)$. With the convergence of z_l and the existence of a lower bound for the sequences y_l^i, $i = 1, ..., n_1+1$ we get the boundedness of the sequences $\{z_l^i\}_{l=1}^{\infty}$. So there is a subsequence $\{l_m\}_{m=1}^{\infty} \subset \mathbb{N}$ such that $z_{l_m}^i \stackrel{m\to\infty}{\longrightarrow} z^i$ and $\lambda_{l_m}^i \stackrel{m\to\infty}{\longrightarrow} \lambda^i$ for $i = 1, ..., n_1+1$. F is continuous, and so $z^i \in \operatorname{epi} F(\cdot, x_2^*)$ for all i and $z \in \operatorname{conv} \operatorname{epi} F(\cdot, x_2^*)$, because $z = \sum \lambda^i z^i$. □

Example 3.4. Lemma 3.3 is not true in general:
We take the graph of the function $f(x_1) = e^{-x_1^2}$ as the set $G \subset \mathbb{R}^2$, and we consider $x_2^* = 0$. Obviously $G(x_2)$ is not uniformly bounded in every open set including x_2^*. Considering the set $\operatorname{epi} F(\cdot, 0)$ we get the open half plane $\{x_1 > 0\}$ by forming the convex hull.

With the help of Lemma 3.3 we can formulate the following helpful assertions.

Lemma 3.5. *If $G(x_2)$ is locally bounded, then the following statements are valid:*
(i) $\min h(\cdot, x_2^*) = \min F(\cdot, x_2^*) \quad \forall\, x_2^* \in \mathbb{R}^{n_2}$
(ii) $x_1^* \in \arg\min F(\cdot, x_2^*) \Longrightarrow \exists\, x_2 \in \mathbb{R}^{n_2} : (x_1^*, x_2) \in G$
(iii) $\arg\min h(\cdot, x_2) = \operatorname{conv} \arg\min F(\cdot, x_2) \quad \forall\, x_2 \in \mathbb{R}^{n_2}$

Proof.
(i) Let x_1^* be a minimizer for $h(\cdot, x_2^*)$. The vector $(x_1^*, h(x_1^*, x_2^*))$ is a convex combination of vectors $(x_1^i, h^i) \in \operatorname{epi} F(\cdot, x_2^*)$. Therefore $h(x_1^*, x_2^*) \geq \min_i F(x_1^i, x_2^*) \geq \min F(\cdot, x_2^*)$ and so $\min h(\cdot, x_2^*) \geq \min F(\cdot, x_2^*)$. With $h(x) \leq F(x) \quad \forall x \in \mathbb{R}^n$, the proof is complete.
(ii) Assuming x_1^* is a minimizer for $F(\cdot, x_2^*)$. We can find a vector $\bar{x} = (\bar{x}_1, \bar{x}_2) \in G$ such that $d(\bar{x}, x^*) = d(x^*, G) \leq d(x^*, x)\, \forall\, x \in G$. Considering $(\bar{x}_1, x_2^*)$, we get for the case $\bar{x}_1 \neq x_1^*$:

$$\begin{aligned} F(\bar{x}_1, x_2^*) = d((\bar{x}_1, x_2^*), G) &\leq d((\bar{x}_1, x_2^*), \bar{x}) \\ &= \|\bar{x}_2 - x_2^*\| \\ &< \sqrt{\|x_1^* - \bar{x}_1\|^2 + \|x_2^* - \bar{x}_2\|^2} \\ &= d(\bar{x}, x^*) = F(x_1^*, x_2^*) \end{aligned}$$

which contradicts the choice of $\bar{x}$. So $\bar{x}_1$ has to be equal to x_1^*.
(iii) With (i) and (3.2 (iii)) we get $\arg\min F(\cdot, x_2^*) \subseteq \arg\min h(\cdot, x_2^*)$. Since $h(\cdot, x_2^*)$ is convex, $\operatorname{conv} \arg\min F(\cdot, x_2^*) \subseteq \arg\min h(\cdot, x_2^*)$ is also true.
If on the other hand x_1^* minimizes $h(\cdot, x_2^*)$, so $(x_1^*, h(x_1^*, x_2^*))$ is a convex combination of $(x_1^i, h^i) \in \operatorname{epi} F(\cdot, x_2^*)$, $i = 1, ..., n_1+1$. With (i) we have $h^i = h(x_1^*, x_2^*)$ for all i. The construction rules lead to $h^i \geq F(x_1^i, x_2^*)$ and $F(x_1^i, x_2^*) \geq h(x_1^*, x_2^*)$, so we get $F(x_1^i, x_2^*) = h^i = \min F(\cdot, x_2^*)$ for all i. Therefore $x_1^i \in \arg\min F(\cdot, x_2^*)$ and finally $x_1^* \in \operatorname{conv} \arg\min F(\cdot, x_2^*)$. □

Corollary 3.6.
(i) $h(x) \geq 0 \quad \forall\, x \in \mathbb{R}^n$
(ii) *If $G(x_2^*) \neq \emptyset$ for $x^* \in \mathbb{R}^n$, then* $\operatorname{conv} G(x_2^*) = \arg\min h(\cdot, x_2^*) \times \{x_2^*\}$.

3.3. A sufficient condition

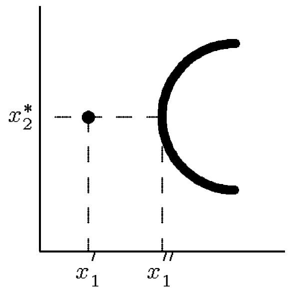


Figure 2

With the results of the last section we are now able to construct convex games based on a given set $G \subset \mathbb{R}^n$. Obviously the set G is always included in the Nash set of the constructed game. Now our aim is to find a sufficiently large class of sets for which the reversed inclusion is true under the condition (B1).

It is clear that the class of compact convex sets belongs to such a class, because the distance function is convex for these sets (see [12, pp. 28, 34]). On the other side there are bad examples. A simple one is shown in Figure 2. Here the condition (B1) is satisfied, but the Nash set resulting from the constructed game (h_1, h_2) includes the whole line segment $[(x_1', x_2^*), (x_1'', x_2^*)]$. ($h_i$ was defined with $\operatorname{epi} h_i(\cdot, x_{-i}) = \operatorname{conv} \operatorname{epi} F(\cdot, x_{-i})$ for $x \in \mathbb{R}^2$, $i = 1, 2$). So we have to avoid the existence of such two points like (x_1', x_2^*) and (x_1'', x_2^*). This can be ensured if all projections G_{-i} of G are convex. We will show, that all sets satisfying this condition are useful.

Theorem 3.7. *Assume $n \in \mathbb{N}$ und $n = n_1 + ... + n_k$ with $n_i \in \mathbb{N}\,(i = 1, ..., k)$. Furthermore let $G \subset \mathbb{R}^n$ be nonempty and compact and let all projections G_{-i} of G onto $\mathbb{R}^{n-i}$ $(i = 1, ..., k)$ be convex.*
Then G is the Nash set of a convex game if and only if the condition (B1) *holds.*

Proof. The necessity of (B1) has already been demonstrated in Lemma 3.1.
Now let $G \subset \mathbb{R}^n$ be a set with the desired properties. As shown above it is possible to construct a convex game by means of the convex hull of the distance function. With $F(x) = d(x, G)$ we define $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $\operatorname{epi} h_i(\cdot, x_{-i}) = \operatorname{conv} \operatorname{epi} F(\cdot, x_{-i})$ for all $x \in \mathbb{R}^n$ and $i = 1, ..., k$.
It remains to show that G is indeed the Nash set of the game $(h_1, ..., h_k)$, that means $x^* \in G$ iff $x_i^* \in \arg\min h_i(\cdot, x_{-i}^*)$ for $i = 1, ..., k$.
If $x^* \in G$ then $F(x^*) = h_i(x^*) = 0$ for all i, and with Corollary 3.6 follows $x_i^* \in \arg\min h_i(\cdot, x_{-i}^*)$.

For the other direction of the proof we consider a vector $x^* \in \mathbb{R}^n$ satisfying the property of a Nash equilibrium. Now two cases are possible.
At first all sets $G(x_{-i}^*)$ could be nonempty. Then we get with Corollary 3.6 $x^* \in \operatorname{conv} G(x_{-i}^*)$ for $i = 1, ..., k$, and with (B1) immediately follows $x^* \in G$.
On the other side there could be an index i_0 with $G(x_{-i_0}^*) = \emptyset$. Then all sets $G(x_{-i}^*)$ must be empty. Assuming that this is not true there is an index j_0 with $G(x_{-j_0}^*) \neq \emptyset$. Due to Corollary 3.6 x^* belongs to $\operatorname{conv} G(x_{-j_0}^*)$. So x^* is a convex combination of $x^1, ..., x^m \in$

$G(x^*_{-j_0})$, whereas $x^l = (x^l_{j_0}, x^*_{-j_0})$ for $l = 1, ..., m$. Hence it follows $x^l_{-i_0} \in G_{-i_0}$, and with the convexity of G_{-i_0} we get $x^*_{-i_0} \in G_{-i_0}$ in contradiction to $G(x^*_{-i_0}) = \emptyset$.
Up to this point we have shown that in the second case all values $r_i = h_i(x^*)$ are greater than zero. Without loss of generality we assume $r_1 <= r_i$ for $i = 1, ..., k$. Considering x^*_1 we get with Lemma 3.5 the existence of subvectors $y^1_1, ..., y^m_1 \in \arg\min F(\cdot, x^*_{-1})$ such that $x^*_1 = \sum_{j=1}^m \lambda_j y^j_1$ with $\lambda_j \geq 0$ and $\sum_{j=1}^m \lambda_j = 1$. Due to Lemma 3.5 there are now y^j_{-1} $(j = 1, ..., m)$ such that $y^j = (y^j_1, y^j_{-1}) \in G$ and $\|y^j_{-1} - x^*_{-1}\| = r_1$ for all j.
Now we fix an arbitrary index $i_0 \in \{2, ..., k\}$. With the convexity of G_{-i_0} we get $\sum_{j=1}^m \lambda_j y^j_{-i_0} \in G_{-i_0}$. Thus

$$\begin{aligned} r_{i_0}^2 &\leq \left\| \sum_{j=1}^m \lambda_j y^j_{-i_0} - x^*_{-i_0} \right\|^2 \\ &= \left\| \sum_{j=1}^m \lambda_j (y^j_{-i_0} - x^*_{-i_0}) \right\|^2 \\ &= \sum_{\substack{i=1 \\ i \neq i_0}}^k \left\| \sum_{j=1}^m \lambda_j (y^j_i - x^*_i) \right\|^2 \\ &= r_1^2 - \left\| \sum_{j=1}^m \lambda_j y^j_{i_0} - x^*_{i_0} \right\|^2 \\ &\leq r_1^2 \;\leq\; r_{i_0}^2 \end{aligned}$$

Hence $\sum_{j=1}^m \lambda_j y^j_i = x^*_i$ for $i = 1, ..., k$. With $y^j \in G$ we finally get $x^*_{-i} \in G_{-i}$ for all i and $h_i(x^*) = 0$, which contradicts the result we had obtained above. □

4. Nash sets in $\mathbb{R}^2$

In the following we want to restrict ourselves to the 2-dimensional case, i.e. we consider two players with one-dimensional strategy spaces. In this special case we try to generalize the condition (B1) to get an survey of all possible Nash sets. As seen in Figure 2 a construction of a game by means of the distance function is not possible here. So we want to use special results from the field of parametric optimization, related with the concept of upper semicontinuous multivalued functions. There are two definitions of upper semicontinuity known to the author, but in the context of multifunctions with nonempty compact images they are equivalent. So we do not have to distinguish between it.

Lemma 4.1. *Assume $n, n_1, n_2 \in \mathbb{N}$ and $n = n_1 + n_2$.*

(i) *If $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function, $h(\cdot, x_2)$ is convex and $\phi(x_2) = \operatorname{argmin} h(\cdot, x_2)$ is nonempty and compact for all $x_2 \in \mathbb{R}^{n_2}$, then $\phi : \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{n_1}$ is a upper semi-continuous multifunction and $\operatorname{graph}(\phi)$ is connected.*

(ii) *If* $\phi : \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{n_1}$ *is a upper semicontinuous multifunction with nonempty, compact and convex images, then there is a continuous function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$*, such that* $f(\cdot, x_2)$ *is convex and* $\operatorname{argmin} f(\cdot, x_2) = \phi(x_2)$ *for all* $x_2 \in \mathbb{R}^{n_2}$*.*

Proof. The proof of (i) can be found in [4, p. 68], (ii) immediately follows from Lemma 3.2 and Lemma 3.5 □

4.1. A further necessary condition

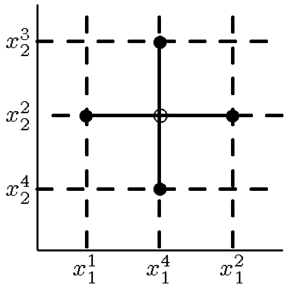


Figure 3

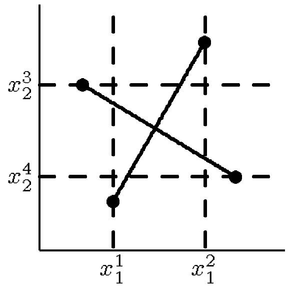


Figure 4

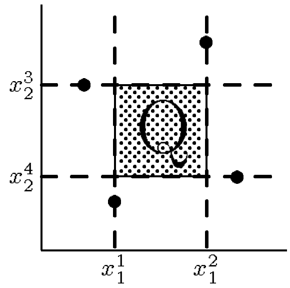


Figure 5

Now we want to use the condition (B1) in the two-dimensional case. If we have 4 points arranged such as in Figure 3 there has to be a 5th point belonging to both line segments drew solid there. But what happens if we turn the whole picture a little bit? We are now in the situation of Figure 4. Obviously the condition (B1) is satisfied here, but every attempt to construct a game with this Nash set will fail. Thus we can formulate a further condition.

$$\begin{array}{c}\text{Assume } x^1, x^2, x^3, x^4 \in G \text{ and } [x^1, x^2] \cap [x^3, x^4] \neq \emptyset. \\ \text{If } Q_i = [x_i^1, x_i^2] \times [x_{-i}^3, x_{-i}^4], \text{ then } Q_i \cap G \neq \emptyset, i = 1, 2\end{array} \tag{B2}$$

Lemma 4.2. *Assume* $n_1 = n_2 = 1$, $\varphi_1 : \mathbb{R}^{n_1} \longrightarrow \mathbf{P}(\mathbb{R}^{n_2})$ *and* $\varphi_2 : \mathbb{R}^{n_2} \longrightarrow \mathbf{P}(\mathbb{R}^{n_1})$ *upper semicontinuous with nonempty, compact, convex image sets. Furthermore let* $x^1, x^2 \in \operatorname{graph}(\varphi_1)$ *and* $x^3, x^4 \in \operatorname{graph}(\varphi_2)$ *with* $[x^1, x^2] \cap [x^3, x^4] \neq \emptyset$*. If* $Q = [x_1^1, x_1^2] \times [x_2^3, x_2^4]$ *and* $x^i \notin Q$ *for* $i = 1, ..., 4$ *then the intersection of the graphs of* φ_1 *and* φ_2 *is nonempty, and one intersection point belongs to* Q*.*

Proof.

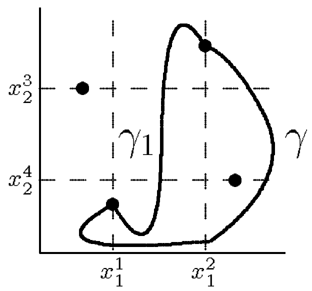


Figure 6

Due to Lemma 4.1 $G^1 = \text{graph}(\varphi_1|_{[x_1^1,x_1^2]})$ and $G^2 = \text{graph}(\varphi_2|_{[x_2^3,x_2^4]})$ are compact and connected sets. Assuming $G^1 \cap G^2 = \emptyset$ we get $d(G^1, G^2) > 0$. Thus there is $\epsilon > 0$ such that $B_\epsilon(G^1) \cap G^2 = \emptyset$. $B_\epsilon(G^1)$ is an open and connected set, therefore it is path connected. Hence there is a continuous path γ_1 in $B_\epsilon(G^1)$ connecting x^1 and x^2, i.e. there is a continuous function $\gamma_1 : [0,1] \longrightarrow \mathbb{R}^2$ with $\gamma_1(0) = x^1$, $\gamma_1(1) = x^2$ and $\gamma_1([0,1]) \subset B_\epsilon(G^1)$. Obviously we can assume γ_1 to be injective and $\gamma_1([0,1]) \subset [x_1^1, x_1^2] \times \mathbb{R}^{n_2}$. Because $[x_2^3, x_2^4] \subset (x_2^1, x_2^2)$ we can complete γ_1 to a closed Jordan curve γ which does not intersect G^2 and encloses only one of the points x^3 and x^4 (see Figure 6). With the theorem of Jordan ([1, p. 39]) we get two open disjoint sets $U, V \subset \mathbb{R}^2$, whereas $G^2 \subset U \cup V$ and $x^3 \in U$ and $x^4 \in V$. Since G^2 is connected, this is not possible. So $G^1 \cap G^2 \neq \emptyset$ and $G^1 \cap G^2 \subset Q$. □

Corollary 4.3. *Assume (f_1, f_2) to be a convex game with Nash set G, $f_1, f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Then G satisfies the condition (B2).*

Example 4.4.

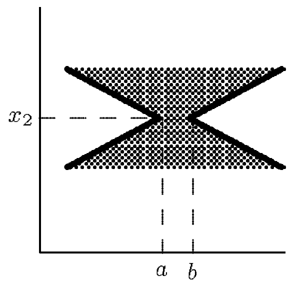


Figure 7

It is easy to see that (B1) holds if (B2) is satisfied. So we have a more general condition, but considering Figure 7 we get immediately, that this condition is not sufficient, yet. Due to Lemma 4.1 it is necessary and sufficient to construct two upper semicontinuous multivalued maps intersecting only in the bold faced set. The graph of the map belonging to player 1 includes the whole shaded area. The graph of the other map especially includes the points (a, x_2) and (b, x_2). Additionally it links these two points without intersecting the shaded area. It is clear that this is not possible. If we avoid such situations we can fully characterize Nash sets in two dimensions (see below).

4.2. Concluding remarks

Coming to the end we want to complete the investigation of 2-dimensional Nash sets. In this simple case a full characterization is possible, but it can be seen that the difficulties will enormously grow in higher dimensions.

At first we want to give some additionally assertions without a proof, then the idea of the proof will be indicated through an example.

Considering such sets like this drawn in Figure 7 it is easy to get an idea why they have so bad properties. But it is difficult to express this idea. One attempt is the following one.

If we have $i = 1, 2$, $a, b \in G_i$ and $\overline{x}_i \in (a, b)$, such that $(a, b) \cap G_i = \emptyset$, then we define $M := \{x_{-i} \in \mathbb{R}^{n-i} : (a, x_{-i}) \in \operatorname{conv} G_1(a) \text{ and } (b, x_{-i}) \in \operatorname{conv} G_1(b)\}$, $G^{>} := \{x \in G : x_i > \overline{x}_i\}$ and $G^{<} := \{x \in G : x_i < \overline{x}_i\}$.

$$\begin{array}{c}\text{If } M \text{ is nonempty, then there are } \delta \in \mathbb{R},\ x \in (\{a, b\} \times M) \cap G, \\ \text{such that } G^{\leq}_{-i} \cap (x_{-i}, x_{-i} + \delta) = \emptyset \text{ or } G^{\geq}_{-i} \cap (x_{-i}, x_{-i} + \delta) = \emptyset\end{array} \qquad \text{(B3)}$$

Lemma 4.5. *Let $G \subset \mathbb{R}^2$ be compact and nonempty, $n_1 = n_2 = 1$.*
G fulfils the conditions (B2) *and* (B3) *iff there is a convex game with Nash set G.*

Lemma 4.6. *Let $G \subset \mathbb{R}^2$ be compact and nonempty, $n_1 = n_2 = 1$.*
The condition (B2) *is sufficient for the existence of a convex game with Nash set G if G consists of a finite number of points or if the projections of the connected components of G onto $\mathbb{R}^{n_i}$ are pairwise disjoint* $(i = 1, 2)$.

The complete proof of Lemma 4.5 can be found in [2], Lemma 4.6 immediately follows.

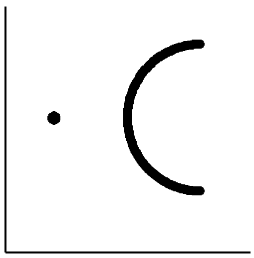


Figure 8

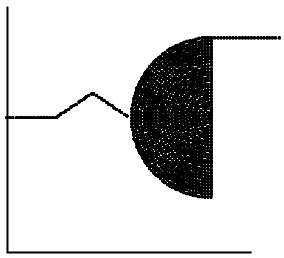


Figure 9

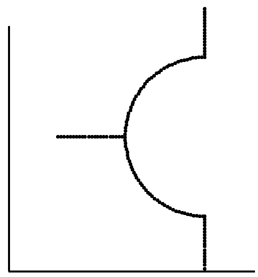
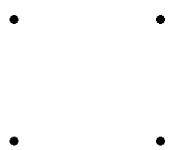


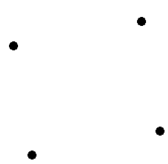
Figure 10

Example 4.7. Now we want to return to the set shown in Figure 8. In section 3.3 we showed that the construction we described there could not be used for this set. But there is really a convex game with this Nash set. The Figures 9 and 10 include the corresponding upper semicontinuous maps belonging to the players. Similar the proof of Lemma 4.5 can be done by piecewise constructing the two maps. This is why a generalization of (B2) and (B3) cannot be done in a similar way in higher dimensions.

4.3. Some Examples

To show how difficult a description of Nash sets in higher dimensions can be we want to give some examples of Nash sets. Minimal changes of the Nash set of a given convex game can cause the loss of this property. On the left we have drawn the Nash sets, on the right we have changed these sets in the way indicated above.





These figures show that there is only one arrangement of 4 points that can be a Nash set. The turned points do not fulfil (B2) (comp. Figures 3, 4, 5).

Here the changed set does not satisfy condition (B3).

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