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Separation by Hyperplanes in Finite-Dimensional Vector Spaces over Archimedean Ordered Fields

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Theorems on the separation of convex sets by hyperplanes are among the basic tools of convex analysis and mathematical programming. The main results of the present paper are new (and in a sense best possible) separation theorems in the setting of a finite-dimensional vector space $\mathbb{X}$ over an archimedean ordered field $\mathbb{F}$. There is an emphasis on the differences between the case in which $\mathbb{F}$ is the real field $\mathbb{R}$ and that in which $\mathbb{F}$ is a proper subfield of $\mathbb{R}$. The rational field $\mathbb{Q}$ is of special importance because of its relevance to computation. A new theorem for $\mathbb{R}^d$ concerns the *free separation* of two convex sets, where this means that there is a separating hyperplane H such that all sufficiently small perturbations of H still separate the two sets. In a sense that is made precise, this is the unique *maximal theorem* for free separation in $\mathbb{R}^d$. A theorem for general $\mathbb{X}$ implies that if a proper convex subset C of $\mathbb{X}$ is *s-closed*, then C is an intersection of open halfspaces. (The condition of s-closedness, defined in the text, is satisfied by all closed convex subsets of $\mathbb{R}^d$. In an arbitrary $\mathbb{X}$, it is satisfied by polyhedra and by many other convex sets, but when $\mathbb{F} \neq \mathbb{R}$ it is stronger than mere closedness.) There is also a study of conditions under which an $\mathbb{F}$-valued convex function on a convex subset C of $\mathbb{X}$ is the supremum of a collection of $\mathbb{F}$-valued affine functions on $\mathbb{X}$. (In $\mathbb{R}^d$, this leads to the usual subdifferentiability of convex functions.) The s-closedness of C is again a relevant condition. In conjunction with the relevance of s-closedness to line-searches, and the related fact that the standard theorems on extremal structure of convex sets in $\mathbb{R}^d$ extend to s-closed subsets of $\mathbb{F}^d$, this suggests that results on the behavior of s-closed sets may eventually provide useful tools in the development of genuinely rational optimization algorithms.

Keywords: archimedean field, finite-dimensional vector space, convex set, s-closed, separation, support, convex function, epigraph, optimization

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1. Introduction

Throughout this paper, $\mathbb{F}$ denotes an arbitrary archimedean ordered field. Recall that whenever $\mathbb{F}$ is an ordered field and $\mathbb{G}$ is the smallest subfield of $\mathbb{F}$ that contains the

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multiplicative unit 1, there is a unique isomorphism of $\mathbb{G}$ (as an ordered field) onto the rational field $\mathbb{Q}$. It is customary to use members of $\mathbb{Q}$ and of the integers $\mathbb{Z}$ to denote the corresponding members of $\mathbb{G}$. An ordered field $\mathbb{F}$ is *archimedean* if for each $a \in \mathbb{F}$ there exists an integer n such that $a < n$; equivalently, $\mathbb{F}$ is isomorphic with a subfield of $\mathbb{R}$. And $\mathbb{F}$ is *Dedekind-complete* if each subset of $\mathbb{F}$ that has an upper bound has a least upper bound; this is equivalent to requiring that $\mathbb{F}$ is isomorphic with $\mathbb{R}$. (For these basic facts, see, e.g. [26].)

We are especially concerned with the case in which $\mathbb{F}$ is $\mathbb{R}$ and with the case in which $\mathbb{F}$ is the rational field $\mathbb{Q}$. The latter concern is motivated by the fact that all digital implementations of algorithms are bound in one way or another to the rational numbers. Of course, algebraic numbers may be encoded by the rational coefficients of their minimal polynomials over $\mathbb{Q}$, so there are ways of handling, in a rational manner, problems over certain field extensions of $\mathbb{Q}$. However, these extensions are still quite small proper subfields of $\mathbb{R}$, and it seems clear that it will never be possible to deal directly with the complete field $\mathbb{R}$ in the binary world of our computers. In practice, this limitation has severe consequences. If one wants to use, at least conceptually, the power of $\mathbb{R}$, then one must resort to *approximations* throughout the computation, and this calls for analyses of stability and error propagation. It seems desirable, in order to avoid some of these difficulties, to devise *genuinely rational* algorithms whenever possible, and this is especially important in the quest for strongly polynomial algorithms. Forcing oneself to work entirely with an arbitrary subfield of $\mathbb{R}$ rules out the application of various analytic tools that are available when working over $\mathbb{R}$, and in so doing it helps to clarify the discrete aspects of the problems at hand. In particular, the setting of an arbitrary subfield excludes the rounding procedures that figure so prominently in the design and analysis of many numerical algorithms for the binary (Turing machine) model of computation. (For comments on this, see especially Megiddo [19, 21].) All of this suggests the need for a better understanding of the basic processes of analysis in an arbitrary $\mathbb{F}$ and in vector spaces over $\mathbb{F}$. See [1, 2], [4], [6, 7], [11], and [20] for some other papers that were motivated in part by the fact that the rational field $\mathbb{Q}$ and some other small subfields of $\mathbb{R}$ are more directly related to numerical optimization than is $\mathbb{R}$ itself.

For many problems in mathematical programming, the geometric notions of *separation, support, extreme points* and *duality* are fundamental. It was shown by Klee [14] that the standard theorems on extremal structure of closed convex subsets of $\mathbb{R}^d$ extend without change to s-closed subsets of $\mathbb{F}^d$, even when the ordered field $\mathbb{F}$ is not required to be archimedean. Mardešić [18] showed by dramatic examples that the standard support properties for closed convex subsets of $\mathbb{R}^d$ *do not extend* to s-closed subsets of $\mathbb{Q}^2$. Carvalho and Trotter [4] and Hartmann and Trotter [11] studied abstract linear duality theory, particularly duality in $\mathbb{Q}^d$, and Onn and Trotter [22] focused on separation. The present paper makes a more detailed study of the separation properties of convex subsets of a finite-dimensional vector space $\mathbb{X}$ over an archimedean ordered field $\mathbb{F}$. In particular, it delineates the very limited extent to which the separation properties that are so familiar and so useful in $\mathbb{R}^d$ can be extended to $\mathbb{X}$. One of our results (Theorem 3.2) is a new separation theorem for $\mathbb{R}^d$ that was developed with the aim of applying it to $\mathbb{F}^d$. It is a *maximal separation theorem* for what we call *free separation*, and it is in a sense the unique such theorem for $\mathbb{R}^d$ (see 3.3). For the case in which $\mathbb{F} \neq \mathbb{R}$, maximal separation theorems are established for free and for strong separation (6.2 and 6.5). Another result

(6.7) shows that in $\mathbb{X}$, a limited but very important separation property is enjoyed by all s-closed sets. Our final separation result (7.4) deals with $\mathbb{F}$-valued continuous convex functions on a convex subset C of $\mathbb{X}$. It gives intrinsic (and sharp) conditions, in terms of properties of the domain C, for the strong separation of epigraphs of convex functions from points not in the epigraph. As a corollary, we see when convex functions are suprema of affine functions.

Here are the section headings:

2. Preliminaries;
3. A maximal theorem for free separation in $\mathbb{R}^d$;
4. Successes and failures in $\mathbb{Q}^d$;
5. Limitations on separation when $\mathbb{F} \neq \mathbb{R}$;
6. Maximal theorems for free and for strong separation when $\mathbb{F} \neq \mathbb{R}$;
7. Convex functions on $\mathbb{X}$ as suprema of affine functions.

2. Preliminaries

In the interest of brevity, we adopt the following

Standing hypotheses:

- $\mathbb{F}$ is an archimedean ordered field;
- $\mathbb{X}$ is a finite-dimensional vector space over $\mathbb{F}$;
- d is the dimension of $\mathbb{X}$;
- B and C are nonempty convex subsets of $\mathbb{X}$.

Since $\mathbb{F}$ is an ordered field, the usual definitions of *segment*, *ray*, *line*, etc., may be used without change in the space $\mathbb{X}$. In particular, a *closed segment* in $\mathbb{X}$ is a set of the form

$$[x, y] = \{(1-\lambda)x + \lambda y : \lambda \in \mathbb{F} \wedge 0 \leq \lambda \leq 1\}$$

for distinct points x and y of $\mathbb{X}$. A subset C of $\mathbb{X}$ is *convex* if $[x, y] \subset C$ for each pair of points $x, y \in C$.

The field $\mathbb{F}$ is topologized in the usual way, taking its open intervals – sets of the form

$$]\alpha, \gamma[= \{\beta : \alpha < \beta < \gamma\}$$

– as a basis for the open sets. For each line L in $\mathbb{X}$ and each choice of two distinct points x and y of L, the parametrization of L as $\{(1-\lambda)x + \lambda y : \lambda \in \mathbb{F}\}$ provides an invertible affine transformation of $\mathbb{F}$ onto L. That gives rise in L to a natural interval topology that is independent of the choice of x and y. The interval topology on lines in $\mathbb{X}$ then leads to a natural topology for $\mathbb{X}$ itself, calling a convex set *open* if its intersection with each line is open in the line's interval topology – and then taking the open convex subsets of $\mathbb{X}$ as a basis for the entire collection of open sets. When the dimension d is finite, this natural topology for $\mathbb{X}$ is the same as the product topology that $\mathbb{X}$ inherits by virtue of being isomorphic with $\mathbb{F}^d$. With respect to this topology, the interior, closure, and boundary of C in $\mathbb{X}$ are denoted by $\mathrm{int}(C)$, $\mathrm{cl}(C)$, and $\mathrm{bd}(C)$ respectively, and the interior of C relative to its own affine hull $\mathrm{aff}(C)$ is denoted by $\mathrm{relint}(C)$.

A subset S of $\mathbb{X}$ is *bounded* if for each neighborhood U of the origin there exists $\mu \in \mathbb{F}$ such that $S \subset \mu U$. When d is finite, this is equivalent to the condition that for some (and hence, in fact, for every) vector basis $b_1, \dots, b_d$ of $\mathbb{X}$, there exists $\alpha \in \mathbb{F}$ such that S is contained in the parallelotope

$$\Big\{\sum_{i=1}^{d} \alpha_i b_i : \alpha_1, \dots, \alpha_d \in [-\alpha, \alpha]\Big\}.$$

The following are well-known elementary properties of convex subsets of $\mathbb{R}^d$, and the usual proofs show that they are valid in $\mathbb{X}$ as well. We use these properties freely, often without specific reference. (In 2.2, $[p, q[$ denotes the segment that is "closed at p, open at q.")

Lemma 2.1. relint(C) *is nonempty.*

Lemma 2.2. *If* $p \in \mathrm{int}(C)$ *and* $q \in \mathrm{cl}(C)$, *then* $[p, q[\subset \mathrm{int}(C)$.

Lemma 2.3. *If* $p \in \mathrm{int}(C)$, *R is a ray issuing from the origin, and the closure* $\mathrm{cl}(C)$ *contains a translate of R, then the ray $p + R$ is contained in* $\mathrm{int}(C)$.

We note in passing that the results 2.1–2.3 hold even when the field $\mathbb{F}$ is nonarchimedean, and only 2.1 requires finite dimensionality. Similar comments apply to several later results. However, we do not discuss this matter further, because we want to focus on finite-dimensional vector spaces over archimedean ordered fields.

If p is a point of a convex subset C of $\mathbb{X}$, and $L_p(C)$ is the set of all points $q \in \mathbb{X}$ such that the set $p + \mathbb{F}q$ is contained in C, then $L_p(C)$ is a linear subspace of $\mathbb{X}$. It follows from 2.3 that when C is fixed, the subspace $L_p(C)$ is the same for all $p \in \mathrm{relint}(C)$; this subspace is called the *lineality space* of C and is denoted by $\mathrm{ls}(C)$. (When the convex set C is closed or open, $L_p(C)$ is the same for all $p \in C$.)

A subset C of $\mathbb{X}$ is called *segmentally closed* (hereafter, *s-closed*) if, for each closed segment $[x, y]$, the intersection $C \cap [x, y]$ is empty or a singleton or a closed segment. This condition implies that C is convex and (with the aid of 2.1 and 2.2) closed. Every closed convex subset of $\mathbb{R}^d$ is s-closed, but this is not true for a general $\mathbb{X}$. For example, if $\alpha \in \mathbb{R} \setminus \mathbb{F}$ and $\alpha > 0$, then the set $\{\gamma \in \mathbb{F} : -\alpha < \gamma < \alpha\}$ is both open and closed in $\mathbb{F}$ but it is not s-closed. For a more interesting example, suppose that $d \geq 2$, let $\alpha_1, \dots, \alpha_d$ be d distinct members of $\mathbb{R} \setminus \mathbb{F}$ such that the set $\{1, \alpha_1, \dots, \alpha_d\}$ is linearly independent over $\mathbb{F}$, and set

$$C_1 = \{(\gamma_1, \dots, \gamma_d) \in \mathbb{F}^d : |\textstyle\sum_{i=1}^{d} \alpha_i \gamma_i| \leq 1\}.$$

Then the convex set C_1 is both closed and open in $\mathbb{F}^d$. Though C_1 is itself unbounded, the intersection $C_1 \cap L$ is bounded for each line L in $\mathbb{F}^d$. However, $C_1 \cap L$ is never a closed segment $[x, y]$ nor an open segment $]x, y[$ (with $x, y \in \mathbb{F}^d$), for in $\mathbb{F}^d$ the "ends are missing." The condition of s-closedness avoids the difficulties associated with "missing ends" and thus facilitates searches over lines that intersect the set.

A *hyperplane* in $\mathbb{X}$ is a flat (affine subspace) of dimension $d - 1$. Equivalently, it is a translate of a $(d-1)$-dimensional vector subspace of $\mathbb{X}$ and hence is of the form $H = \{x : f(x) = \gamma\}$, where $\gamma \in \mathbb{F}$ and the functional $f : \mathbb{X} \to \mathbb{F}$ is linear and not identically zero. The sets $\{x : f(x) \leq \gamma\}$ and $\{x : f(x) \geq \gamma\}$ are the *closed halfspaces* bounded by H, and the corresponding *open halfspaces* are defined by using $<$ and $>$ in place of $\leq$ and $\geq$.

Two sets B and C are *weakly* ⟨resp. *strictly*⟩ *separated* by a hyperplane H if B lies in one of the two closed ⟨resp. open⟩ halfspaces bounded by H and C lies in the other one.

Note, for the set C_1 described above, that even though C_1 is closed and convex, no point of $\mathbb{F}^d \setminus C_1$ is even weakly separated from C_1. By contrast, we show in Section 6 that if C is a convex subset of $\mathbb{F}^d$ that is bounded and closed, or that is s-closed, then each point of $\mathbb{F}^d \setminus C$ is strictly (in fact, strongly) separated from C.

Two sets B and C are *strongly separated* by a hyperplane $H = \{x : f(x) = \gamma\}$ if there exists $\epsilon > 0$ such that one of B and C lies in the halfspace $\{x : f(x) \leq \gamma - \epsilon\}$ and the other lies in the halfspace $\{x : f(x) \geq \gamma + \epsilon\}$. Let $\Sigma(B, C)$ denote the set of all $f \in \mathbb{X}^*$ (the conjugate space of $\mathbb{X}$) for which there exist γ and ϵ as described. Then $\Sigma(B, C)$ is a convex subset of $\mathbb{X}^*$, and B and C are strongly separated if and only if $\Sigma(B, C) \neq \emptyset$. We say that B and C are *freely separated* provided the set $\Sigma(B, C)$ has nonempty interior in $\mathbb{X}^*$. This is a stronger condition than strong separation. For example, two parallel but disjoint flats in $\mathbb{X}$ are strongly separated but not freely separated.

For surveys of theorems dealing with weak, strict, strong, and some other sorts of separation, see [16] and [5]. The notion of a *maximal separation theorem* was introduced in [15], and fourteen such theorems (all in $\mathbb{R}^d$) were proved there. Here we refine the terminology of [15] and then prove some new maximal separation theorems for $\mathbb{X}$ – one for the case $\mathbb{F} = \mathbb{R}$ and two for the case in which $\mathbb{F}$ is a proper subfield of $\mathbb{R}$.

Suppose that V is a vector space over an ordered field, $\mathcal{K}$ is the class of all nonempty convex proper subsets of V, $\mathcal{B}$ and $\mathcal{C}$ are nonempty subclasses of $\mathcal{K}$, and "S-" denotes a type of separation in V. We say that the pair $\{\mathcal{B}, \mathcal{C}\}$ is *$\mathcal{B}$-maximal for S-separation* in V provided that the following conditions are satisfied:

1. The sets B and C are S-separated whenever $B \in \mathcal{B}$ and $C \in \mathcal{C}$ with $B \cap C = \emptyset$.

$2_{\mathcal{B}}$. For each $B \in \mathcal{B}$ there exists $C \in \mathcal{C}$ such that $B \cap C = \emptyset$.

$3_{\mathcal{B}}$. For each $Y \in \mathcal{K} \setminus \mathcal{B}$ there exists $C \in \mathcal{C}$ such that $Y \cap C = \emptyset$ but Y and C are not S-separated.

Similarly, the pair $\{\mathcal{B}, \mathcal{C}\}$ is *$\mathcal{C}$-maximal for S-separation* in V provided that condition (1) holds, and also

$2_{\mathcal{C}}$. For each $C \in \mathcal{C}$ there exists $B \in \mathcal{B}$ such that $B \cap C = \emptyset$.

$3_{\mathcal{C}}$. For each $Z \in \mathcal{K} \setminus \mathcal{C}$ there exists $B \in \mathcal{B}$ such that $B \cap Z = \emptyset$ but B and Z are not S-separated.

Finally, we say that the pair $\{\mathcal{B}, \mathcal{C}\}$ *specifies a maximal theorem for S-separation in V* provided that $\{\mathcal{B}, \mathcal{C}\}$ is both $\mathcal{B}$-maximal and $\mathcal{C}$-maximal for S-separation.

3. A maximal theorem for free separation in $\mathbb{R}^d$

To prepare for the main result of this section, we need the following.

Lemma 3.1 (Line-free convex sets). *If B and C are line-free convex subsets of $\mathbb{R}^d$, the vector difference $B - C$ contains a line if and only if there is a ray R issuing from the origin such that each of B and C contains a translate of R.*

Proof. At several points in the proof, Lemma 2.3 is used without specific reference.

It is obvious that $B - C$ contains a line if the stated condition is satisfied. Suppose, conversely, that $B - C$ contains a line $p + \mathbb{R}q$, so that for each $\tau \in \mathbb{R}$ there exist $b_\tau \in B$

and $c_\tau \in C$ with $p + \tau q = b_\tau - c_\tau$. Then for each $\tau \geq 0$,

$$p = \bar{b}_\tau - \bar{c}_\tau$$

where

$$\bar{b}_\tau = \tfrac{1}{2}(b_\tau + b_{-\tau}) \in B \qquad \text{and} \qquad \bar{c}_\tau = \tfrac{1}{2}(c_\tau + c_{-\tau}) \in C.$$

Since the fixed point p is equal to $\bar{b}_\tau - \bar{c}_\tau$ for each real τ, the sets $\{\bar{b}_\tau : \tau \geq 0\}$ and $\{\bar{c}_\tau : \tau \geq 0\}$ are both bounded or both unbounded.

Now suppose first that the sets $\{\bar{b}_\tau : \tau \geq 0\}$ and $\{\bar{c}_\tau : \tau \geq 0\}$ are both bounded. If the set $\{b_\tau : \tau \geq 0\}$ is also bounded, then so is the set $\{b_{-\tau} : \tau \geq 0\}$, and since $p + \tau q = b_\tau - c_\tau$ for all τ it follows that the sets $\{c_\tau : \tau \geq 0\}$ and $\{c_{-\tau} : \tau \geq 0\}$ are both unbounded. A routine argument then shows (using compactness of the unit sphere) that there is a line in the set C, contradicting the assumption that C is line-free. In the remaining case, the sets $\{b_\tau : \tau \geq 0\}$ and $\{b_{-\tau} : \tau \geq 0\}$ are both unbounded, and this leads to the conclusion that B contains a line.

There remains the situation in which the sets $\{\bar{b}_\tau : \tau \geq 0\}$ and $\{\bar{c}_\tau : \tau \geq 0\}$ are both unbounded. Then there exists a sequence $\tau_i \to \infty$ such that $\|\bar{b}_{\tau_i}\| \to \infty$. With $\mu_i = \|\bar{b}_{\tau_i}\|^{-1}$, we may assume (again using compactness of the unit sphere) that the sequence $\mu_i \bar{b}_{\tau_i}$ converges to a vector u of unit norm. It is then true that the sequence $\mu_i \bar{c}_{\tau_i}$ also converges to u, and with b_0 and c_0 denoting relative interior points of B and C it follows by routine arguments (similar to those in [8] and [13]) that B and C contain the rays $b_0 + [0, \infty[u$ and $c_0 + [0, \infty[u$ respectively. □

When C is a convex subset of $\mathbb{R}^d$ and F is a ray or a flat in $\mathbb{R}^d$, we say that F is an *asymptote* of C provided that $F \cap \mathrm{cl}(C) = \emptyset$ and $0 \in \mathrm{cl}(C - F)$. (For an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^d$, the second condition is equivalent to the assertion that $\inf\{\|x - y\| : x \in C \wedge y \in F\} = 0$.) When F is a flat of dimension k, it is called a *k-asymptote*, and when F is a ray it is called a *ray-asymptote*. A convex set C is said to be *continuous* when C is closed and nonempty, $\mathrm{bd}(C)$ does not contain any ray, and C does not admit any ray-asymptote. This terminology in [8] was motivated by the fact that the "continuity" of a closed convex set is equivalent to the condition that its support function (mapping the unit sphere into $[0, \infty]$) is continuous. It is easily seen that a line L is a 1-asymptote of a closed convex set C if and only if L contains a ray-asymptote of C. It was shown in [13] that a continuous set C does not admit asymptotes of any dimension, which amounts to saying that all linear images of C are closed. An independent proof of this appears in [3].

Theorem 3.2 (A maximal theorem for free separation in $\mathbb{R}^d$). *With $d \geq 2$, suppose that $\mathcal{B}$ denotes the class of all continuous convex subsets of $\mathbb{R}^d$ and $\mathcal{C}$ denotes the class of all line-free closed convex subsets of $\mathbb{R}^d$. Then for each disjoint pair $B \in \mathcal{B}$ and $C \in \mathcal{C}$, the sets B and C are freely separated. The pair $\{\mathcal{B}, \mathcal{C}\}$ specifies a maximal theorem for free separation.*

Proof. Theorem 1.3 of [8] implies that if B and C are disjoint closed convex sets and at least one of them is continuous, then B and C are strongly separated. That applies here, and from the fact that B and C are strongly separated it follows that the origin does not belong to the closure of the vector difference $B - C$.

We claim also that $B - C$ is line-free. For this it suffices, in view of 3.1 and the fact that neither B nor C contains a line, to rule out the existence of a ray R issuing from the origin and of points $b \in B$ and $c \in C$ such that $b + R \subset B$ and $c + R \subset C$. Suppose that R, b, and c do exist, and consider the intersection of the set B with the 2-dimensional set

$$S = \operatorname{conv}((b + R) \cup (c + R)).$$

Since $b + R$ is contained in B while $c + R$ is disjoint from B, it is easy to verify that for some $p \in [b, c]$ the ray $p + R$ either contains a ray in B's boundary or determines a line that is an asymptote of B. Each of these possibilities contradicts the assumption that the set B is continuous.

From the fact that $B - C$ is line-free it follows by a routine argument (using, for example, 2.3) that the same is true of the closed convex cone $K = \operatorname{cl}([0, \infty[\,(B - C))$. Hence $K \cap -K = \{0\}$. Now let $F = \{f \in (\mathbb{R}^d)^* : f(k) \geq 0 \text{ for all } k \in K\}$. The fact that K is line-free implies, by a standard result on polars of convex cones, that the set F is not contained in any hyperplane. Hence, by 2.1, F has nonempty interior.

The easy constructions that establish maximality are left to the reader. They are similar to those in [15]. □

Recall that a point q of a closed convex subset C of $\mathbb{X}$ is called a *support point* of C if C is contained in a closed halfspace whose bounding hyperplane H passes through q; H itself is then called a *supporting hyperplane* of C. Further, q is an *exposed point* of C if C is supported at q by a hyperplane that intersects C only at q, and q is a *smooth point* of C if there is a unique hyperplane supporting C at q. In proving the next result, we need the fact that each line-free closed convex subset of $\mathbb{R}^d$ has an exposed point. That was proved in the bounded case by Straszcewicz [25] and extended to unbounded sets in [12].

For each type of separation considered in [15], two or more maximal separation theorems are established there. In each case, the classes $\mathcal{B}$ and $\mathcal{C}$ are *invariant* in the sense that they are closed under nonsingular affine mappings of the containing space. However, the following result shows that for free separation in $\mathbb{R}^d$, there is only one maximal separation theorem in which at least one of the classes is invariant.

Theorem 3.3 (Uniqueness of the maximal theorem for free separation in $\mathbb{R}^d$). *With $d \geq 2$, suppose that $\mathbb{R}^d$ is equipped with the usual Euclidean distance, that the pair $\{\mathcal{B}, \mathcal{C}\}$ specifies a maximal theorem for free separation, and that at least one of the classes $\mathcal{B}$ and $\mathcal{C}$ is closed under the proper rigid motions of $\mathbb{R}^d$. Then one of $\mathcal{B}$ and $\mathcal{C}$ is the class of all continuous convex subsets of $\mathbb{R}^d$ and the other is the class of all nonempty line-free closed convex subsets of $\mathbb{R}^d$.*

Proof. Let $\mathcal{B}' = \{\operatorname{cl}(B) : B \in \mathcal{B}\}$ and $\mathcal{C}' = \{\operatorname{cl}(C) : C \in \mathcal{C}\}$. Then each member of $\mathcal{B}' \cup \mathcal{C}'$ is line-free, for if a set contains a line no set can be freely separated from it. Now suppose that some member C of $\mathcal{C}$ fails to be closed. Let $q \in \operatorname{cl}(C) \setminus C$, let J be a hyperplane that supports $\operatorname{cl}(C)$ at q, let J^+ be the closed halfspace that is bounded by J and contains C, and let J^- denote the other closed halfspace that is bounded by J. Let B be an arbitrary member of $\mathcal{B}$, p an exposed point of $\operatorname{cl}(B)$, H a hyperplane that supports $\operatorname{cl}(B)$ at p and has $H \cap \operatorname{cl}(B) = \{p\}$, and H^+ a closed halfspace that is bounded by H and contains B. Then produce a pair (B'', C'') as follows:

- if $\mathcal{B}$ is invariant under proper rigid motions, let B'' denote the image of B under a motion that carries p onto q and H^+ onto J^-; and let $C'' = C$;

- if $\mathcal{C}$ is invariant under proper rigid motions, let C'' denote the image of C under a motion that carries q onto p and J^- onto H^+; and let $B'' = B$.

In each case, $B'' \in \mathcal{B}$, $C'' \in C$, $B'' \cap C'' = \emptyset$, and B'' and C'' are not freely separated. It follows from this contradiction that each member of $\mathcal{B} \cup \mathcal{C}$ is a closed set.

Now we claim that at most one of $\mathcal{B}$ and $\mathcal{C}$ can include a set that is not continuous. For suppose that both $\mathcal{B}$ and $\mathcal{C}$ include noncontinuous members. Then there are sets $B \in \mathcal{B}$ and $C \in \mathcal{C}$, hyperplanes H and J that bound halfspaces H^+ and J^+ containing B and C respectively, and rays $R \subset H$ and $S \subset J$ such that R is an asymptote or boundary ray of B and S is an asymptote or boundary ray of C. Moving the appropriate one of B and C by a suitable rigid motion, we can obtain a member of $\mathcal{B}$ and a member of $\mathcal{C}$ such that the two members lie in disjoint halfspaces and the bounding hyperplanes of these halfspaces contain parallel rays, one a boundary ray or asymptote of one of the sets, the other a boundary ray or asymptote of the other set. Then, even though the two sets are strongly separated, they are not freely separated.

The above argument shows that at most one of the families $\mathcal{B}$ and $\mathcal{C}$ can have a member that is not continuous. Hence at least one – say $\mathcal{B}$ – consists exclusively of continuous sets. It then follows, from the maximality assumption in conjunction with 3.2, that $\mathcal{C}$ includes all line-free closed convex sets. But then, again from maximality and 3.2, $\mathcal{B}$ must consist of all continuous sets. □

4. Successes and failures in $\mathbb{Q}^d$

In attempting to carry analytic machinery from $\mathbb{R}^d$ to $\mathbb{Q}^d$, one naturally hopes for success but must be prepared for striking failures. In addition to the easy examples in [9], we mention an impressive example of Körner [17]. He first observes that if $f : \mathbb{Q} \to \mathbb{Q}$ is a differentiable bijection with nonzero derivative, and if the inverse f^{-1} is continuous, then f^{-1} is differentiable. However, he then constructs a differentiable bijection whose derivative is everywhere equal to 1 but whose inverse is everywhere discontinuous.

Another dramatic example is provided by V. Mardešić [18]. Suppose that C is a bounded closed convex subset of $\mathbb{X}$ and that $\mathrm{int}(C) \neq \emptyset$. For the case in which $\mathbb{F} = \mathbb{R}$, it is well-known that $\mathrm{bd}(C)$ is topologically a $(d-1)$-sphere, that each point of $\mathrm{bd}(C)$ is a support point, that the smooth points in $\mathrm{bd}(C)$ constitute "almost all" of $\mathrm{bd}(C)$ (both in the measure-theoretic sense and in the sense associated with the Baire category theorem), and that C is the closed convex hull of its exposed points. When $\mathbb{F}$ is a proper subfield of $\mathbb{R}$, the set $\mathrm{bd}(C)$ may be empty, but of course $\mathrm{bd}(C)$ is infinite when $d \geq 2$ and C is bounded and s-closed. Mardešić shows that in $\mathbb{Q}^2$ (indeed, in $\mathbb{F}^2$ for any countable field $\mathbb{F} \subset \mathbb{R}$) there are bounded s-closed convex bodies K_1, K_2, K_3 and K_4 that behave as follows:

- K_1 has no support points;
- each boundary point of K_2 is exposed but not smooth;
- each boundary point of K_3 is smooth but not exposed;
- each boundary point of K_4 is both smooth and exposed.

Although the above examples are discouraging, we may take some encouragement from the results of [14] on extremal structure. These show that the usual results on extremal structure (e.g., the fact that a bounded closed convex set is the convex hull of its set of extreme points) extend to a finite-dimensional vector space over an arbitrary (even non-archimedean) ordered field when the condition of closedness is replaced by that of s-closedness.

The situation for polyhedra is also encouraging. A *polyhedron* is the intersection of a finite number of closed halfspaces, and a *polytope* is a bounded polyhedron. Equivalently, a polytope is the convex hull of a finite set. It is evident that each polyhedron is s-closed. With respect to separation properties, polyhedra and polytopes in $\mathbb{X}$ behave just as they do in $\mathbb{R}^d$:

- if B is a polytope, C is a line-free polyhedron, and $B \cap C = \emptyset$, then B and C are freely separated;
- if B and C are disjoint polyhedra, then B and C are strongly separated;
- if B and C are polyhedra whose relative interiors are disjoint, then B and C are weakly separated.

In fact, the usual proofs of these and many other properties of polyhedra (see [10, 24, 27]) either apply immediately in a finite-dimensional vector space over an arbitrary (even nonarchimedean) ordered field or they can be easily modified so as to apply in that setting.

5. Limitations on separation when $\mathbb{F} \neq \mathbb{R}$

We turn now to the separation situation in $\mathbb{X}$ when $\mathbb{F} \neq \mathbb{R}$. The goal is to find conditions on the structure of individual convex subsets B and C of $\mathbb{X}$ that assure some sort of hyperplane separation when B and C happen to be disjoint. This section contains examples showing that, by comparison with the situation in $\mathbb{R}^d$, only very limited separation results can be expected when $\mathbb{F} \neq \mathbb{R}$. Section 6 contains maximal theorems for free and for strong separation when $\mathbb{F} \neq \mathbb{R}$.

Since $\mathbb{X}$, as a vector space over the field $\mathbb{F}$, is isomorphic with $\mathbb{F}^d$, we assume for convenience that $\mathbb{X} = \mathbb{F}^d$ and we work with the natural embedding of $\mathbb{F}^d$ in $\mathbb{R}^d$. When C is a convex subset of $\mathbb{X}$, it is essential to distinguish C's role as a subset of $\mathbb{X}$ from its role as a subset of $\mathbb{R}^d$. To do this, we make frequent use of the subscript $_\mathbb{F}$ and the prefix $\mathbb{F}$- to refer to operations in $\mathbb{X}$, and of the subscript $_\mathbb{R}$ and the prefix $\mathbb{R}$- to refer to operations in $\mathbb{R}^d$. For example, C's closures in $\mathbb{X}$ and in $\mathbb{R}^d$ are denoted by $\mathrm{cl}_\mathbb{F}(C)$ and $\mathrm{cl}_\mathbb{R}(C)$, respectively, and they are referred to as the $\mathbb{F}$-*closure* and the $\mathbb{R}$-*closure* of C. Similarly, the lineality space of C in $\mathbb{X}$ is denoted by $\mathrm{ls}_\mathbb{F}(C)$ and the lineality space of $\mathrm{cl}_\mathbb{R}(C)$ in $\mathbb{R}^d$ is denoted by $\mathrm{ls}_\mathbb{R}(C)$. Other uses of $\mathbb{F}$ and $\mathbb{R}$ as subscripts or prefixes should be clear from context.

The following two properties of C prove to be relevant:

(LIF) the set $\mathrm{cl}_\mathbb{R}(C)$ is line-free;

(LSD) $\mathrm{cl}_\mathbb{R}(\mathrm{ls}_\mathbb{F}(C)) = \mathrm{ls}_\mathbb{R}(\mathrm{cl}_\mathbb{R}(C))$; i.e., the set $\mathrm{ls}_\mathbb{F}(C)$ is dense in the set $\mathrm{ls}_\mathbb{R}(C)$ ($= \mathrm{ls}_\mathbb{R}(\mathrm{cl}_\mathbb{R}(C))$).

(Think of "LIF" and "LSD" as standing respectively for "line-free" and "lineality space dense.")

For convenience, the properties LIF and LSD are expressed in terms of the natural embedding of $\mathbb{X}$ in $\mathbb{R}^d$, but they are in fact independent of the embedding. To express this independence more precisely, suppose that $B_\mathbb{F}$ is a basis for $\mathbb{X}$ as a vector space over $\mathbb{F}$, that $B_{\mathbb{R}}$ is a basis for $\mathbb{R}^d$ as a vector space over $\mathbb{R}$, that the mapping $\psi : B_{\mathbb{F}} \to B_{\mathbb{R}}$ is a bijection, and that $\varphi : \mathbb{X} \to \mathbb{R}^d$ is the injection of $\mathbb{X}$ into $\mathbb{R}^d$ that extends ψ and is an $\mathbb{F}$-vector-space isomorphism of $\mathbb{X}$ into $\mathbb{R}^d$. Then each of the properties LIF and LSD holds for the natural embedding if and only if it holds for the embedding φ. A similar independence of the embedding holds for the other properties and conditions, used in this and the next two sections, that involve subsets of $\mathbb{X}$ in their relationship to $\mathbb{R}^d$.

Theorem 5.1 (A limitation on weak separation). *If C is a convex subset of $\mathbb{X}$ such that each point of $\mathbb{X} \setminus \mathrm{cl}_{\mathbb{F}}(C)$ is weakly separated from C, then C has the property LSD.*

Proof. Let $S = \mathrm{ls}_{\mathbb{F}}(C)$, $C' = \mathrm{cl}_{\mathbb{R}}(C)$, and $S' = \mathrm{ls}_{\mathbb{R}}(C')$. By 2.1, we may assume without loss of generality that $0 \in \mathrm{int}_{\mathbb{F}}(C)$, whence $0 \in \mathrm{int}_{\mathbb{R}}(C')$. Note also that if $S' = \mathbb{R}^d$, then $C' = \mathbb{R}^d$ and $C = \mathbb{X}$. Thus it suffices to consider the case in which $\dim_{\mathbb{R}}(S') < d$, whence $\dim_{\mathbb{F}}(S) < d$ and there is a nondegenerate subspace T of $\mathbb{X}$ such that $S \cap T = \{0\}$ and $S + T = \mathbb{X}$. Setting $k = \dim_{\mathbb{F}}(T)$ and assuming that each point of $T \setminus \mathrm{cl}_{\mathbb{F}}(C)$ is weakly separated from C, we will show that $\dim_{\mathbb{R}}(S') \leq d - k$. Since $\dim_{\mathbb{F}}(T) = d - \dim_{\mathbb{F}}(S)$, this will imply that $\dim_{\mathbb{R}}(S') \leq \dim_{\mathbb{F}}(S)$, whence $S' = \mathrm{cl}_{\mathbb{R}}(S)$ and C has the property LSD.

For each point $t \in T \setminus \{0\}$, there is a nonzero $\mu_t \in \mathbb{F}$ such that the point $\bar{t} = \mu_t t$ does not belong to $\mathrm{cl}_{\mathbb{F}}(C)$. By assumption, there is an $\mathbb{F}$-hyperplane H_t through the origin in the space $\mathbb{X}$ such that C is contained in the $\mathbb{F}$-closed halfspace J_t that consists of all points of $\mathbb{X}$ on the origin's side of the $\mathbb{F}$-hyperplane $H_t + \bar{t}$. Since $0 \in \mathrm{int}_{\mathbb{F}}(C)$, it is clear that $t \notin H_t$. Note also that the $\mathbb{R}$-closed halfspace $\mathrm{cl}_{\mathbb{R}}(J_t)$ contains C', and hence S' is contained in the $\mathbb{R}$-hyperplane $H_t' = \mathrm{cl}_{\mathbb{R}}(H_t)$.

Now set $L_0 = \mathbb{X}$ and choose $t_1 \in (T \cap L_0) \setminus \{0\}$. Then set $L_1 = H_{t_1} = L_0 \cap H_{t_1}$, thus obtaining $\dim_{\mathbb{F}}(L_1) = d-1$ and $S' \subset \mathrm{cl}_{\mathbb{R}}(L_1)$. If $k \geq 2$, we may choose $t_2 \in (L_1 \cap T) \setminus \{0\}$ and then set $L_2 = L_1 \cap H_{t_2}$, a $(d-2)$-dimensional subspace of $\mathbb{X}$ that contains S'. At the start of stage j of this process (where $j \leq k$), we have a $(d-j+1)$-subspace L_{j-1}, we choose $t_j \in (L_{j-1} \cap T) \setminus \{0\}$, and then we set $L_j = L_{j-1} \cap H_{t_j}$. At the end of the kth stage, the subspace L_k of $\mathbb{X}$ is of dimension $d-k$, and since

$$S' \subset \bigcap_{i=1}^{k} H_i = L_k,$$

it follows that $\dim_{\mathbb{R}}(S') \leq d - k$. That completes the proof. □

The following lemma is used in establishing additional limitations on separation.

Lemma 5.2 (A construction lemma). *With $d \geq 2$, suppose that G is a Euclidean ball in $\mathbb{R}^d$, that $q \in \mathrm{bd}(G)$, and that V is a dense subset of $\mathrm{int}(G)$. Then G contains a set W that has the following properties:*

- *W is a compact convex set with nonempty interior;*
- *q is an exposed point of W;*
- *with the exception of q, the extreme points of W all belong to V, and they form a sequence converging to q;*
- *the hyperplane H that supports G at q is the unique hyperplane in $\mathbb{R}^d$ that weakly separates q from W.*

Proof. We assume without loss of generality that $q = 0$ and $G = \{x \in \mathbb{R}^d : \|x - z\| \leq 1\}$, where $\|z\| = 1$ and $\langle z, h\rangle = 0$ for all $h \in H$.

For each $\epsilon > 0$, define the convex cone

$$K(\epsilon) = \{x : \langle z, x\rangle \geq \epsilon\|x\|\}.$$

Then $K(\epsilon)$ increases as ϵ decreases, and

$$\bigcup_{\epsilon>0} K(\epsilon) = \{0\} \cup \{x : \langle z, x\rangle > 0\}.$$

Note also that for each $x \in G \setminus K(\epsilon)$ it is true that

$$\langle x, x\rangle = \langle z - x, z - x\rangle - \langle z, z\rangle + 2\langle z, x\rangle < 1 - 1 + 2\epsilon\|x\| \leq 2\epsilon.$$

Now let T be a d-simplex such that $z \in \mathrm{int}(T)$, q is a vertex of T, and T's other vertices all belong to V. Let $\delta_1, \delta_2, \ldots$ and $\epsilon_1, \epsilon_2, \ldots$ be sequences of positive numbers, both converging to 0, and for each positive integer n let V_n be a finite subset of the intersection $V \cap (G \setminus K(\epsilon_n))$ such that each point of this intersection is within δ_n of some point of V_n. Then let

$$W = \mathrm{conv}(T \cup \bigcup_{n=1}^{\infty} V_n).$$

The first three properties claimed for W are now easily verified, using the final inequality of the preceding paragraph to see that the extreme points of W form a sequence converging to q. To complete the proof, we show that the fourth property also holds when the deltas and epsilons are chosen appropriately.

Let $U = \{x \in H : \|x\| = 1\}$. Other than H, each hyperplane J through the origin is of the form

$$J = J(\beta, u) = \{x : \langle \beta z + u, x\rangle = 0\}$$

for some $\beta \neq 0$ and $u \in U$, with $\beta > 0$ if and only if z is interior to the closed halfspace

$$J^+(\beta, u) = \{x : \langle \beta z + u, x\rangle \geq 0\}.$$

It suffices, for our purpose, to show that the deltas and epsilons can be chosen so that the constructed set W intersects the open halfspace $J^-(\beta, u) = \{x : \langle \beta z + u, x\rangle < 0\}$ for each $\beta > 0$ and $u \in U$.

We claim that $K(1/\sqrt{1+\beta^2}) \subset J^+(\beta, u)$ for each $u \in U$. To establish this, it suffices to consider $x = z + \gamma v$ with $v \in U$ and $\gamma \geq 0$, and to note that if $x \in K(1/\sqrt{1+\beta^2})$ then

$$\langle z, z + \gamma v\rangle \geq \frac{\sqrt{1+\gamma^2}}{\sqrt{1+\beta^2}},$$

so $\gamma^2 \leq \beta^2$ and we have $\langle \beta z + u, z + \gamma v\rangle = \beta + \gamma\langle u, v\rangle \geq 0$.

Now for each $u \in U$ and $\beta > 0$, let

$$p(\beta, u) = \frac{1}{1+\beta^2}(z - \beta u).$$

Then $p(\beta, u) \in J(\beta, u)$ and

$$\begin{aligned} \|z - p(\beta, u)\|^2 &= \left\| \left(1 - \frac{1}{1+\beta^2}\right) z + \frac{\beta}{1+\beta^2} \right\|^2 = \frac{1}{(1+\beta^2)^2} \|\beta^2 z + \beta u\|^2 \\ &= \frac{1}{(1+\beta^2)^2}(\beta^4 + \beta^2) = \frac{\beta^2}{1+\beta^2}, \end{aligned}$$

so

$$\|z - p(\beta, u)\| = \beta/\sqrt{1+\beta^2} < 1$$

and the ball of center $p(\beta, u)$ and radius

$$\rho(\beta) = 1 - (\beta/\sqrt{1+\beta^2})$$

is contained in G. From this it follows that the set $G \cap J^-(\beta, u)$ contains an open ball of radius $\rho(\beta)/2$, and hence, since $K(1/\sqrt{1+\beta^2}) \subset J^+(\beta, u)$, this same open ball is contained in the set $G \setminus K(1/\sqrt{1+\beta^2})$.

Because of the above observations, it is now clear that if we choose $\epsilon_n = 1/\sqrt{1+n^2}$, then $K(\epsilon_n) \subset J^+(n, u)$ for all $u \in U$ and $G \cap J^-(n, u)$ contains an open ball of radius $1 - n\epsilon_n$. Setting $\delta_n = (1 - n\epsilon_n)/2$ assures that the constructed set W intersects the set $G \cap J^-(n, u)$ for each n and u, and hence completes the proof. □

When $\mathbb{F} \neq \mathbb{R}$, a subset P of $\mathbb{X}$ is here called a *pseudotope* if it satisfies the following four conditions:

- P is a subset of $\mathbb{X}$ that is bounded, closed, and convex, and has nonempty interior;
- P is the $\mathbb{F}$-convex hull of the set $\text{ext}(P)$ of all extreme points of P;
- the set $\text{ext}(P)$ forms a sequence that converges to a point $q \in \mathbb{R}^d \setminus \mathbb{X}$;
- q is both an exposed point and a smooth point of the subset $\text{cl}_{\mathbb{R}}(P)$ of $\mathbb{R}^d$.

The point $q \in \mathbb{R}^d \setminus \mathbb{X}$ will be called the *pseudovertex* of the pseudotope P.

The following result is an immediate consequence of Theorem 5.2.

Theorem 5.3 (Existence of pseudotopes). *With hypotheses as in* 5.2, *suppose also that* $\mathbb{F} \neq \mathbb{R}$, $q \notin \mathbb{X}$, *and* $V = \mathbb{X} \cap \text{int}(G)$. *Then the set* $\mathbb{X} \cap W$ *is a pseudotope in* $\mathbb{X}$, *with* q *as its pseudovertex.*

Theorem 5.4 (s-closedness of pseudotopes). *With* $\mathbb{F} \neq \mathbb{R}$, *suppose that* q *is the pseudovertex of a pseudotope* P *in* $\mathbb{X}$, *that* L *is a line in* $\mathbb{X}$, *and that* $L' = \text{cl}_{\mathbb{R}}(L)$. *If* L' *misses* q, *then the intersection* $L \cap P$ *is empty, a singleton, or a closed segment in* $\mathbb{X}$. *Hence the set* P *is s-closed if and only if* q *does not lie on any line in* $\mathbb{R}^d$ *that passes through two points of* P.

Proof. The second statement is an immediate consequence of the first statement and the relevant definitions. Now, with $P' = \text{cl}_{\mathbb{R}}(P)$, suppose that L' misses q and yet the intersection $L \cap P$ is not as claimed. Then there is a point

$$y \in \text{bd}_{\mathbb{R}}(P') \setminus (\mathbb{X} \cup \{q\})$$

such that y belongs to the $\mathbb{R}$-closure of $L \cap P$. Since y is not an extreme point of P but must be an $\mathbb{R}$-convex combination of extreme points of P', it follows that the $\mathbb{R}$-hyperplane supporting P' at y must pass through q. However, that contradicts the exposedness of the pseudovertex q. □

Theorem 5.5 (A limitation on strong separation). *With $\mathbb{F} \neq \mathbb{R}$ and $d \geq 2$, suppose that B is a proper convex subset of $\mathbb{X}$ and is not a singleton. Let $B' = \mathrm{cl}_{\mathbb{R}}(B)$. Then there exist $w \in \mathbb{R}^d \setminus \mathbb{X}$ and $q \in B' \setminus \mathbb{X}$ such that q is the unique point of B' nearest to w. For each such pair (w, q), the Euclidean ball $G = \{x \in \mathbb{R}^d : \|x - w\| \leq \|w - q\|\}$ contains a set P that satisfies the following conditions:*

- *P is a pseudotope in $\mathbb{X}$;*
- *B is disjoint from P;*
- *B is not strongly separated from P;*
- *if no nonzero multiple of $w - q$ belongs to $\mathbb{X}$, then B is not even weakly separated from P.*

Proof. We assume as usual that $\mathbb{X} = \mathbb{F}^d \subset \mathbb{R}^d$. For each point $w \in \mathbb{R}^d$, let $g(w)$ denote the unique point of B' that is nearest to w with respect to the usual Euclidean distance in $\mathbb{R}^d$. Then the function $g : \mathbb{R}^d \to \mathrm{bd}_{\mathbb{R}}(B')$ is a continuous surjection. Since (with $\mathbb{F} \neq \mathbb{R}$) the space $\mathbb{R}^d \setminus \mathbb{X}$ is connected, the same is true of the image $g(\mathbb{R}^d \setminus \mathbb{X})$. Since the space $\mathbb{X}$ is totally disconnected, the same is true of the intersection $\mathbb{X} \cap \mathrm{bd}_{\mathbb{R}}(B')$. It follows, since $\mathrm{bd}_{\mathbb{R}}(B')$ is not a singleton, that there exists $w \in \mathbb{R}^d \setminus \mathbb{X}$ such that the point $g(w)$ does not belong to $\mathbb{X}$. Now set $q = g(w)$, define G as in the statement of 5.2, and let H' denote the hyperplane in $\mathbb{R}^d$ that is perpendicular to the segment $[w, q]$ at the point b. Then H' is the unique hyperplane in $\mathbb{R}^d$ that weakly separates G from B'.

Now let P be the pseudotope whose existence is guaranteed by 5.2–5.3, set $H = H' \cap \mathbb{X}$, a flat in $\mathbb{X}$, and consider two cases.

(a) If $\dim(H) = d - 1$, then H is a hyperplane in $\mathbb{X}$ and is the only hyperplane in $\mathbb{X}$ that weakly separates B and P. Since $\mathrm{dist}(B, P) = 0$, there is no hyperplane that strongly separates B and P. It is also clear that some nonzero multiple of $w - q$ belongs to $\mathbb{X}$.

(b) If $\dim(H) < d - 1$, then there is no hyperplane in $\mathbb{X}$ that weakly separates B and P. For if there were such a hyperplane in $\mathbb{X}$, its closure would be a hyperplane that weakly separates B' and P in $\mathbb{R}^d$ – that is, its closure would, by the uniqueness of H', be equal to H'. However, that implies $\dim(H) = d - 1$. □

To see that the condition in the following result cannot be entirely abandoned, see Theorem 6.9 at the end of the next section.

Theorem 5.6 (A limitation on weak separation). *Let C be a convex subset of $\mathbb{X}$, let $C' = \mathrm{cl}_{\mathbb{R}}(C)$, and suppose that the following condition is satisfied:*

- *there exists a point $q \in \mathrm{bd}_{\mathbb{R}}(C') \setminus \mathbb{X}$ that is contained in a supporting hyperplane of C' in $\mathbb{R}^d$ normal to a point of N, where*

$$N = \{y = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d : \exists i, j \in \{1, \ldots, d\} : \eta_i \in \mathbb{F} \wedge \eta_j \notin \mathbb{F}\}.$$

Then there is a pseudotope in $\mathbb{X} \setminus C$ (with pseudovertex q) that is not even weakly separated from C.

Proof. The proof follows immediately from the last statement in Theorem 5.5 once it is established that each point y of N has the property that $\mathbb{R}y \cap \mathbb{X} = \{0\}$. So, suppose that $y = (\eta_1, \ldots, \eta_d) \in N$ – say $\eta_1 \in \mathbb{F}$, $\eta_2 \notin \mathbb{F}$ – and there exists $\lambda \in \mathbb{R}$ such that $\lambda y \in \mathbb{X}$. Then $\lambda\eta_1 \in \mathbb{F}$ implies that $\lambda \in \mathbb{F}$, whence $\lambda\eta_2 \in \mathbb{F}$ leads to the contradiction that $\eta_2 \in \mathbb{F}$. □

Corollary 5.7. *Let C be a convex subset of $\mathbb{X}$, let $C' = \mathrm{cl}_{\mathbb{R}}(C)$, and suppose that $\mathbb{F}$ is countable. If the set $\mathrm{bd}_{\mathbb{R}}(C)$ contains uncountably many nonsmooth points, or the set of normal directions to supporting hyperplanes at smooth points of $\mathrm{bd}_{\mathbb{R}}(C)$ is uncountable, then there is a pseudotope in $\mathbb{X} \setminus C$ that is not even weakly separated from C.*

Proof. Since $\mathbb{F}$ is countable and $\mathbb{X}$ is finite-dimensional, the set $\mathbb{X}$ is countable and hence so is the intersection $\mathbb{X} \cap \mathrm{bd}_{\mathbb{R}}(C')$. If the set $\mathrm{bd}_{\mathbb{R}}(C)$ contains uncountably many nonsmooth points, it includes a nonsmooth point q that is not in $\mathbb{X}$. Among the hyperplanes that support C' at q, there are uncountably many normal directions and hence there is a normal that belongs the set N. Thus the conclusion follows from 5.6.

If the set of normal directions to supporting hyperplanes at smooth points of $\mathrm{bd}_{\mathbb{R}}(C')$ is uncountable, then the same is true of the directions of normals to supporting hyperplanes at smooth points that do not belong to $\mathbb{X}$, and among these normals there must be one that belongs to N. Again, 5.6 is applicable. □

A convex subset S of $\mathbb{X}$ will be called a *slab* if there exists a nonzero linear functional $f : \mathbb{X} \to \mathbb{F}$ and there exists an s-closed proper subset I of $\mathbb{F}$ such that $S = \{x \in \mathbb{X} : f(x) \in I\}$.

Corollary 5.8. *Suppose that $\mathbb{F} \neq \mathbb{R}$, $d \geq 3$, and C is a polyhedron in $\mathbb{X}$ that is neither a singleton nor a slab. Then there is a pseudotope in $\mathbb{X} \setminus C$ that is not weakly separated from C.*

Proof. Under the hypotheses, C has a t-face G with $1 \leq t \leq d-2$. We assume without loss of generality that $0 \in G$. Since the orthogonal complement of $\mathrm{aff}_{\mathbb{R}}(G)$ is the $\mathbb{R}$-closure of a linear subspace of $\mathbb{X}$, the set N defined in 5.6 must include a vector that is normal to G. Further, the set $\mathrm{cl}_{\mathbb{R}}(G) \setminus G$ is nonempty, and hence the stated conclusion follows from 5.6. □

At the end of the next section there is a separation result that is valid only when $\dim(\mathbb{X}) \leq 2$. Limitations on improving that result are indicated by the following examples.

Example 5.9. For each of the following statements there are disjoint closed convex sets B and C of $\mathbb{Q}^2$ that have the indicated properties:

- B is bounded and C is a closed segment, but B and C are not weakly separated;
- B is closed and has bounded intersection with each line, and C is a singleton, but B and C are not weakly separated;
- B and C are both pseudotopes but they are not weakly separated;
- B is bounded and s-closed, C is a line or a segment, but B and C are not strictly separated.

Proof. For the first example, let B' be a circular disk in $\mathbb{R}^2$ such that the boundary of B' includes a point p of $\mathbb{Q}^2$ for which the slope of the tangent to B' at p does not belong to

$\mathbb{Q}$. Let q be a point of $\mathbb{Q}^2$ such that the segment $[p, q]$ misses B'. Finally, set $B = B' \cap \mathbb{Q}^2$ and let C be the segment in $\mathbb{Q}^2$ whose ends are p and q.

For the second example let B be the set C_1 of Section 2.

For the third example, start with two circular disks B' and C' in $\mathbb{R}^2$ such that $B' \cap C'$ is a point $p \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and the common tangent line L in $\mathbb{R}^2$ has slope that does not belong to $\mathbb{Q}$. Approximate B' and C' by pseudotopes B and C, using the method of 5.2.

The final example can be constructed as follows. Let $C'' = \{(x, 0) : x \in \mathbb{R}\}$ or $C'' = \{(x, 0) : 0 \leq x \leq 1\}$, choose $\mu \in [0, 1] \setminus \mathbb{Q}$, and let B' be a circular disk in $\mathbb{R}^2$ that is tangent to C'' at the point $p = (\mu, 0)$. Then produce B'' from B' in the manner of the preceding example. Again, set $B = B'' \cap \mathbb{Q}^2$ and $C = C'' \cap \mathbb{Q}^2$. □

6. Maximal theorems for free and for strong separation when $\mathbb{F} \neq \mathbb{R}$

Assuming that $\mathbb{F} \neq \mathbb{R}$, we will obtain a maximal theorem for free separation and a maximal theorem for strong separation.

Theorem 6.1 (Free separation in $\mathbb{X}$). *If B and C are convex subsets of $\mathbb{X}$ such that $\mathrm{cl}_{\mathbb{R}}(B)$ and $\mathrm{cl}_{\mathbb{R}}(C)$ are freely separated in $\mathbb{R}^d$, then B and C are freely separated in $\mathbb{X}$.*

Proof. Let $e_1, \ldots, e_d$ be the standard basis for $\mathbb{R}^d$ and hence for $\mathbb{X}$ as well. Then, of course, a functional $f \in \mathbb{X}^*$ is uniquely determined by its values on the points e_i. It is therefore clear (since $\mathbb{X}$ is a dense subset of $\mathbb{R}^d$) that the set of all functionals whose values on $e_1, \ldots, e_d$ are in $\mathbb{F}$ is dense in $(\mathbb{R}^d)^*$. By hypothesis, the set of linear functionals $f \in (\mathbb{R}^d)^*$ for which $\sup f(B) < \inf f(C)$ has nonempty interior in $(\mathbb{R}^d)^*$. Hence $\Sigma(B, C)$ has nonempty interior in $\mathbb{X}^*$, and that yields the desired conclusion. □

Theorem 6.2 (A maximal theorem for free separation in $\mathbb{X}$). *Let $\mathcal{B}$ denote the class of all singletons in $\mathbb{X}$ and let $\mathcal{C}$ denote the class of all closed convex subsets C of $\mathbb{X}$ that have property LIF. If $\mathbb{F} \neq \mathbb{R}$ then the pair $\{\mathcal{B}, \mathcal{C}\}$ is $\mathcal{B}$-maximal for strong separation and $\mathcal{C}$-maximal for free separation. Hence $\{\mathcal{B}, \mathcal{C}\}$ specifies a maximal theorem for free separation in $\mathbb{X}$.*

Proof. Suppose that $C \in \mathcal{C}$ and $b \in \mathbb{X} \setminus C$, and let $C' = \mathrm{cl}_{\mathbb{R}}(C)$. Then C' is line-free and $\{b\}$ is continuous, so it follows from Theorem 3.2 that the sets $\{b\}$ and C' are freely separated in $\mathbb{R}^d$. But then Theorem 6.1 implies that $\{b\}$ and C are freely separated in $\mathbb{X}$. It is evident that if a convex subset C of $\mathbb{X}$ omits a point b of the closure $\mathrm{cl}_{\mathbb{F}}(C)$, or if $\mathrm{cl}_{\mathbb{R}}(C)$ is not line-free and b is an arbitrary point of $\mathbb{X} \setminus C$, then $\{b\}$ and C are not freely separated in $\mathbb{X}$. Hence the pair $\{\mathcal{B}, \mathcal{C}\}$ is $\mathcal{C}$-maximal for free separation in $\mathbb{X}$. To verify the $\mathcal{B}$-maximality of the pair for strong separation, use Theorem 5.5. □

The following special cases of Theorem 6.2 were noted by Hartmann and Trotter [11] and Onn and Trotter [22].

Theorem 6.3 (Separation for bounded sets and cones). *Suppose that C is a bounded closed convex subset of $\mathbb{X}$, or that C is a pointed closed convex cone of the form $[0, \infty[D$ where D is a bounded closed convex subset of a hyperplane in $\mathbb{X} \setminus \{0\}$. Then each point of $\mathbb{X} \setminus C$ is freely separated from C.*

Proof. In each case, the set C has property LIF, so the stated conclusion follows from 6.2. □

Note that 6.3 need not hold when C is merely a line-free closed convex cone in $\mathbb{X}$. For example, suppose that $\mathbb{F} = \mathbb{Q}$, let $\alpha_1, \ldots, \alpha_d$ denote real numbers that are linearly independent over $\mathbb{Q}$, and set

$$C_2 = \{(\gamma_1, \ldots, \gamma_d) \in \mathbb{F}^d : \textstyle\sum_{i=1}^d \alpha_i \gamma_i \geq 0\}$$

Then C_2 is a line-free closed convex cone in $\mathbb{Q}^d$ and yet no point of $\mathbb{Q}^d$ is weakly separated from C_2.

Next comes a relative of another observation of Onn and Trotter [22].

Theorem 6.4 (Strong separation of certain pairs of sets). *With $\mathbb{X}$ embedded in $\mathbb{R}^d$, suppose that C' is a closed convex subset of $\mathbb{R}^d$, B' is a continuous convex subset of $\mathbb{R}^d$ that is disjoint from C', and let $L' = \mathrm{ls}_{\mathbb{R}}(C')$. Suppose also that B' intersects $\mathbb{X}$ and that L' is the closure of a subspace L of $\mathbb{X}$. Then the subsets $B = B' \cap \mathbb{X}$ and $C = C' \cap \mathbb{X}$ are strongly separated by a hyperplane in $\mathbb{X}$.*

Proof. In $\mathbb{X}$, let M denote a subspace complementary to L, and let M' denote the closure of M in $\mathbb{R}^d$. Then $\mathbb{R}^d$ is the direct sum of L' and M', and $\mathbb{X}$ is the direct sum of L and M. Let Π denote the projection of $\mathbb{R}^d$ onto M' whose nullspace is L'. Then the closed convex subsets $\Pi(B')$ and $\Pi(C')$ of M' are respectively continuous and line-free. (To verify the former statement, use the fact (from [13]) that a continuous set does not admit asymptotes of any dimension.) Hence by 3.2 the sets $\Pi(B')$ and $\Pi(C')$ are freely separated by a hyperplane in M'. From Theorem 6.2 it follows that the intersections $\Pi(B') \cap M$ and $\Pi(C') \cap M$ are freely (and hence strongly) separated by a hyperplane H in M. Then the set $\Pi^{-1}(H) \cap \mathbb{X}$ is a hyperplane in $\mathbb{X}$, and it strongly separates B and C. □

The set C_2 introduced after 6.3 shows that 6.4 may fail when the set L' is not required to be the closure of a subspace of $\mathbb{X}$.

Theorem 6.5 (A maximal theorem for strong separation in $\mathbb{X}$). *Suppose that $\mathbb{F} \neq \mathbb{R}$. Let $\mathcal{B}$ denote the collection of all singletons in $\mathbb{X}$, and let $\mathcal{C}$ denote the collection of all closed convex subsets of $\mathbb{X}$ that have property LSD. Then the pair $\{\mathcal{B}, \mathcal{C}\}$ specifies a maximal theorem for strong separation in $\mathbb{X}$.*

Proof. If $B \in \mathcal{B}$ and $C \in \mathcal{C}$, it follows from Theorem 6.4 that B and C are strongly separated. If Y is a nonempty proper convex subset of $\mathbb{X}$ that is not a singleton, then Theorem 5.5 guarantees the existence of a pseudotope $C \subset \mathbb{X} \setminus Y$ that is not strongly separated from Y. Since C is bounded, so is $\mathrm{cl}_{\mathbb{R}}(C)$, and hence C and $\mathrm{cl}_{\mathbb{R}}(C)$ have the same lineality space $\{0\}$. Hence C has the property LSD and it follows that the pair $\{\mathcal{B}, \mathcal{C}\}$ is $\mathcal{B}$-maximal for strong separation.

To show that the pair $\{\mathcal{B}, \mathcal{C}\}$ is $\mathcal{C}$-maximal for weak separation, we must show that if Z is a nonempty convex subset of $\mathbb{X}$ that fails to be closed or that lacks the property LSD, then there is a singleton in $\mathbb{X} \setminus Z$ that is not strongly separated from Z. That is obvious when Z is not closed. If Z lacks the property LSD, then Theorem 5.1 guarantees the existence of a singleton in $\mathbb{X} \setminus \mathrm{cl}_{\mathbb{F}}(Z)$ that is not even weakly separated from C. □

Theorem 6.6 (Lineality spaces in $\mathbb{X}$ and $\mathbb{R}^d$). *If C is a convex subset of $\mathbb{X}$ that is s-closed or has property LIF, then C has property LSD.*

Proof. Let $S = \mathrm{ls}_{\mathbb{F}}(C)$, $C' = \mathrm{cl}_{\mathbb{R}}(C)$, and $S' = \mathrm{ls}_{\mathbb{R}}(C')$. We want to show that $\mathrm{cl}_{\mathbb{R}}(S) = S'$. That is obvious if C has LIF, for then $S = \{0\} = S'$. To complete the proof, we derive a contradiction from the assumption that C is s-closed but $\mathrm{cl}_{\mathbb{R}}(S)$ is a proper subset of S'. By 2.1 we may assume that C has nonempty interior in $\mathbb{X}$. It is also clear that S' is not the whole space $\mathbb{R}^d$, and hence there exists $c' \in \mathrm{bd}_{\mathbb{R}}(C')$. Then $c' + S' \subset \mathrm{bd}_{\mathbb{R}}(C')$ and there exists a linear subspace T' of $\mathbb{R}^d$ that is a complement of S' and is $\mathbb{F}$-generated in the sense that the intersection $T = T' \cap \mathbb{X}$ is a subspace of $\mathbb{X}$ whose $\mathbb{F}$-dimension is equal to $\dim_{\mathbb{R}}(T')$.

Let U be an open subset of C. Then $\mathrm{cl}_{\mathbb{R}}(U + T)$ meets $c' + S'$ in a set of dimension $\dim_{\mathbb{R}}(S')$. Hence there is a point $p \in \mathrm{int}_{\mathbb{F}}(C)$ such that the set $\mathrm{cl}_{\mathbb{R}}(p + T) \cap \mathrm{bd}_{\mathbb{R}}(C')$ is not contained in $c' + \mathrm{cl}_{\mathbb{R}}(S)$, whence the two sets have empty intersection. But then there exists a point $q \in \mathbb{X} \setminus C$ such that the segment $[p, q]_{\mathbb{R}}$ meets $c' + S'$ in a point z that does not belong to $c + S$. Clearly the point z does not belong to C, but it certainly belongs to the set $\mathrm{cl}_{\mathbb{R}}([p, q]_{\mathbb{X}})$. This implies that C is not s-closed, and the contradiction completes the proof. ☐

The next result now follows immediately from 6.5 and 6.6.

Theorem 6.7 (s-closed subsets of $\mathbb{X}$ as intersections of halfspaces). *If C is an s-closed subset of $\mathbb{X}$, then each point of $\mathbb{X} \setminus C$ is strongly separated from C by a hyperplane. In particular, C is an intersection of open halfspaces in $\mathbb{X}$.*

Theorem 6.8. *If $d \geq 3$ and C is a polyhedron in $\mathbb{X}$, then the following two conditions are equivalent:*

- *C is a singleton or a slab;*
- *C is weakly separated from each pseudotope in $\mathbb{X} \setminus C$.*

Proof. If C is a slab, then C contains a hyperplane H and it is clear that each convex subset of $\mathbb{X} \setminus C$ is weakly separated from C by some translate of H. If C is a singleton, then by 6.3 each bounded closed convex subset of $\mathbb{X}$ (and hence each pseudotope) in $\mathbb{X} \setminus C$ is freely separated from C. Thus, the first condition in 6.8 implies the second. By 5.8, the second condition implies the first. ☐

Mardešić [18] treated the special case of the following result in which C is a singleton and B is s-closed. Note, in connection with 5.6, that the following condition on B is satisfied when B is a pseudotope.

Theorem 6.9 (Separation in $\mathbb{F}^2$). *Suppose that B and C are nonempty disjoint convex subsets of $\mathbb{F}^2$, that C is a polyhedron, and that at most one point of $\mathrm{bd}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(B) \setminus B$ is collinear with two points of B. Then B and C are weakly separated by a line in $\mathbb{F}^2$.*

Proof. The proof consists of verifying the following statements, where B' and C' denote the closures in $\mathbb{R}^2$ of B and C respectively.

If the intersection $B' \cap C'$ is nonempty, it consists of a single inner point of an edge of C'. If L denotes the line in $\mathbb{R}^2$ that contains this edge, then $L \cap \mathbb{F}^2$ is a line in $\mathbb{F}^2$ that weakly separates B and C.

If the sets B' and C' do not intersect but the infimum of the distances from a point of B' to a point of C' is zero, the reasoning is similar to that in the preceding case.

If $\mathrm{bd}_{\mathbb{R}}(B')$ or $\mathrm{bd}_{\mathbb{R}}(C')$ contains a line L, then the intersection $L \cap \mathbb{F}^2$ is a line in $\mathbb{F}^2$ and there exists a vector $v \in \mathbb{F}^2$ such that $v + L$ separates B and C. Thus we may assume that both B' and C' are line-free.

If a ray in B' is parallel to a ray in C' (i.e. one is a translate of the other), then B and C are separated by a line in $\mathbb{F}^2$ that is parallel to this ray.

In the remaining case, the sets B' and C' are at positive distance from each other, are both line-free, and no ray in one is parallel to any ray in the other. Hence by 3.1 the vector difference is line-free and is at positive distance from the origin. But then the origin is freely separated from the vector difference $B' - C'$, and it follows that B and C are strongly separated by a line in $\mathbb{F}^2$. □

Theorem 6.8 shows that 6.9 does not extend to the case $d \geq 3$. Limitations on extending 6.9 when $d = 2$ are indicated by Examples 5.9.

7. Convex functions on $\mathbb{X}$ as suprema of affine functions

From both a theoretical and an algorithmic viewpoint, one of the most important properties of a convex function φ on an open convex subset C of $\mathbb{R}^d$ is the fact that φ is the supremum of a collection of affine functions on C. (In $\mathbb{R}^d$ this leads to the existence of a supporting hyperplane at each point of φ's graph, and hence to *subdifferentiability*; see e.g. [23].) The present section studies the extent to which this result carries over to general $\mathbb{X}$.

If C is a convex subset of $\mathbb{X}$ and $\varphi : C \to \mathbb{F}$ is an $\mathbb{F}$-valued function on C, then φ is *convex* if

$$\varphi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for all $x, y \in C$ and $\lambda \in \mathbb{F}$ with $0 < \lambda < 1$. The function φ is *affine* if

$$\varphi\big(\lambda x + (1-\lambda)y\big) = \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for all x, y, λ as described, and it is then a routine matter to extend φ to an affine function whose domain is the entire space $\mathbb{X}$.

If a convex set $C \subset \mathbb{X}$ is closed and the function $\varphi : C \to \mathbb{F}$ is continuous and convex, then of course the epigraph

$$\mathrm{epi}(\varphi) = \{(x, \alpha) : x \in C,\ \alpha \in \mathbb{F},\ \varphi(x) \geq \alpha\}$$

is a closed convex subset of $\mathbb{X} \times \mathbb{F}$ and the set $D_\gamma = \{x \in C : \varphi(x) \leq \gamma\}$ is closed and convex for each $\gamma \in \mathbb{F}$. However, the s-closedness of C does not imply that of $\mathrm{epi}(\varphi)$ and D_γ. To see this, let $\mathbb{X} = \mathbb{Q}$, $C = [-2, 2]$ and $\varphi(x) = x^2$ for all $x \in C$. Then the intersection of $\mathrm{epi}(\varphi)$ with the closed segment $[(-2, 2), (2, 2)]$ is not a closed segment, and the set D_2 is not s-closed.

The following result is a consequence of Theorems 6.5 and 6.7.

Theorem 7.1 (Convex functions as suprema of affine functions). *Suppose that φ is a continuous convex function on a closed convex subset C of $\mathbb{X}$. Then φ is the supremum of a collection of $\mathbb{F}$-valued affine functions on C if either of the following conditions is satisfied:*

(i) $\text{epi}(\varphi)$ *is s-closed;*

(ii) *the subset* $\text{ls}_{\mathbb{R}}(\text{cl}_{\mathbb{R}}(\text{epi}(\varphi)))$ *of* $\mathbb{R}^{d+1}$ *is the* $\mathbb{R}$*-closure of a subspace of* $\mathbb{X} \times \mathbb{F}$.

Proof. It follows from 6.7 in case (i), and from 6.5 in case (ii), that for each point $y \in C$ and each $\beta < \varphi(y)$, there is a hyperplane H in the space $\mathbb{X} \times \mathbb{F}$ that strongly separates the point (y, β) and the convex set $\text{epi}(\varphi)$. Now let the functional $\psi : \mathbb{X} \to \mathbb{F}$ be defined by the condition that $(x, \psi(x)) \in H$ for each $x \in \mathbb{X}$. Then it follows by a routine argument that ψ is affine, $\beta < \psi(y)$ and $\psi(x) < \varphi(s)$ for all $x \in C$. □

Each of the conditions (i) and (ii) is less than satisfactory. The second uses the embedding space while the first, though intrinsic, makes assumptions on the epigraph of φ that cannot easily be characterized in terms of natural assumptions on C and φ. A more natural set than an interval and a more natural convex function than the square $\varphi(x) = x^2$ can hardly be imagined, and yet, as we have seen, (i) does not hold. The rest of this section is concerned with finding some natural intrinsic conditions on C and φ that still force φ to be the supremum of a collection of affine functions.

We begin with two lemmas. The first is very well-known, at least in the case of $\mathbb{R}^d$; its proof is included as a service to the reader.

Lemma 7.2 (Continuity of convex functions). *If C is an open convex subset of $\mathbb{X}$ and $\varphi : C \to \mathbb{F}$ is a convex function, then φ is continuous.*

Proof. The assertion that φ is upper semicontinuous is equivalent to the assertion that the set $F = \{(x, \alpha) : x \in C \wedge \alpha > \varphi(x)\}$ is open. Let $E = \text{epi}(\varphi)$. Clearly, $\text{int}(E)$ is contained in F. To see that each point of F belongs to $\text{int}(E)$, it suffices (ignoring obvious cases) to consider an arbitrary pair of points (x, α) and (y, β) in $C \times \mathbb{F}$ with $x \neq y$ and

$$\beta - \varphi(y) < 0 < \alpha - \varphi(x).$$

Then with $\delta = \alpha - \varphi(x)$ and $\eta = \varphi(y) - \beta$, it is true for all λ with $0 < \lambda < \frac{\delta}{\delta + \eta}$ that

$$(1 - \lambda)(x, \alpha) + \lambda(y, \beta) \in \text{int}(E).$$

(Consider the two segments $\text{conv}\{(x, \varphi(x)), (y, \varphi(y))\}$ and $\text{conv}\{(x, \alpha), (y, \beta)\}$ and use the convexity of φ.) It follows that the set $F = \text{int}(\text{epi}(\varphi))$ is open.

To establish the lower semicontinuity of φ, consider an arbitrary point $q \in C$ and an arbitrary sequence $(p_i)_{i \in \mathbb{N}}$ in $C \cap (2q - C)$ converging to q. For each index i, the point $r_i = 2q - p_i$ also belongs to C, and by convexity,

$$\varphi(q) \leq \frac{1}{2}\varphi(p_i) + \frac{1}{2}\varphi(r_i).$$

Since $\limsup \varphi(r_i) \leq \varphi(q)$ by upper semicontinuity, it follows that $\liminf \varphi(p_i) \geq \varphi(q)$. Hence φ is also lower semicontinuous at q. □

Next we prove an extendability result.

Lemma 7.3 (Unique extensions of convex functions). *Suppose that C is an open convex subset of $\mathbb{X}$, and $\varphi : C \to \mathbb{F}$ is a convex function. Let $C' = \mathrm{int}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(C))$. Then there is a unique convex function $\varphi' : C' \to \mathbb{R}$ such that $\varphi'|_C = \varphi$.*

Proof. Let $E = \mathrm{epi}(\varphi)$ and $E' = \mathrm{cl}_{\mathbb{R}}(E) \cap (C' \times \mathbb{R})$. We show that E' is a convex subset of $\mathbb{R}^{d+1}$. Let $(x, \alpha), (y, \beta) \in E'$. Since C' is open and $\mathbb{F}^d$ is dense in $\mathbb{R}^d$, there are points $a_0, \ldots, a_d, b_0, \ldots, b_d \in C$ and positive numbers $\lambda_0, \ldots, \lambda_d, \mu_0 \ldots, \mu_d \in \mathbb{F}$ with $\sum_{i=0}^d \lambda_i = \sum_{i=0}^d \mu_i = 1$ such that $x = \sum_{i=0}^d \lambda_i a_i$ and $y = \sum_{i=0}^d \mu_i b_i$. Now, let

$$K = \mathrm{conv}_{\mathbb{F}}(\cup_{i=0}^d \{(a_i, \varphi(a_i)), (b_i, \varphi(b_i))\}).$$

By the convexity of φ, $K + R \subset E$, where R is the ray $\{0\}^d \times [0, \infty[_{\mathbb{F}}$. This implies that $\mathrm{conv}_{\mathbb{R}}\{(x, \alpha), (y, \beta)\} \subset E'$, and hence E' is a convex subset of $\mathbb{R}^{d+1}$.

It is now easy to define an extension of φ to $\mathbb{R}^d$; just set

$$\varphi'(c') = \min\{\alpha : (c', \alpha) \in E'\}.$$

for each $c' \in C'$. A routine argument shows that φ' has the desired properties. Since, by 7.2, each convex extension of φ is continuous, there is at most one such extension. Hence φ' is the unique convex extension of φ to the set C'. □

When, as in 7.3, the convex domain C of φ is open, then φ is necessarily continuous. However, when C is a closed convex subset of $\mathbb{X}$, the continuity and convexity of φ on C does not guarantee the existence of a convex extension of φ to the $\mathbb{R}$-closure $\mathrm{cl}_{\mathbb{R}}(C)$. For instance, let $\mu \in \mathbb{R} \setminus \mathbb{Q}$, $p = (0, \mu)$, and let B' be a circular disk in $[0, \infty[_{\mathbb{R}} \times \mathbb{R}$ that is tangent to the y-axis at the point p. Approximate B' by a suitable pseudotope $B'' \subset B'$ where the vertices of B'' all belong to $\mathbb{Q}^2$ and converge to p. Then set $C = B'' \cap \mathbb{Q}^2$, $C' = \mathrm{cl}_{\mathbb{R}}(C)$, and let $\varphi : C \to \mathbb{Q}$ be defined by $\varphi(\xi, \eta) = 1/\xi$. It is clear that C is s-closed and φ is a continuous convex function on C; however, there is no convex extension of φ to C'.

Let us now reconsider 7.1. If C is s-closed and $\mathrm{ls}_{\mathbb{F}}(C) = \{0\}$, then (with the aid of 6.6) $\mathrm{ls}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(\mathrm{epi}(\varphi))) = \{0\}$. Hence condition (ii) of 7.1 is satisfied and φ is the supremum of a collection of affine functions on $\mathbb{X}$. Our final separation results will allow C to have a (barely) non-trivial lineality space. However, we first describe some examples that dampen too-high expectations by showing that even when $\mathbb{F} = \mathbb{Q}$, there are continuous convex functions that are not the supremum of a collection of affine functions on $\mathbb{X}$, and that this can happen both when C is s-closed and $\mathrm{ls}_{\mathbb{F}}(C)$ is 2-dimensional and when C is closed but not s-closed and $\mathrm{ls}_{\mathbb{F}}(C)$ is 0- or 1-dimensional.

Let $\beta \in \mathbb{R} \setminus \mathbb{Q}$, and let V denote the vector space over $\mathbb{Q}$ with basis 1 and β. Identifying V with a subset of $\mathbb{R}$, we begin with the construction of a certain convex function $\tau : V \to \mathbb{Q}$. Note that V is countable, choose a bijection $\gamma : \mathbb{N} \to V \setminus \mathbb{Z}$, and then set $\gamma_i = \gamma(i)$ for $i \in \mathbb{N}$. Further, let $Z_0 = \mathbb{Z}$ and $Z_i = \mathbb{Z} \cup \{\gamma_j : j = 1, \ldots, i\}$. We will now construct a sequence $(\tau_i)_{i \in \mathbb{N} \cup \{0\}}$ of piecewise linear convex functions $\tau_i : \mathbb{R} \to \mathbb{R}$ with the property that $\tau_i(\mu) \in \mathbb{Q}$ for all $\mu \in Z_i$. The functions are most easily described in terms of their epigraphs E_i, $i \in \mathbb{N} \cup \{0\}$. The set of all extreme points of E_i will be denoted by V_i.

Let $V_0 = \{(\mu, \mu^2) : \mu \in \mathbb{Z}\}$, and $E_0 = \operatorname{conv}(V_0)$. Clearly, $V_0 = \operatorname{ext}(E_0)$, the function τ_0 corresponding to E_0 is piecewise linear and convex, and τ_0 is rational on Z_0. Now suppose that τ_i and hence E_i and V_i have already been constructed for some $i \in \mathbb{N} \cup \{0\}$, and let $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in Z_i$ such that $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, $[\alpha_0, \alpha_3] \cap Z_i \setminus \{\alpha_j : j = 0, 1, 2, 3\} = \emptyset$, and $\alpha_1 < \gamma_{i+1} < \alpha_2$. Further, for $j = 0, 1, 2$ let L_j denote the line spanned by $(\alpha_j, \tau_i(\alpha_j))$ and $(\alpha_{j+1}, \tau_i(\alpha_{j+1}))$, let H_0^+, H_1^- and H_2^+ denote the corresponding open halfplanes such that H_0^+ and H_2^+ contain $\operatorname{int}_{\mathbb{R}} E_i$, while H_1^- is disjoint from E_i. The set $D = H_0^+ \cap H_1^- \cap H_2^+$ is an open triangle. Now choose a rational number ζ such that $v = (\gamma_{i+1}, \zeta) \in D$, set $V_{i+1} = V_i \cup \{v\}$, $E_{i+1} = \operatorname{conv}(V_{i+1})$, and let τ_{i+1} denote the function corresponding to E_{i+1}. Since the set V_{i+1} is convexly independent, $V_{i+1} = \operatorname{ext}(E_{i+1})$, and hence $\tau_{i+1}(\mu) = \tau_i(\mu)$ for all $\mu \in Z_i$. Also, $\tau_{i+1}(\gamma_{i+1}) = \zeta \in \mathbb{Q}$.

Finally, we define the function $\tau : V \to \mathbb{Q}$ by setting

$$\tau(\mu) = \min_{i=0,\ldots,\infty} \tau_i(\mu) \qquad \text{for } \mu \in V.$$

Having constructed the auxiliary function τ, we now define the desired function $\varphi : \mathbb{Q}^2 \to \mathbb{Q}$ by setting

$$\varphi(\xi, \eta) = \tau(\eta\beta - \xi), \qquad \text{for } (\xi, \eta) \in \mathbb{Q}^2.$$

Note first that $\eta\beta - \xi \in V$, whence φ is indeed rational on the whole rational plane. Clearly, φ is convex, hence by 7.3 admits a unique convex extension φ' to $\mathbb{R}^2$. Note that the function φ' is constant on each line that is parallel to the line $L = \mathbb{R}(\beta, 1)$; in particular, L is mapped to 0. Hence $L \subset \operatorname{ls}_{\mathbb{R}}(\operatorname{epi}_{\mathbb{R}}(\varphi'))$, and since φ' is not linear, actually $L = \operatorname{ls}_{\mathbb{R}}(\operatorname{epi}_{\mathbb{R}}(\varphi'))$. On the other hand, $\operatorname{ls}_{\mathbb{Q}}(\operatorname{epi}_{\mathbb{Q}}(\varphi)) = \{0\}$.

Now let H be a hyperplane in $\mathbb{Q}^3$ that is disjoint from $\operatorname{int}_{\mathbb{Q}}(\operatorname{epi}_{\mathbb{Q}}(\varphi))$. Then $\operatorname{cl}_{\mathbb{R}}(H)$ is parallel to L, and it follows easily that $H = \mathbb{Q}^2 \times \{\delta\}$ with $\delta \leq 0$. Hence the set $\operatorname{epi}_{\mathbb{Q}}(\varphi)$ cannot be weakly separated from any point with positive last coordinate. This implies in particular that φ is not the supremum of any collection of affine functions on $\mathbb{Q}^2$.

It is evident that C can be restricted to the rational points of a strip parallel to L (just like the set C_1 in Section 2), whence the construction extends to the situation where $\operatorname{ls}_{\mathbb{F}}(C) = \{0\}$ and C is closed but not s-closed. It is also not hard to generalize the example to higher dimensions and, in particular, to include an example (in $\mathbb{Q}^d$ with $d \geq 3$) with $\dim(\operatorname{ls}_{\mathbb{F}}(C)) = 1$.

The next result closes the gap between the examples just given and the trivial conclusions of 7.1.

Theorem 7.4. *If φ is a continuous convex function on an s-closed subset C of $\mathbb{X}$, and if $\operatorname{ls}_{\mathbb{F}}(C)$ is at most of dimension 1, then each point of $(\mathbb{X} \times \mathbb{F}) \setminus \operatorname{epi}(\varphi)$ is strongly separated from C by a nonvertical hyperplane. If $d = 2$ and $\dim(\operatorname{ls}_{\mathbb{F}}(C)) = 1$, the conclusion holds even when the set C is merely closed.*

Proof. Let $\mathbb{X}$ be canonically embedded in $\mathbb{R}^d$, and suppose without loss of generality that $0 \in C$. Let $E = \operatorname{epi}(\varphi)$ and $E' = \operatorname{cl}_{\mathbb{R}}(E)$. Note that the orthogonal projection S of $\operatorname{ls}_{\mathbb{F}}(E)$ on $\mathbb{X}$ is contained in $\operatorname{ls}_{\mathbb{F}}(C)$, and the orthogonal projection S' of $\operatorname{ls}_{\mathbb{R}}(E')$ on $\mathbb{R}^d$ is contained in $\operatorname{ls}_{\mathbb{R}}(C')$.

Since C is s-closed, $\operatorname{ls}_{\mathbb{F}}(C)$ is dense in $\operatorname{ls}_{\mathbb{R}}(C')$ by 6.6, hence $\operatorname{ls}_{\mathbb{R}}(C')$ is $\mathbb{F}$-generated. Since $\operatorname{ls}_{\mathbb{F}}(C)$ is of dimension at most 1, we have either $\dim_{\mathbb{R}}(S') = 0$ or $\dim_{\mathbb{R}}(S') = 1$. In the former case, $S = S' = \{0\}$, while in the latter case, $S' = \mathrm{ls}_{\mathbb{R}}(C') = \mathrm{cl}_{\mathbb{R}}(\mathrm{ls}(C))$. Since, by 7.3, the restriction $\varphi|_{\mathrm{ls}_{\mathbb{F}}(C)}$ has a unique convex extension φ' to S', and the latter is linear, so is the former. Hence $\mathrm{cl}_{\mathbb{R}}(S) = S'$. It follows from 6.5 that each point (p, λ) of $(\mathbb{X} \times \mathbb{F}) \setminus \mathrm{epi}(\varphi)$ is strongly separated from $\mathrm{epi}(\varphi)$ by a nonvertical hyperplane.

For the final assertion, note that when $d = 2$ and $\dim(\mathrm{ls}_{\mathbb{F}}(C)) = 1$, $\mathrm{ls}_{\mathbb{F}}(C)$ is trivially dense in $\mathrm{ls}_{\mathbb{R}}(C')$; then continue as in the general situation. □

As a simple corollary of 7.4 we obtain the following final result.

Corollary 7.5 (Convex functions as suprema of affine functions). *If φ is a continuous convex function on an s-closed subset C of $\mathbb{X}$, and if $\mathrm{ls}_{\mathbb{F}}(C)$ is of dimension at most 1, then φ is the supremum of a collection of affine functions on $\mathbb{X}$.*

Proof. By 7.4 each point (p, λ) of the set $(\mathbb{X} \times \mathbb{F}) \setminus \mathrm{epi}(\varphi)$ is strongly separated from $\mathrm{epi}(\varphi)$ by a nonvertical hyperplane. For each $p \in C$ and $\lambda < \varphi(p)$, a strongly separating nonvertical hyperplane generates (in a standard way) an affine function $f : \mathbb{X} \to \mathbb{F}$ such that $f < \varphi$ and $\lambda < f(p)$. Hence the stated conclusion follows. □

Now suppose, finally, that we have a convex function φ on an *open* convex subset C of $\mathbb{X}$ (whose lineality space is of dimension at most 1). Then, even though φ is necessarily continuous by 7.2, the examples given before the statement of 7.4 show that unless $\mathrm{cl}(C)$ is s-closed, we still cannot be sure that φ is the supremum of a collection of affine functions on C.

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