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Asymptotic Analysis to a Phase-Field Model with a Nonsmooth Memory Kernel

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A phase-field model based on the Gurtin-Pipkin heat flux law is considered. The resulting system has been investigated by Colli and Laurençot who proved existence and uniqueness results, when the time relaxation coefficient is strictly positive. The aim of this paper is the study of the asymptotic behaviour of such a solution, as the time relaxation goes to zero, and of the related limit problem.

1. Introduction

Consider a two-phase material located in a bounded domain $\Omega \subset \mathbb{R}^3$ until a given time $T > 0$. Denote by ϑ its relative temperature (fixed in order that $\vartheta = 0$ is the equilibrium temperature between the two phases) and by χ an order parameter (usually named *phase-field*) which could represent the local proportion of one phase. To describe the evolution of the pair (ϑ, χ) we consider the following system

$$\partial_t(\vartheta + \lambda(\chi)) - k * \Delta\vartheta = f \quad \text{in } Q := \Omega\times]0,T[, \tag{1.1}$$

$$\mu\partial_t\chi - \nu\Delta\chi + \beta(\chi) \ni -\sigma'(\chi) + \lambda'(\chi)\vartheta \quad \text{in } Q, \tag{1.2}$$

$$\vartheta(\cdot,0) = \vartheta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega, \tag{1.3}$$

$$\partial_n\chi = \partial_n(k * \vartheta) = 0 \quad \text{on } \partial\Omega\times]0,T[, \tag{1.4}$$

where $\partial_t = \partial/\partial t$, $*$ denotes the usual convolution product with respect to the time over $]0,t[$, Δ the Laplacian (in space variables), ∂_n the outer normal derivative on $\partial\Omega$. The coefficients μ and ν are positive constants, k is the so-called heat kernel, f is a source term, ϑ_0 and χ_0 are given; moreover, the maximal monotone graph β and the given functions σ and λ entail a nonlinear constrained dynamics for the phase fraction χ.

It must be noted that, in the energy balance equation (1.1), the Gurtin-Pipkin [19] heat flux law is considered, accounting for memory effects. On the other hand, relation (1.2) can be derived following the Ginzburg-Landau theory (see [7], [16], [17], [20]). The system (1.1)–(1.2) is also considered in other different applications in Material Sciences.

Now we are going to recall some existence and uniqueness results of solutions to problem (1.1)–(1.4), outlining the assumptions taken on the heat kernel k and the function λ. Since the main issue of this paper is the asymptotic analysis, when the time relaxation constant μ tends to 0^+, we will also have to mention the results where, instead of (1.2), a relation close to the following is handled in some sense

$$-\nu\Delta\chi + \beta(\chi) \ni -\sigma'(\chi) + \lambda'(\chi)\vartheta \quad \text{in } Q. \tag{1.5}$$

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We start our brief survey by describing the papers [10, 11, 12, 13] by Colli, Gilardi, and Grasselli. In [10], by assuming $k \in W^{2,1}(0,T)$ and λ Lipschitz continuous, they obtained existence, uniqueness, and regularity results of strong solutions to problem (1.1)–(1.4). Assuming $\lambda(\chi) = \lambda\chi$, with λ constant (and hence $\lambda(\chi)$ is linear), in [11] they proved existence results for weak solutions; in [12] they extend the previous existence and uniqueness results for strong solution to the case where $\partial_t(\vartheta + \lambda\chi + \varphi * \vartheta + \psi * \chi)$ occurs in (1.1) instead of $\partial_t(\vartheta + \lambda\chi)$ (φ and ψ being further memory kernels); still in [12] they studied the asymptotic behaviour of the above obtained solutions, when the coefficient ν of the interfacial energy term vanishes. Finally, by assuming also a positivity condition (see (2.14) below), in [13] they carried out the asymptotic analysis of the solution obtained in [12], when $\mu \to 0^+$. They actually got existence, uniqueness and regularity results for problem (1.1)–(1.5)–(1.3)–(1.4) together with some error estimates.

On the other hand, Aizicovici and Barbu [1] studied (1.1)–(1.3) together with Dirichlet boundary conditions, in the particular case where: $\lambda(\chi) = \lambda\chi$, $\beta(\chi) = \chi^3$, $\sigma'(\chi) = -\chi$. It must be noted that they dealt with k just integrable and of positive type (see (2.3) below), obtaining the existence of a solution which is unique provided that k is a nonnegative decreasing and convex function, by using semigroup techniques. By assuming a stronger positivity condition on k, they also investigate the asymptotic behaviour as $t \to +\infty$. The asymptotic analysis, when $\mu \to 0^+$, to the problem investigated in [1] is performed in [8] by Chiusano and Colli; by requiring $\lambda > 1/4$, they actually proved the existence and the uniqueness of the solution to the limit problem.

In [14, 15] Colli and Laurençot considered problem (1.1)–(1.4) in the case where: the heat kernel k is just integrable and of positive type, β and σ are general, and λ is a nonlinear function. In particular, they assumed only $\lambda'' \in L^\infty(\mathbb{R})$, allowing λ to have a quadratic growth; such a behaviour entails that the system (1.1)–(1.2) is a model not only for solid-liquid phase transitions, but also for ferromagnetic transformations (see [14, Sec. 1]). In [14] they obtained the existence of weak solutions and studied the asymptotic behaviour of such solutions as $t \to +\infty$. Under a further assumption on the datum χ_0, in [15] they improved the regularity of the solutions obtained in [14], and then they got a uniqueness result.

The aim of the present paper is to address the study of the asympotic analysis, as $\mu \to 0^+$, of the solutions to problem (1.1)–(1.4) in the general setting considered by Colli and Laurençot, complementing the results of [8] and [13]. By taking a positivity assumption (see (2.14) below), we deal with k just integrable and of positive type (while $k \in W^{2,1}(0,T)$ is assumed in [13]), β maximal monotone graph and σ general (while $\beta = \chi^3$ and $\sigma' = -\chi$ are taken in [8]). We are able to handle just the case where $\lambda(\chi) = \lambda\chi$; on the other hand, we can observe that this is the case considered in [8] and [13] too.

The outline of the paper is as follows. The next section is devoted to the assumptions, the notation, and the statements of the results. Section 3 is concerned with the proof of Theorem 2.2. In Section 4 we deal with the proof of Theorem 2.1: we obtain some *a priori* estimates independently of μ and then we pass to the limit as $\mu \to 0^+$. Theorems 2.1 and 2.2 actually imply an existence and uniqueness result to the "limit" problem (1.1)–(1.5)–(1.3)–(1.4). In Section 5 we prove a regularity result (Theorem 2.3) for the solution just mentioned, by deriving the corresponding estimates; then we deduce some error estimates (Theorems 2.5 and 2.6). Throughout the paper some comments and remarks are given.

2. Statements of the results

Let $\Omega \subset \mathbb{R}^3$, be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{2,1}$. We set $Q_t := \Omega\times]0,t[$ for $t \in]0,T[$, $\mathrm{H} := L^2(\Omega)$ and $\mathrm{V} := H^1(\Omega)$. $\mathrm{W} := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\}$. Identifying, as usual, H with its dual space H′, we recall that $\mathrm{W} \subset \mathrm{V} \subset \mathrm{H} \subset \mathrm{V}' \subset \mathrm{W}'$ with dense and continuous injections. Henceforth, we denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between W′ and W, by $\langle\cdot,\cdot\rangle$ the duality pairing between V′ and V, and by $(\cdot,\cdot)$ the scalar product in H. The norm both in H or in H^3 is simply indicated by $\|\cdot\|$. Besides, we denote by $((\cdot,\cdot))$ the scalar product in V. Then, the associated Riesz isomorphism $J:\mathrm{V}\to\mathrm{V}'$ and scalar product in V′ can be specified by

$$\langle Jv_1, v_2\rangle := ((v_1,v_2)), \quad ((u_1,u_2))_* := \langle u_1, J^{-1}u_2\rangle \quad \text{for } v_i \in \mathrm{V}, \ u_i \in \mathrm{V}', \ i=1,2.$$

We decide to take $\|\cdot\|_{\mathrm{V}} := ((\cdot,\cdot))^{1/2}$ and $\|\cdot\|_{\mathrm{V}'} := ((\cdot,\cdot))_*^{1/2}$.

Hereafter we take $\lambda(\chi) = \lambda\chi$, with λ constant. Concerning the data, we assume the following hypotheses

$$\lambda,\ \nu \in]0,+\infty[; \tag{2.1}$$
$$k \in L^1(0,T); \tag{2.2}$$
$$\int_0^t (v(s),(k*v)(s))ds \geq 0, \quad \forall\, t\in[0,T], \quad \forall\, v \in L^2(0,T;\mathrm{H}); \tag{2.3}$$
$$\beta = \partial\hat{\beta} : \mathbb{R}\to 2^{\mathbb{R}} \text{ with domain } D(\beta) \text{ such that } \mathrm{int}(D(\beta)) \text{ is nonempty and } \beta(0) \ni 0; \tag{2.4}$$
$$\hat{\beta} : \mathbb{R} \to [0,+\infty] \text{ is a proper, convex, lower semicontinuous function;} \tag{2.5}$$
$$\sigma \in C^2(\mathbb{R}) \text{ with } \sigma'(0)=0 \text{ and } \sigma'' \in L^\infty(\mathbb{R}); \tag{2.6}$$
$$\hat{\beta}+\sigma \text{ is nonnegative.} \tag{2.7}$$

Now, by introducing the auxiliary unknown

$$e = \vartheta + \lambda\chi$$

(which corresponds to the enthalpy), problem (1.1)–(1.4) becomes

$$\partial_t e - k*\Delta e + \lambda k * \Delta\chi = f \quad \text{in } Q; \tag{2.8}$$
$$\mu\partial_t\chi - \nu\Delta\chi + \alpha(\chi) \ni \lambda e \quad \text{in } Q; \tag{2.9}$$
$$e(\cdot,0) = e_0 \quad \text{and} \quad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega; \tag{2.10}$$
$$\partial_n(k*e) = \partial_n\chi = 0 \quad \text{on } \partial\Omega\times]0,T[; \tag{2.11}$$

where $e_0 = \vartheta_0 + \lambda\chi_0$ and α is related to β, σ, and λ by

$$\alpha(z) = \beta(z) + \sigma'(z) + \lambda^2 z, \quad \forall\, z \in \mathbb{R}.$$

Now, we assume on α the following conditions

$$\alpha = \partial j : \mathbb{R}\to 2^{\mathbb{R}} \quad \text{with} \quad \alpha(0)\ni 0; \tag{2.12}$$

$$j : \mathbb{R} \to [0, +\infty] \text{ is a proper, convex, lower semicontinuous function with } j(0) = 0; \tag{2.13}$$

$$\begin{aligned} &\exists l > 0 \text{ such that} \\ &(\eta_1 - \eta_2)(z_1 - z_2) \geq l(z_1 - z_2)^2 \quad \forall\, z_i \in D(\alpha), \quad \forall\, \eta_i \in \alpha(z_i), \quad i = 1, 2 \end{aligned} \tag{2.14}$$

(recall that $D(\alpha)$ denotes the domain of α). Through (2.14) we assume on α a monotonicity condition which is surely satisfied if the Lipschitz constant of σ' is smaller than λ^2. From the physical point of view, this corresponds to require the latent heat to be large enough.

Now, thanks to (2.1)–(2.7) and to (2.15)–(2.16) below, the results of Colli and Laurençot [14, Thm 2.3], [15, Thm 2.2-Rem 4.3] ensure that the following problem has a unique solution.

Problem (P_μ). Let $\mu > 0$ be given and (2.1)–(2.7) hold. Moreover, let f_μ, $e_{0\mu}$, and $\chi_{0\mu}$ satisfy

$$f_\mu \in L^1(0, T; \mathrm{H}) \tag{2.15}$$

$$e_{0\mu} \in \mathrm{H}, \quad \chi_{0\mu} \in \mathrm{V} \quad \text{and} \quad \hat{\beta}(\chi_{0\mu}) \in L^1(\Omega). \tag{2.16}$$

Find $(e_\mu, \chi_\mu, \xi_\mu)$ such that

$$e_\mu \in C^0([0, T]; \mathrm{V}') \cap L^\infty(0, T; \mathrm{H}); \tag{2.17}$$

$$\chi_\mu \in H^1(0, T; \mathrm{H}) \cap C^0([0, T]; \mathrm{V}) \cap L^2(0, T; \mathrm{W}); \tag{2.18}$$

$$\xi_\mu \in L^2(0, T; \mathrm{H}); \tag{2.19}$$

$$\partial_t e_\mu - f_\mu \in L^2(0, T; \mathrm{W}'); \tag{2.20}$$

and

$$\langle\langle \partial_t e_\mu, v \rangle\rangle - (k * e_\mu, \Delta v) + \lambda(k * \Delta\chi_\mu, v) = (f_\mu, v)\,, \forall\, v \in \mathrm{W}\,, \text{ a.e. in }]0, T[; \tag{2.21}$$

$$\mu\partial_t \chi_\mu - \nu\Delta\chi_\mu + \xi_\mu = \lambda e_\mu \quad \text{a.e. in } Q; \tag{2.22}$$

$$\xi_\mu \in \alpha(\chi_\mu) \quad \text{a.e. in } Q; \tag{2.23}$$

$$e_\mu(\cdot, 0) = e_{0\mu} \quad \text{in } \mathrm{V}'; \tag{2.24}$$

$$\chi_\mu(\cdot, 0) = \chi_{0\mu} \quad \text{in } \mathrm{V}. \tag{2.25}$$

Now, we study the asymptotic behaviour of the solutions of the previous problem, as $\mu \to 0^+$. Hereafter, we let μ vary in $]0, 1[$ and denote by $(e_\mu, \chi_\mu, \xi_\mu)$ the unique solution to problem (P_μ), corresponding to the data f_μ, $e_{0\mu}$, $\chi_{0\mu}$ satisfying (2.15)–(2.16). In order to pass to the limit in (2.21)–(2.25), as $\mu \to 0^+$, we assume that the following convergences hold

$$f_\mu \to f \quad \text{in } L^1(0, T; \mathrm{H}) \tag{2.26}$$

$$e_{0\mu} \to e_0 \quad \text{in H} \tag{2.27}$$

$$\mu^{1/2}\chi_{0\mu} \to 0 \quad \text{in H}. \tag{2.28}$$

Moreover, we suppose that

$$\begin{aligned} &\exists C > 0 \text{ such that} \\ &\mu^{1/2}\|\chi_{0\mu}\|_V + \mu\|j(\chi_{0\mu})\|_{L^1(\Omega)} \leq C \quad \text{for any } \mu \in]0, 1[. \end{aligned} \tag{2.29}$$

Theorem 2.1. *Let* (2.1)–(2.7) *and* (2.12)–(2.14) *hold. Moreover, let* f_μ, $e_{0\mu}$, $\chi_{0\mu}$, f, *and* e_0 *satisfy* (2.26)–(2.29). *Then there exists a triplet* (e, χ, ξ) *such that*

$$e_\mu \to e \quad in \ C^0([0,T];\mathrm{V}') \tag{2.30}$$

$$e_\mu \overset{*}{\rightharpoonup} e \quad in \ L^\infty(0,T;\mathrm{H}) \tag{2.31}$$

$$\chi_\mu \to \chi \quad in \ L^2(0,T;\mathrm{V}) \tag{2.32}$$

$$\chi_\mu \rightharpoonup \chi \quad in \ L^2(0,T;\mathrm{W}) \tag{2.33}$$

$$\mu\chi_\mu \rightharpoonup 0 \quad in \ H^1(0,T;\mathrm{H}) \tag{2.34}$$

$$\mu^{1/2}\chi_\mu \overset{*}{\rightharpoonup} 0 \quad in \ L^\infty(0,T;\mathrm{V}) \tag{2.35}$$

$$\xi_\mu \rightharpoonup \xi \quad in \ L^2(0,T;\mathrm{H})\,. \tag{2.36}$$

Moreover, the triplet (e, χ, ξ) *solves the problem*

$$\langle\langle \partial_t e, v\rangle\rangle - (k * e, \Delta v) + \lambda(k * \Delta\chi, v) = (f, v)\,,\ \forall\ v \in \mathrm{W}\,,\ \ a.e.\ in\ ]0,T[; \tag{2.37}$$

$$-\nu\Delta\chi + \xi = \lambda e \quad a.e.\ in\ Q; \tag{2.38}$$

$$\xi \in \alpha(\chi) \quad a.e.\ in\ Q; \tag{2.39}$$

$$e(\cdot, 0) = e_0 \quad in\ \mathrm{V}'. \tag{2.40}$$

Note that Theorem 2.1 ensures, in particular, the existence of a solution to problem (2.37)–(2.40), whenever f and e_0 satisfy (2.41) and (2.42) below. Actually, the uniqueness result holds.

Theorem 2.2. *Let* (2.1)–(2.7) *and* (2.12)–(2.14) *hold. Moreover, let* f *and* e_0 *satisfy*

$$f \in L^1(0,T;\mathrm{H}) \tag{2.41}$$

$$e_0 \in \mathrm{H}. \tag{2.42}$$

Then the problem (2.37)–(2.40) *has a unique solution.*

Hereafter, (e, χ, ξ) denotes the solution to problem (2.37)–(2.40). By assuming some stronger hypotheses on the sequence of the data, we can obtain uniform bounds which yield further regularity for e and χ.

Theorem 2.3. *Let* (2.1)–(2.7) *and* (2.12)–(2.14) *hold. Let* f_μ, $e_{0\mu}$, $\chi_{0\mu}$, f, *and* e_0 *satisfy* (2.26)–(2.29). *Assume moreover*

$$\|f_\mu\|_{L^1(0,T;\mathrm{V})\cap L^2(0,T;\mathrm{V}')} + \|e_{0\mu} - \lambda\chi_{0\mu}\|_V \le c' \tag{2.43}$$

$$\chi_{0\mu} \in W \tag{2.44}$$

$$\chi_{0\mu} \in D(\alpha)\ a.e.\ in\ \Omega\ and\ \alpha^0(\chi_{0\mu}) \in \mathrm{H} \tag{2.45}$$

(where, for $y \in D(\alpha)$, $\alpha^0(y)$ *denotes the element of* $\alpha(y)$ *having minimum modulus),*

$$\begin{aligned}&there\ exists\ \xi_{0\mu} \in \mathrm{H}\ such\ that\ \xi_{0\mu} \in \alpha(\chi_{0\mu})\ a.e.\ in\ \Omega\ and\\ &\mu^{-1/2}\|\nu\Delta\chi_{0\mu} - \xi_{0\mu} + \lambda e_{0\mu}\| \le c''\,,\\ &for\ some\ c', c'' > 0\ and\ any\ \mu \in]0,1[\,.\end{aligned} \tag{2.46}$$

Then

$$\|e_\mu\|_{L^\infty(0,T;\mathrm{V})\cap H^1(0,T;\mathrm{V}')} \le c_1 \tag{2.47}$$

$$\mu^{1/2}\|\partial_t\chi_\mu\|_{L^\infty(0,T;\mathrm{H})} + \|\chi_\mu\|_{L^\infty(0,T;\mathrm{W})\cap H^1(0,T;\mathrm{V})} \le c_2 \tag{2.48}$$

for some $c_1,\ c_2 > 0$ *and any* $\mu \in]0,1[$.

Remark 2.4. In view of Theorems 2.2 and 2.3, by (2.47) and (2.48), we deduce the following convergences, besides (2.30)–(2.36).

$$e_\mu \overset{*}{\rightharpoonup} e \quad \text{in } L^\infty(0,T;\mathrm{V}) \tag{2.49}$$
$$e_\mu \rightharpoonup e \quad \text{in } H^1(0,T;\mathrm{V}') \tag{2.50}$$
$$\chi_\mu \rightharpoonup \chi \quad \text{in } H^1(0,T;\mathrm{V}) \tag{2.51}$$
$$\chi_\mu \overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(0,T;\mathrm{W}) \tag{2.52}$$
$$\mu^{1/2}\chi_\mu \overset{*}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}(0,T;\mathrm{H}). \tag{2.53}$$

Using the generalized Ascoli theorem (see Simon [21, Cor. 4, Sec. 8]), from (2.49)–(2.52), we deduce moreover the strong convergences

$$e_\mu \to e \quad \text{in } C^0([0,T];\mathrm{H}) \tag{2.54}$$
$$\chi_\mu \to \chi \quad \text{in } C^0([0,T];\mathrm{V})\,. \tag{2.55}$$

Finally, we note that equation (2.37) and the initial condition (2.40) may be meant in V′ and H, respectively, thanks to (2.50) and (2.54).

Now, we establish some error estimates.

Theorem 2.5. *Let all the assumptions of* Theorem 2.3 *hold. Then*

$$\|e_\mu - e\|_{C^0([0,T];\mathrm{V}')} + \|\chi_\mu - \chi\|_{L^2(0,T;\mathrm{V})} \le c_3\,\{\mu + \varepsilon_\mu\} \tag{2.56}$$

$$\|\chi_\mu - \chi\|_{C^0([0,T];\mathrm{V})} \le c_4\,\{\mu^{1/2} + \varepsilon_\mu\} \tag{2.57}$$

where

$$\varepsilon_\mu := \|e_{0\mu} - e_0\| + \|f_\mu - f\|_{L^1(0,T;\mathrm{H})}\,, \tag{2.58}$$

for some $c_3,\ c_4 > 0$ *and any* $\mu \in]0,1[$.

Theorem 2.6. *Let all the assumptions of* Theorem 2.3 *hold and moreover*

$$\alpha \in C^1(\mathbb{R}) \ \textit{non decreasing}. \tag{2.59}$$

Then

$$\|e_\mu - e\|_{C^0([0,T];\mathrm{H})} + \|\chi_\mu - \chi\|_{L^2(0,T;\mathrm{W})} \le c_5\,\{\mu^{1/2} + (\varepsilon_\mu)^{1/2}\} \tag{2.60}$$

$$\|\chi_\mu - \chi\|_{L^\infty(0,T;\mathrm{W})} \le c_6\,\{\mu^{1/4} + (\varepsilon_\mu)^{1/2}\}\,, \tag{2.61}$$

for some $c_5,\ c_6 > 0$ *and any* $\mu \in]0,1[$.

Remark 2.7. We warn that, in the proofs in the following sections, we will employ the same symbol c for different constants, even in the same formula, in regard of simplicity.

3. Proof of Theorem 2.2

We proceed by contradiction. Let (e_1, χ_1, ξ_1) and (e_2, χ_2, ξ_2) be two solutions to problem (2.37)–(2.40), provided by Theorem 2.1. Set now $\tilde{e} := e_1 - e_2$, $\tilde{\chi} := \chi_1 - \chi_2$, and $\tilde{\xi} := \xi_1 - \xi_2$. Writing (2.37)–(2.38) for both the triplets and taking the differences, we have

$$\langle\langle \partial_t \tilde{e}, v \rangle\rangle - (k * \tilde{e}, \Delta v) + \lambda (k * \Delta \tilde{\chi}, v) = 0, \quad \forall v \in W, \qquad \text{a.e. in }]0, T[\tag{3.1}$$

$$-\nu \Delta \tilde{\chi} + \tilde{\xi} = \lambda \tilde{e} \quad \text{a.e. in } Q. \tag{3.2}$$

Next, we rewrite (3.1) by the means of the isomorphism J and by inserting the additional term $k * (\tilde{e} - \lambda \tilde{\chi})$ in order to guarantee coerciveness. Thus, we obtain

$$\begin{aligned} &\langle\langle \partial_t \tilde{e}, v \rangle\rangle + (k * \tilde{e}, Jv) - (k * \tilde{e}, v) - \lambda (J(k * \tilde{\chi}), v) + \lambda (k * \tilde{\chi}, v) = 0, \\ &\quad \forall v \in W, \qquad \text{a.e. in }]0, T[. \end{aligned} \tag{3.3}$$

Take now $v = J^{-1} \tilde{e}$ in (3.3) and integrate from 0 to $t \in]0, T[$. We get

$$\frac{1}{2} \|\tilde{e}(t)\|_{V'}^2 + \int_0^t ((k * \tilde{e})(s), \tilde{e}(s))\, ds = \sum_{j=1}^3 I_j(t), \tag{3.4}$$

where

$$I_1(t) := \int_0^t (k * \tilde{e}, J^{-1} \tilde{e}) \tag{3.5}$$

$$I_2(t) := \lambda \int_0^t (J(k * \tilde{\chi}), J^{-1} \tilde{e}) \tag{3.6}$$

$$I_3(t) := -\lambda \int_0^t (k * \tilde{\chi}, J^{-1} \tilde{e}). \tag{3.7}$$

In order to estimate these integrals, we recall here some properties, which will be useful in the sequel: the elementary inequality

$$ab \le (\delta/2) a^2 + (2\delta)^{-1} b^2 \quad \forall\, a, b \in \mathbb{R}, \ \forall\, \delta > 0 \tag{3.8}$$

and the Young theorem implying

$$\|a * b\|_{L^2(0,T;X)} \le \|a\|_{L^1(0,T)} \|b\|_{L^2(0,T;X)} \quad \forall\, a \in L^1(0,T), \ b \in L^2(0,T;X), \tag{3.9}$$

for any real Banach space X.

Now, owing to Hölder inequality, (3.9) and the definition of J, we get

$$|I_1(t)| \le \|k * \tilde{e}\|_{L^2(0,t;V')} \|J^{-1} \tilde{e}\|_{L^2(0,t;V)} \le \|k\|_{L^1(0,T)} \|\tilde{e}\|_{L^2(0,t;V')}^2 \tag{3.10}$$

$$\begin{aligned} |I_2(t)| &\le \lambda \|J(k * \tilde{\chi})\|_{L^2(0,t;V')} \|J^{-1} \tilde{e}\|_{L^2(0,t;V)} \le \\ &\le \lambda \|k\|_{L^1(0,T)} \|\tilde{\chi}\|_{L^2(0,t;V)} \|\tilde{e}\|_{L^2(0,t;V')} \end{aligned} \tag{3.11}$$

$$|I_3(t)| \le \lambda \|k * \tilde{\chi}\|_{L^2(0,t;V')} \|J^{-1}\tilde{e}\|_{L^2(0,t;V)} \le \\ \le c\lambda \|k\|_{L^1(0,T)} \|\tilde{\chi}\|_{L^2(0,t;V)} \|\tilde{e}\|_{L^2(0,t;V')} . \tag{3.12}$$

Then, multiplying (3.2) by $\tilde{\chi}$ and integrating over Q_t, we obtain

$$\nu \int_0^t \|\nabla \tilde{\chi}(s)\|^2 \, ds + \int_0^t (\tilde{\xi}(s), \tilde{\chi}(s)) \, ds = \lambda \int_0^t (\tilde{e}(s), \tilde{\chi}(s)) \, ds. \tag{3.13}$$

Using (2.14) and Hölder inequality, by (3.13) we deduce that

$$\min\{\nu, l\} \|\tilde{\chi}\|^2_{L^2(0,t;V)} \le \lambda \|\tilde{\chi}\|_{L^2(0,t;V)} \|\tilde{e}\|_{L^2(0,t;V')} . \tag{3.14}$$

Finally, we add (3.4) and (3.14), by taking (2.3) and (3.10)–(3.12) into account. Using (3.8), we deduce the existence of a positive constant c (depending also on ν and l) such that

$$\frac{1}{2}\|\tilde{e}(t)\|^2_{V'} + \min\{\nu, l\} \|\tilde{\chi}\|^2_{L^2(0,t;V)} \le \frac{1}{2} \min\{\nu, l\} \|\tilde{\chi}\|^2_{L^2(0,t;V)} + c\|\tilde{e}\|^2_{L^2(0,t;V')} , \\ \text{for a.a. } t \in]0, T[. \tag{3.15}$$

Thus, Gronwall lemma implies $\tilde{e} = \tilde{\chi} = 0$ a.e. in Q and a comparison in (3.2) enables us to conclude the proof. □

4. Proof of Theorem 2.1

The present proof consists of the derivation of *a priori* estimates (independently of μ) and the passage to the limit when μ tends to zero. For the sake of brevity, in some of the estimates, we proceed only in a formal way, but we are able to make this procedure rigorous (also see Remark 4.2 below). From the *a priori* bounds we deduce some weak or weak* convergences and (2.39). At the end of the proof we also deduce (2.32). Some comments and remarks complete this section.

First a priori estimate

We take $v = J^{-1}e_\mu$ in (2.21) and integrate from 0 to $t \in]0, T[$. Proceeding as in the first part of the previous section, we obtain

$$\frac{1}{2}\|e_\mu(t)\|^2_{V'} + \int_0^t ((k * e_\mu)(s), e_\mu(s)) \, ds \le \\ \le \frac{1}{2}\|e_{0\mu}\|^2_{V'} + \|k\|_{L^1(0,T)} \left\{ \|e_\mu\|^2_{L^2(0,t;V')} + c\lambda \|\chi_\mu\|_{L^2(0,t;V)} \|e_\mu\|_{L^2(0,t;V')} \right\} + \\ + c \int_0^t \|f_\mu(s)\| \|e_\mu(s)\|_{V'} \, ds. \tag{4.1}$$

Next, we multiply (2.22) by χ_μ and we integrate over Q_t. Using, as usual, Hölder inequality, we get

$$\frac{\mu}{2}\|\chi_\mu(t)\|^2 + \nu \int_0^t \|\nabla \chi_\mu(s)\|^2 \, ds + \int_0^t (\xi_\mu(s), \chi_\mu(s)) \, ds \le \\ \le \frac{\mu}{2}\|\chi_{0\mu}\|^2 + \lambda \|\chi_\mu\|_{L^2(0,t;V)} \|e_\mu\|_{L^2(0,t;V')}. \tag{4.2}$$

Now we add (4.1) and (4.2), by taking (2.3), (2.12) and (2.14) into account. By (3.8), there exists a positive constant $\tilde{c}$ (depending also on ν and l) such that

$$\begin{aligned}&\|e_\mu(t)\|^2_{\mathrm{V}'} + \mu\|\chi_\mu(t)\|^2 + \min\{\nu, l\}\, \|\chi_\mu\|^2_{L^2(0,t;\mathrm{V})} \le \\ &\le \|e_{0\mu}\|^2_{\mathrm{V}'} + \mu\|\chi_{0\mu}\|^2 + \tilde{c}\|e_\mu\|^2_{L^2(0,t;\mathrm{V}')} + c\int_0^t \|f_\mu(s)\|\|e_\mu(s)\|_{\mathrm{V}'}\, ds.\end{aligned} \tag{4.3}$$

Thanks to (2.26)–(2.28), an extended version of Gronwall lemma (see, e.g., Baiocchi [2]) enables us to deduce that

$$\|e_\mu\|_{L^\infty(0,T;\mathrm{V}')} \le c \tag{4.4}$$

$$\|\chi_\mu\|_{L^2(0,T;\mathrm{V})} \le c \tag{4.5}$$

$$\mu^{1/2}\|\chi_\mu\|_{L^\infty(0,T;\mathrm{H})} \le c. \tag{4.6}$$

Second a priori estimate

We choose $v = e_\mu$ in (2.21) and integrate from 0 to $t \in]0, T[$. Using Hölder inequality and (3.9), we have

$$\begin{aligned}&\frac{1}{2}\|e_\mu(t)\|^2 + \int_0^t ((k * \nabla e_\mu)(s), \nabla e_\mu(s))\, ds \le \frac{1}{2}\|e_{0\mu}\|^2 + \\ &+ \lambda\|k\|_{L^1(0,T)}\|\Delta\chi_\mu\|_{L^2(0,t;\mathrm{H})}\|e_\mu\|_{L^2(0,t;\mathrm{H})} + \int_0^t \|f_\mu(s)\|\|e_\mu(s)\|\, ds.\end{aligned} \tag{4.7}$$

Next, we multiply (2.22) by $-\Delta\chi_\mu$ and we integrate over Q_t. Suppose, for the moment, that the graph α is a Lipschitz continuous function. Then $\xi_\mu = \alpha(\chi_\mu)$. Using again Hölder inequality, we get

$$\begin{aligned}&\frac{\mu}{2}\|\nabla\chi_\mu(t)\|^2 + \nu\|\Delta\chi_\mu\|^2_{L^2(0,t;\mathrm{H})} + \iint_{Q_t} \alpha'(\chi_\mu)|\nabla\chi_\mu|^2 \le \\ &\le \frac{\mu}{2}\|\nabla\chi_{0\mu}\|^2 + \lambda\|\Delta\chi_\mu\|_{L^2(0,t;\mathrm{H})}\|e_\mu\|_{L^2(0,t;\mathrm{H})}.\end{aligned} \tag{4.8}$$

Now we add (4.7) and (4.8), by taking (2.3) and (2.12) into account. Using (3.8), we deduce that there exists a positive constant $c(\nu)$ (depending also on ν) such that

$$\begin{aligned}&\frac{1}{2}\|e_\mu(t)\|^2 + \frac{\mu}{2}\|\nabla\chi_\mu(t)\|^2 + \frac{\nu}{2}\|\Delta\chi_\mu\|^2_{L^2(0,t;\mathrm{H})} \le \\ &\le \frac{1}{2}\|e_{0\mu}\|^2 + \frac{\mu}{2}\|\nabla\chi_{0\mu}\|^2 + c(\nu)\|e_\mu\|^2_{L^2(0,t;\mathrm{H})} + \int_0^t \|f_\mu(s)\|\|e_\mu(s)\|\, ds.\end{aligned} \tag{4.9}$$

Inequality (4.9) still holds when α is a graph satisfying (2.12)–(2.14). One can prove that by approximating α by its Yosida regularization (see, e.g., [6], [13]). Finally, thanks to (2.26)–(2.27) and (2.29), we apply again an extended version of Gronwall lemma to deduce that

$$\|e_\mu\|_{L^\infty(0,T;\mathrm{H})} \le c \tag{4.10}$$

$$\|\chi_\mu\|_{L^2(0,T;\mathrm{W})} \le c \tag{4.11}$$

$$\mu^{1/2}\|\chi_\mu\|_{L^\infty(0,T;\mathrm{V})} \le c\,. \tag{4.12}$$

Third a priori estimate

Arguing as before, we can assume that α is a Lipschitz continuous function (otherwise, we introduce its Yosida approximation). Then we multiply (2.22) by $\xi_\mu = \alpha(\chi_\mu)$ and we integrate over Q_t. Taking (2.12) into account, using Hölder inequality and (3.8), we get

$$\begin{aligned} &\mu\|j(\chi_\mu(t))\|_{L^1(\Omega)} + \iint_{Q_t} \alpha'(\chi_\mu)|\nabla\chi_\mu|^2 + \|\xi_\mu\|^2_{L^2(0,t;\mathrm{H})} \leq \\ &\leq \mu\|j(\chi_{0\mu})\|_{L^1(\Omega)} + \frac{1}{2}\|\xi_\mu\|^2_{L^2(0,t;\mathrm{H})} + \frac{\lambda^2}{2}\|e_\mu\|^2_{L^2(0,t;\mathrm{H})}\,. \end{aligned} \tag{4.13}$$

Owing to (2.12), (2.29) and (4.10), we apply Gronwall lemma and we deduce the following upper bounds.

$$\|\xi_\mu\|_{L^2(0,T;\mathrm{H})} \leq c \tag{4.14}$$

$$\mu\|j(\chi_\mu)\|_{L^\infty(0,T;L^1(\Omega))} \leq c\,. \tag{4.15}$$

Fourth a priori estimate

Taking (4.10)–(4.11) and (4.14) into account, a comparison in (2.22) leads to

$$\mu\|\partial_t\chi_\mu\|_{L^2(0,T;\mathrm{H})} \leq c\,. \tag{4.16}$$

Now, by (2.21), we have

$$\langle\langle \partial_t e_\mu - f_\mu, v\rangle\rangle \leq (\|k*e_\mu\| + \lambda\|k*\chi_\mu\|)\|v\|_{\mathrm{W}}, \quad \forall v\in W, \qquad \text{a.e. in }]0,T[\tag{4.17}$$

hence

$$\|\partial_t e_\mu - f_\mu\|_{\mathrm{W}'} \leq \|k*e_\mu\| + \lambda\|k*\chi_\mu\| \qquad \text{a.e. in }]0,T[. \tag{4.18}$$

Finally, thanks to (3.9), (4.5) and (4.10), from (4.18), we deduce that

$$\|\partial_t e_\mu - f_\mu\|_{L^2(0,T;\mathrm{W}')} \leq c\,. \tag{4.19}$$

Passage to the limit

Thanks to (4.10), (4.11), (4.14), (4.19), (4.12), and (4.16), well known weak and weak* compactness results allow us to deduce the existence of (e,χ,ξ) such that, at least for a subsequence of $\mu\to 0^+$,

$$e_\mu \overset{*}{\rightharpoonup} e \quad \text{in } L^\infty(0,T;\mathrm{H}) \tag{4.20}$$

$$\chi_\mu \rightharpoonup \chi \quad \text{in } L^2(0,T;\mathrm{W}) \tag{4.21}$$

$$\xi_\mu \rightharpoonup \xi \quad \text{in } L^2(0,T;\mathrm{H}) \tag{4.22}$$

$$\partial_t e_\mu - f_\mu \rightharpoonup \partial_t e - f \quad \text{in } L^2(0,T;\mathrm{W}') \tag{4.23}$$

$$\mu^{1/2}\chi_\mu \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;\mathrm{V}) \tag{4.24}$$

$$\mu\chi_\mu \rightharpoonup 0 \quad \text{in } H^1(0,T;\mathrm{H})\,. \tag{4.25}$$

The above convergences are enough in order to pass to the limit in (2.21)–(2.22), when $\mu \to 0^+$, obtaining (2.37)–(2.38). Next, we observe that (2.26) and (4.20) imply

$$e_\mu - 1 * f_\mu \overset{*}{\rightharpoonup} e - 1 * f \quad \text{in} \quad L^\infty(0,T;\mathrm{H}) \tag{4.26}$$

(where $1 * u$ denotes the time integral primitive of u).

Thanks to [21, Cor.4, Sec.8], from (4.23) and (4.26), we deduce the strong convergence

$$e_\mu - 1 * f_\mu \to e - 1 * f \quad \text{in} \quad C^0([0,T];\mathrm{V}') \tag{4.27}$$

and hence (still thanks to (2.26))

$$e_\mu \to e \quad \text{in} \quad C^0([0,T];\mathrm{V}'). \tag{4.28}$$

We want to show that (2.39) holds. Toward this aim, we multiply (2.22) by χ_μ and we integrate over Q_t; we obtain

$$\iint_{Q_t} \xi_\mu \chi_\mu = -\nu \iint_{Q_t} |\nabla \chi_\mu|^2 - \frac{\mu}{2}\|\chi_\mu(t)\|^2 + \frac{\mu}{2}\|\chi_{0\mu}\|^2 + \lambda \iint_{Q_t} e_\mu \chi_\mu. \tag{4.29}$$

We take the lim sup, as $\mu \to 0^+$, of both sides of (4.29); using (4.21), (2.28), and (4.28), we get

$$\limsup_{\mu\to 0} \iint_{Q_t} \xi_\mu \chi_\mu \le -\nu \iint_{Q_t} |\nabla \chi|^2 + \lambda \iint_{Q_t} e\chi = \iint_{Q_t} (\nu\Delta\chi + \lambda e)\chi. \tag{4.30}$$

A comparison in (2.38) gives

$$\limsup_{\mu\to 0} \iint_{Q_t} \xi_\mu \chi_\mu \le \iint_{Q_t} \xi\chi; \tag{4.31}$$

in view of [3, prop.1.1, p.42] (4.31) implies (2.39). Now, we only have to prove (2.32). We take the difference between (2.22) and (2.38); we multiply the resulting equation by $\chi_\mu - \chi$ and we integrate over Q_t; by using (2.14), Hölder inequality and (3.8), we get

$$\begin{aligned} &\frac{\mu}{2}\|\chi_\mu(t)\|^2 + \frac{1}{2}\min\{\nu, l\}\, \|\chi_\mu - \chi\|^2_{L^2(0,t;\mathrm{V})} \le \\ &\le \frac{\mu}{2}\|\chi_{0\mu}\|^2 + \mu \iint_{Q_t} \partial_t\chi_\mu \chi + \frac{\lambda^2}{2(\min\{\nu, l\})}\|e_\mu - e\|^2_{L^2(0,t;\mathrm{V}')}\,; \end{aligned} \tag{4.32}$$

since (2.28), (4.25), (4.28) hold, we deduce (2.32). □

Remark 4.1. Let us consider the following equation:

$$\partial_t(\vartheta + \lambda(\chi)) - k_0\Delta\vartheta - k * \Delta\vartheta = f \quad \text{in } Q, \tag{4.33}$$

where the Coleman-Gurtin [9] heat flux law is taken into account and k_0 is a positive constant. In [4, 5], existence, uniqueness and regularity results were proved for the system (4.33)–(1.2) supplemented by Cauchy and homogeneous Neumann boundary conditions. Remark that the assumptions on the heat kernel are only (2.2) and a positivity condition

—which comes from Thermodynamics, see, e.g., [18], [4, Sec. 1]— for $k_0\delta + k$ (where δ denotes the Dirac mass). Moreover, λ is a nonlinear function which can have a quadratic growth in [4], while it is Lipschitz continuous in [5]. It must be noted that the presence of the term $-k_0\Delta\vartheta$ in (4.33) entails that the solutions of the mixed problem related to (4.33)–(1.2) are more regular than the ones of the problem (1.1)–(1.4), obtained in [14, 15]. Hence, the asymptotic analysis, as $\mu \to 0^+$, of the solutions found in [4, 5] is simpler (using the same tools) than the one performed in the present paper. For the sake of brevity, we don't give here neither a precise statement, nor a proof, concerning such an asymptotic result.

Remark 4.2. Since e_μ is only H-valued (see (2.17) above), the second *a priori* estimate has been actually derived in a formal way. The rigorous procedure should had been the following: multiply (2.21) by a regularization of e_μ, said $e_{\varepsilon\mu} \in \mathrm{W}$, satifying $e_{\varepsilon\mu} + \varepsilon A e_{\varepsilon\mu} = e_\mu$, with A suitable operator from H to W (also see [13] for a similar procedure); then pass to the limit when $\varepsilon \to 0$, obtaining the bounds (4.10)–(4.12). Another method to make our argument rigorous is the following: consider a regularized problem $(P_{\varepsilon\mu})$, instead of (P_μ), by adding the term $(-\varepsilon\Delta e_\mu, v)$ in (2.21); consider, moreover, $(e_{\varepsilon\mu}, \chi_{\varepsilon\mu}, \xi_{\varepsilon\mu})$ the corresponding (more regular) solution to $(P_{\varepsilon\mu})$, satisfying, in particular, $e_{\varepsilon\mu} \in H^1(0,T;\mathrm{V}') \cap L^\infty(0,T;\mathrm{V})$ (thanks to [4, Thm. 2.1]); derive the same bounds (now in a rigorous way); pass to the limit with ε and then with μ.

5. Regularity and error estimates

In the first part of the present section, we deal with the proof of Theorem 2.3; in the second one, we prove the error estimates established by Theorems 2.5 and 2.6.

Proof of Theorem 2.3. As in the previous section, for the sake of brevity, we derive some uniform bounds only in a formal way. A rigorous procedure can be performed arguing as in Remark 4.2.

We differentiate (2.22) in time, we multiply by $\partial_t\chi_\mu$ and we integrate over Q_t. Recalling that the formal derivative α' is bounded from below by l thanks to (2.14), we have

$$\begin{aligned}&\frac{\mu}{2}\|\partial_t\chi_\mu(t)\|^2 + \nu\|\nabla\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{H})} + l\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{H})} \leq \\ &\leq \frac{\mu}{2}\|\partial_t\chi_\mu(0)\|^2 + \lambda\iint_{Q_t}\partial_t e_\mu\partial_t\chi_\mu\,.\end{aligned} \tag{5.1}$$

We recover the initial value of $\partial_t\chi_\mu$ from (2.22), taking (2.24)-(2.25) and (2.46) into account. Applying Hölder inequality, from (5.1) we get

$$\frac{\mu}{2}\|\partial_t\chi_\mu(t)\|^2 + \min\{\nu,l\}\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{V})} \leq c + \lambda\|\partial_t e_\mu\|_{L^2(0,t;\mathrm{V}')}\|\partial_t\chi_\mu\|_{L^2(0,t;\mathrm{V})}. \tag{5.2}$$

By (3.8), there exists a positive constant $\tilde{c}$ (depending also on ν and l) such that

$$\mu\|\partial_t\chi_\mu(t)\|^2 + \min\{\nu,l\}\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{V})} \leq c + \tilde{c}\|\partial_t e_\mu\|^2_{L^2(0,t;\mathrm{V}')}. \tag{5.3}$$

Next, we take $v = J^{-1}\partial_t e_\mu$ in (2.21) and we integrate from 0 to t. We obtain

$$\|\partial_t e_\mu\|^2_{L^2(0,t;\mathrm{V}')} = \sum_{j=4}^{8} I_j(t), \tag{5.4}$$

where

$$I_4(t) := -\int_0^t (k * e_\mu, \partial_t e_\mu) \tag{5.5}$$

$$I_5(t) := \int_0^t (k * e_\mu, J^{-1}\partial_t e_\mu) \tag{5.6}$$

$$I_6(t) := \lambda \int_0^t (J(k * \chi_\mu), J^{-1}\partial_t e_\mu) \tag{5.7}$$

$$I_7(t) := -\lambda \int_0^t (k * \chi_\mu, J^{-1}\partial_t e_\mu) \tag{5.8}$$

$$I_8(t) := \int_0^t (f_\mu, J^{-1}\partial_t e_\mu). \tag{5.9}$$

Owing to Hölder inequality, (3.9) and the definition of J, we estimate the above integrals as follows.

$$|I_4(t) + I_6(t)| \le c\|k\|_{L^1(0,T)} \left\{\|e_\mu\|_{L^2(0,t;\mathrm{V})} + \|\chi_\mu\|_{L^2(0,t;\mathrm{V})}\right\} \|\partial_t e_\mu\|_{L^2(0,t;\mathrm{V}')} \tag{5.10}$$

$$|I_5(t) + I_7(t)| \le c\|k\|_{L^1(0,T)} \left\{\|e_\mu\|_{L^2(0,t;\mathrm{V}')} + \|\chi_\mu\|_{L^2(0,t;\mathrm{V}')}\right\} \|\partial_t e_\mu\|_{L^2(0,t;\mathrm{V}')} \tag{5.11}$$

$$|I_8(t)| \le \|f_\mu\|_{L^2(0,t;\mathrm{V}')}\|\partial_t e_\mu\|_{L^2(0,t;\mathrm{V}')}\,. \tag{5.12}$$

Now, we use (3.8) taking (4.4)–(4.5) and (2.43) into account. We deduce that

$$\|\partial_t e_\mu\|^2_{L^2(0,t;\mathrm{V}')} \le c\left\{1 + \|e_\mu\|^2_{L^2(0,t;\mathrm{V})}\right\}, \tag{5.13}$$

for some positive constant c.

Next, we consider equation (2.21) rewritten in terms of ϑ_μ and χ_μ, recalling that $e_\mu = \vartheta_\mu + \lambda\chi_\mu$. Proceeding formally, we test it by $v = -\Delta\vartheta_\mu$ and we integrate from 0 to t. We obtain

$$\begin{aligned} &\frac{1}{2}\|\nabla\vartheta_\mu(t)\|^2 + \int_0^t ((k * \Delta\vartheta_\mu)(s), \Delta\vartheta_\mu(s))\,ds = \frac{1}{2}\|\nabla\vartheta_{0\mu}\|^2 + \\ &- \lambda\int_0^t (\nabla\partial_t\chi_\mu(s), \nabla\vartheta_\mu(s))\,ds + \int_0^t (\nabla f_\mu(s), \nabla\vartheta_\mu(s))\,ds, \end{aligned} \tag{5.14}$$

where $\vartheta_{0\mu} = e_{0\mu} - \lambda\chi_{0\mu}$.

Now, we apply Hölder inequality and (3.8), recalling (5.3) and (5.13). We have

$$\begin{aligned} &- \lambda\int_0^t (\nabla\partial_t\chi_\mu(s), \nabla\vartheta_\mu(s))\,ds \le c\left\{\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{V})} + \|\nabla\vartheta_\mu\|^2_{L^2(0,t;\mathrm{H})}\right\} \le \\ &\le c\left\{1 + \|e_\mu\|^2_{L^2(0,t;\mathrm{V})} + \|\nabla\vartheta_\mu\|^2_{L^2(0,t;\mathrm{H})}\right\} \le c\left\{1 + \|\nabla\vartheta_\mu\|^2_{L^2(0,t;\mathrm{H})}\right\}, \end{aligned} \tag{5.15}$$

thanks also to (4.5) and (4.10).

Using again Hölder inequality, we obtain

$$\int_0^t (\nabla f_\mu(s), \nabla\vartheta_\mu(s))\, ds \leq \int_0^t \|\nabla f_\mu(s)\|\|\nabla\vartheta_\mu(s)\|\, ds. \tag{5.16}$$

Finally, recalling (5.15)–(5.16) and owing to (2.3) and (2.43), we can apply to (5.14) an extended version of Gronwall lemma and deduce the following bound

$$\|\nabla\vartheta_\mu\|_{L^\infty(0,T;\mathrm{H})} \leq c\,. \tag{5.17}$$

Now, by (5.17), (4.5) and (4.10), we have

$$\|e_\mu\|_{L^2(0,T;\mathrm{V})} \leq c\,. \tag{5.18}$$

Next, by (5.18) and (5.13), we recover

$$\|\partial_t e_\mu\|_{L^2(0,T;\mathrm{V}')} \leq c\,. \tag{5.19}$$

Finally, by (5.19) and (5.3), we get

$$\mu^{1/2}\|\partial_t\chi_\mu\|_{L^\infty(0,T;\mathrm{H})} \leq c \tag{5.20}$$

$$\|\partial_t\chi_\mu\|_{L^2(0,T;\mathrm{V})} \leq c\,. \tag{5.21}$$

Next, we multiply (2.22) by $-\Delta\chi_\mu$ and we integrate by parts only in space. Thanks to (2.14), (5.20) and (4.10) we can deduce

$$\|\chi_\mu\|_{L^\infty(0,T;\mathrm{W})} \leq c\,. \tag{5.22}$$

Finally, thanks to (5.22), (5.17) and (4.10), it holds

$$\|e_\mu\|_{L^\infty(0,T;\mathrm{V})} \leq c \tag{5.23}$$

and (2.47)–(2.48) are completely proved. □

Proof of Theorem 2.5. We argue like in the derivation of the first *a priori* estimate, in the previous section. We consider the difference between equations (2.21) and (2.37), we take $v = J^{-1}(e_\mu - e)$ and we integrate from 0 to $t \in]0,T[$. Next, we multiply the difference between (2.22) and (2.38) by $\chi_\mu - \chi$ and we integrate over Q_t. We detail only the term coming from $\mu\partial_t\chi_\mu$ because we have to handle it in a different way with respect to the procedure adopted in the previous section. Considering it on the right hand side, we get

$$-\mu\iint_{Q_t} \partial_t\chi_\mu(\chi_\mu - \chi) \leq \frac{l}{2}\|\chi_\mu - \chi\|^2_{L^2(0,t;\mathrm{H})} + \frac{\mu^2}{2l}\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{H})}. \tag{5.24}$$

Now, we insert (5.24) in the formula corresponding to (4.2), owing to (2.48); we add the inequalities corresponding to (4.1) and (4.2); we apply an extended version of Gronwall lemma and we obtain

$$\begin{aligned} &\|e_\mu - e\|^2_{C^0([0,T];\mathrm{V}')} + \|\chi_\mu - \chi\|^2_{L^2(0,T;\mathrm{V})} \leq \\ &\leq c\left\{\|e_{0\mu} - e_0\|^2 + \|f_\mu - f\|^2_{L^1(0,T;\mathrm{H})}\right\} + c\mu^2. \end{aligned} \tag{5.25}$$

Thus, on account of (5.25) and (2.58), we deduce (2.56).

Next, we multiply the difference between (2.22) and (2.38) by $\chi_\mu - \chi$ and we integrate only in space. Arguing, in particular, as for (5.24), we get

$$\begin{aligned} &\nu\|\nabla(\chi_\mu - \chi)(s)\|^2 + \frac{l}{2}\|(\chi_\mu - \chi)(s)\|^2 \leq \\ &\leq \frac{\mu^2}{2l}\|\partial_t\chi_\mu(s)\|^2 + \lambda\|(\chi_\mu - \chi)(s)\|_{\mathrm{V}}\|(e_\mu - e)(s)\|_{\mathrm{V}'}, \qquad \text{for a.a. } s \in]0,T[. \end{aligned} \tag{5.26}$$

By using suitably (3.8), we deduce that

$$\|(\chi_\mu - \chi)(s)\|^2_{\mathrm{V}} \leq c\mu^2\|\partial_t\chi_\mu(s)\|^2 + c\|(e_\mu - e)(s)\|^2_{\mathrm{V}'}, \qquad \text{for a.a. } s \in]0,T[. \tag{5.27}$$

Finally, since $\mu\|\partial_t\chi_\mu\|^2_{L^\infty(0,T;\mathrm{H})}$ is uniformly bounded thanks to (2.48), by (5.27) and (2.56) follows (2.57) (recall that $\chi_\mu - \chi$ is continuous from $[0,T]$ into V). □

Proof of Theorem 2.6. We consider the difference between equations (2.21) and (2.37), we take $v = e_\mu - e$ and we integrate from 0 to $t \in]0,T[$. Note that this choice of test function is admissible because Theorem 2.3 ensures that both equations (2.21) and (2.37) may be meant in V′ (see also Remark 2.4); moreover, $e_\mu - e \in L^\infty(0,T;\mathrm{V})$ thanks to (2.49). Next, we consider the difference between (2.22) and (2.38), we multiply by $-\Delta(\chi_\mu - \chi)$ and we integrate over Q_t. We proceed as in the derivation of the second *a priori* estimate, in the previous section. Clearly, some modifications are needed due to the terms coming from $\mu\partial_t\chi_\mu$ and $\xi_\mu - \xi$. We recall that, by assuming (2.59), we have here $\xi_\mu = \alpha(\chi_\mu)$ and $\xi = \alpha(\chi)$. Similarly to the formula (4.8), we obtain

$$\begin{aligned} &\nu\|\Delta(\chi_\mu - \chi)\|^2_{L^2(0,t;\mathrm{H})} + \iint_{Q_t} \alpha'(\chi_\mu)|\nabla(\chi_\mu - \chi)|^2 \leq \\ &\leq \mu\iint_{Q_t} \partial_t\chi_\mu\Delta(\chi_\mu - \chi) + \iint_{Q_t} (\alpha'(\chi) - \alpha'(\chi_\mu))\nabla\chi\cdot\nabla(\chi_\mu - \chi) + \\ &+ \frac{\nu}{4}\|\Delta(\chi_\mu - \chi)\|^2_{L^2(0,t;\mathrm{H})} + \frac{\lambda^2}{\nu}\|e_\mu - e\|^2_{L^2(0,t;\mathrm{H})}. \end{aligned} \tag{5.28}$$

Using again Hölder inequality and (3.8), we get

$$\mu\iint_{Q_t} \partial_t\chi_\mu\Delta(\chi_\mu - \chi) \leq \frac{\nu}{4}\|\Delta(\chi_\mu - \chi)\|^2_{L^2(0,t;\mathrm{H})} + \frac{\mu^2}{\nu}\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{H})}. \tag{5.29}$$

Now, we note that (2.52) implies $\chi_\mu, \chi \in L^\infty(Q)$ thanks to the three-dimensional injection $H^2(\Omega) \subset C^0(\bar{\Omega})$. Thus, owing to (2.59) and (2.32), we have

$$\begin{aligned} &\iint_{Q_t} (\alpha'(\chi) - \alpha'(\chi_\mu))\nabla\chi\cdot\nabla(\chi_\mu - \chi) \leq c\iint_{Q_t} |\nabla\chi\cdot\nabla(\chi_\mu - \chi)| \leq \\ &\leq c\|\nabla\chi\|_{L^2(0,t;\mathrm{H})}\|\nabla(\chi_\mu - \chi)\|_{L^2(0,t;\mathrm{H})} \leq c\|\chi_\mu - \chi\|_{L^2(0,t;\mathrm{V})}. \end{aligned} \tag{5.30}$$

Taking (5.29)–(5.30) and (2.59) into account, from (5.28), we deduce that

$$\frac{\nu}{2}\|\Delta(\chi_\mu - \chi)\|^2_{L^2(0,t;\mathrm{H})} \leq \frac{\mu^2}{\nu}\|\partial_t\chi_\mu\|^2_{L^2(0,t;\mathrm{H})} + \frac{\lambda^2}{\nu}\|e_\mu - e\|^2_{L^2(0,t;\mathrm{H})} + c\|\chi_\mu - \chi\|_{L^2(0,t;\mathrm{V})}. \tag{5.31}$$

Next, we add (5.31) with the inequality corresponding to (4.7) (after a suitable use of (3.8)), owing to (2.48) and (2.56). Applying an extended version of Gronwall lemma, we infer that

$$\|e_\mu - e\|_{C^0([0,T];\mathrm{H})} \le c\left\{\mu^{1/2} + (\varepsilon_\mu)^{1/2}\right\} \tag{5.32}$$

and then

$$\|\Delta(\chi_\mu - \chi)\|_{L^2(0,T;\mathrm{H})} \le c\left\{\mu^{1/2} + (\varepsilon_\mu)^{1/2}\right\}. \tag{5.33}$$

Thus, (2.60) is proved. Next, we multiply the difference between (2.22) and (2.38) by $-\Delta(\chi_\mu - \chi)$ and we integrate only in space. Arguing similarly to (5.29)–(5.30), we get

$$\begin{aligned}&\frac{\nu}{2}\|\Delta(\chi_\mu - \chi)(s)\|^2 \le \frac{\mu^2}{\nu}\|\partial_t\chi_\mu(s)\|^2 + \\ &+ \frac{\lambda^2}{\nu}\|(e_\mu - e)(s)\|^2 + c\|\chi\|_{L^\infty(0,T;\mathrm{V})}\|(\chi_\mu - \chi)(s)\|_{\mathrm{V}}, \quad \text{for a.a. } s \in]0,T[.\end{aligned} \tag{5.34}$$

Finally, recalling (2.48), (2.60), (2.52) and (2.57), by (5.34), we deduce

$$\|\Delta(\chi_\mu - \chi)\|_{L^\infty(0,T;\mathrm{H})} \le c\left\{\mu^{1/4} + (\varepsilon_\mu)^{1/2}\right\} \tag{5.35}$$

and (2.61) is proved. □

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