Journal of Convex Analysis
Volume 6 (1999), No. 2, 395–398

The Barrier Cone of a Convex Set and the Closure of the Cover

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Received October 20, 1997
Revised manuscript received April 7, 1999

For an arbitrary non-empty closed convex set A in $\mathbb{R}^n$, we prove that the polar of the difference between the barrier cone $\mathbb{B}(A)$ and its interior $\operatorname{int}\mathbb{B}(A)$ coincides with the recession cone $0^+(\operatorname{cl}\mathbb{G}(A))$ of the closure of the cover $\mathbb{G}(a)$.

Keywords: convex set, barrier cone, recession cone, cover, polar cone

1991 Mathematics Subject Classification: 52A20

1. Introduction

It is well-known that the recession cone of a non-empty closed convex set A in $\mathbb{R}^n$ is equal to $[\mathbb{B}(A)]^\circ$, the polar of the barrier cone $\mathbb{B}(A)$ of A [11, 14.2.1].

The principal result of this note is a similar equality involving the recession cone of $\operatorname{cl}\mathbb{G}(A)$, instead of $0^+(A)$.

The cover $\mathbb{G}(A)$ of a closed convex set A was introduced in 1983 by Bair and Jongmans [4] in order to solve a problem posed in 1962 by Braunschweiger and Clark in a paper about the Farkas Lemma [8]; their question was the following: when is the conical hull of a closed convex set closed?

This concept of cover plays an important role in many applications like the separation of convex sets or the optimization theory (see [5, 6, 7] and some references in these three papers).

2. Notations

Throughout this paper, A is a non-empty closed convex subset of $\mathbb{R}^n$ (with $n \geq 2$); Δ is a closed half-line with the origin as endpoint, while ${}^s\Delta$ is the line including Δ; if $x \in \mathbb{R}^n \setminus \{0\}$, Δ_x is the ray $\Delta_x = \{\lambda x : \lambda \geq 0\} = \{y \mid \exists \lambda \geq 0,\ y = \lambda x\}$.

For a non-empty C in $\mathbb{R}^n$, we denote by $\operatorname{cl} C$ its closure, $\operatorname{ri} C$ its relative interior, $\operatorname{int} C$ its

ISSN 0944-6532 / $ 2.50

interior: when C is not closed, $C \setminus \operatorname{int} C$ is different from the boundary $(\operatorname{cl} C) \setminus \operatorname{int} C$ of C.

$$0^+(C) = \{y \in \mathbb{R}^n : \forall(\lambda, x) \in [0, +\infty[\times C,\ x + \lambda y \in C\};$$

recall that $0^+(\operatorname{ri} C) = 0^+(\operatorname{cl} C)$ (see [11, 8.3.1, p. 63]).

$$\mathbb{B}(C) = \{u \in \mathbb{R}^n : \exists \alpha \in \mathbb{R},\ \forall x \in C, \langle x, u \rangle \leq \alpha\}$$

is the barrier cone of C. If $u \in \mathbb{B}(C) \setminus \{0\}$, we set

$$H_u(C) = \{x \in \mathbb{R}^n : \langle x, u \rangle = \sup_{y \in C} \langle y, u \rangle\};$$

$H_u(C)$ is a hyperplane; we also put

$$F_u(C) = C \cap H_u(C).$$

If C is a non-empty convex cone with 0 as apex,

$$C^\circ = \{u \in \mathbb{R}^n : \forall x \in C,\ \langle x, u \rangle \leq 0\}$$

is the polar cone of C.

The cover $\mathbb{G}(A)$ of A can be defined as follows:

$$\mathbb{G}(A) = \bigcap_{\Delta \subset 0^+(A)} (A - \Delta)$$

[6, prop. 1]; equivalently, $\mathbb{G}(A)$ is the union of A and the set of all points x such that the conical hull of A from x is closed. In general, $\mathbb{G}(A)$ is not closed, even if A is closed.

Lemma 2.1. *Let u be a point of $\mathbb{B}(A) \setminus \{0\}$; $u \in \operatorname{int} \mathbb{B}(A)$ if and only if $F_u(A)$ is a non-empty bounded face of A.*

Proof. We know that $u \in \operatorname{int} \mathbb{B}(A)$ if and only if u belongs to the interior of the effective domain of the support function $\delta^*(\cdot \mid A)$ of A, thus if and only if the subdifferential of $\delta^*(\cdot \mid A)$ at u is non-empty and bounded (see [11, pp. 28 and 112, and 23.4, p. 217]). Now, this subdifferential at u is equal to $F_u(A)$ (see [11, 23.5.3, p. 219]). □

Lemma 2.2. $\mathbb{B}(A) \setminus [\{0\} \cup \operatorname{int} \mathbb{B}(A)] \subset [\mathbb{B}(\mathbb{G}(A))] \setminus \operatorname{int} \mathbb{B}(\mathbb{G}(A))$.

Proof. Let u be a point belonging to $\mathbb{B}(A) \setminus \operatorname{int} \mathbb{B}(A)$ and different from 0.

From Lemma 2.1, $H_u(A)$ is a support hyperplane of A with $F_u(A)$ unbounded or an asymptote for A; in these two cases, there exist $\Delta \subset 0^+(A)$ and $x \in H_u(A)$ such that $x + \Delta \subset H_u(A)$ [10, prop. 2, p. 357].

Consider the closed halfspace $\sum$ limited by $H_u(A)$ and which includes A; clearly, $A - \Delta \subset \sum$, therefore $\operatorname{cl} \mathbb{G}(A) \subset \sum$ and thus $u \in \mathbb{B}(\mathbb{G}(A))$.

Moreover, $A \subset \operatorname{cl} \mathbb{G}(A)$, whence $H_u(\operatorname{cl} \mathbb{G}(A)) = H_u(A)$; since $F_u(\operatorname{cl} \mathbb{G}(A))$ is empty or is unbounded, u does not belong to $\operatorname{int} \mathbb{B}(\mathbb{G}(A))$ by Lemma 2.1. □

Theorem 2.3. $[\mathbb{B}(A) \setminus \operatorname{int} \mathbb{B}(A)]^\circ = 0^+(\operatorname{cl} \mathbb{G}(A))$ *for all unbounded closed convex subsets* A *of* $\mathbb{R}^n$ $(n \geq 2)$.

Proof. By Lemma 2.2, $B = \mathbb{B}(A) \setminus \operatorname{int} \mathbb{B}(A)$ is included in $\mathbb{B}(\mathbb{G}(A))$, whence [11, 14.2.1, p. 123]

$$B^\circ \supset [\mathbb{B}(\mathbb{G}(A))]^\circ = 0^+(\operatorname{cl} \mathbb{G}(A));$$

note that B° exists because A is unbounded and consequently $\mathbb{B}(A) \neq \mathbb{R}^n$, $\operatorname{int} \mathbb{B}(A) \not\ni 0$, $B \neq \emptyset$.

Conversely, suppose that x belongs to $B^\circ \setminus 0^+(c\ell\, \mathbb{G}(A))$. Let a be a point of $\operatorname{ri} A \subset \operatorname{ri} \mathbb{G}(A)$. Since $x \notin 0^+(\operatorname{cl} \mathbb{G}(A))$ and $x \neq 0$, the set $(a + \Delta_x) \cap \mathbb{G}(A)$ is bounded and in that case, it is possible to find $\lambda > 0$ such that $a + \lambda x \notin \mathbb{G}(A)$.

By the definition of the cover, there exists $\Delta \subset 0^+(A)$ such that $(a + \Delta + \lambda x) \cap A = \emptyset$; since $\Delta \subset 0^+(A)$,

$$(a + {}^s\Delta + \lambda x) \cap A = \emptyset.$$

By Hahn-Banach's Theorem, there exist $u \neq 0$ and $\alpha \in \mathbb{R}$ such that $\langle y, u \rangle = \alpha$ for each $y \in a + {}^s\Delta + \lambda x$, in particular for $y = a + \lambda x$, and $\langle z, u \rangle < \alpha$ for each $z \in \operatorname{ri} A$, in particular for $z = a$.

It follows that $u \in \mathbb{B}(A)$. Moreover, $F_u(A)$ is not a non-empty bounded face of A, otherwise $(F_u(A)) \cap A \neq \emptyset$, so that $F_u(A)$ contains a translate of Δ. Thus, by Lemma 2.1, $u \in B$ and, since $x \in B^\circ$, $\langle x, u \rangle \leq 0$. But, $\langle a, u \rangle < \alpha = \langle a + \lambda x, u \rangle$ implies $\langle x, u \rangle > 0$, which is a contradiction. □

Here are some consequences and remarks about this Theorem.

When A is hyperbolic, i.e. there exists a bounded set B such that $A \subset B + 0^+(A)$ [2, p. 182], the barrier cone $\mathbb{B}(A)$ is closed [2, prop. 5, p. 183], so that $0^+(\operatorname{cl} \mathbb{G}(A))$ coincides with the polar of the boundary ${}^\bullet\mathbb{B}(A) = (\operatorname{cl} \mathbb{B}(A)) \setminus \operatorname{int} \mathbb{B}(A)$ of $\mathbb{B}(A)$; especially, if A is a convex polyhedron but is not bounded and is different from a ray, then the polar of $\mathbb{B}(A)$ is equal to the polar of the boundary of $\mathbb{B}(A)$ because, in this case, [5, prop. 11, p. 292]

$$[\mathbb{B}(A)]^\circ = 0^+(A) = 0^+(\operatorname{cl} \mathbb{G}(A)) = ({}^\bullet\mathbb{B}(A))^\circ.$$

Note that the polar of the boundary of $\mathbb{B}(A)$ can be different from $0^+(\operatorname{cl} \mathbb{G}(A))$. For example, if A is continuous, i.e. the support function of A is continuous [9, p. 379], or equivalently, if A is parabolic, i.e. for each $z \notin A$ and each $\Delta \subset 0^+(A)$, $z + \Delta$ meets A, then $\mathbb{B}(A) \setminus \{0\}$ is open [1, Theor. 2, p. 238], whence

$$\mathbb{R}^n = \{0\}^\circ \subset [\mathbb{B}(A) \setminus \operatorname{int} \mathbb{B}(A)]^\circ = 0^+(\operatorname{ri} \mathbb{G}(A)),$$

so that we obtain again the equality $\mathbb{G}(A) = \mathbb{R}^n$ of [9, 1.3, p. 381], but the polar of the boundary of $\mathbb{B}(A)$ can be different from the whole space $\mathbb{R}^n$, as it is shown by the convex hull A of the parabola $y = x^2$ in the plane.

Acknowledgements. The authors thank the referee for some helpful suggestions which have simplified the presentation of this paper. Especially, they are grateful for his analytical and short proof of Lemma 2.1, which was previously proved with a geometric point of view.

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