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Compactly epi-Lipschitzian Convex Sets and Functions in Normed Spaces*

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We provide several characterizations of compact epi-Lipschitzness for closed convex sets in normed vector spaces. In particular, we show that a closed convex set is compactly epi-Lipschitzian if and only if it has nonempty relative interior, finite codimension, and spans a closed subspace.

Next, we establish that all boundary points of compactly epi-Lipschitzian sets are proper support points. We provide the corresponding results for functions by using inf-convolutions and the Legendre–Fenchel transform. We also give an application to constrained optimization with compactly epi-Lipschitzian data via a generalized Slater condition involving relative interiors.

Keywords: Compactly epi-Lipschitzian set, convex set

1. Introduction

The concept of compactly epi-Lipschitzian (CEL) sets in locally convex topological spaces was introduced by Borwein and Strojwas [6]. It is an extension of Rockafellar's concept of epi-Lipschitzian sets [36]. An advantage of the CEL property is that it always holds in finite dimensional spaces and, in contrast to its epi-Lipschitzian predecessor, makes it possible to recapture much of the detailed information available in finite dimensions. The original motivation for introducing the CEL concept was to select class of closed sets in infinite dimensions (primarily in Banach spaces) for which the Clarke tangent and normal cones [11] adequately measure boundary behavior. A number of strong results were obtained in this direction; see [3], [6], [7], [8], and references therein. At the same time it was clarified that the CEL property is not sufficient for the (weak-star) locally compactness of the Clarke normal cone at boundary points [3, Example 4.1]. To get

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the latter result, Borwein introduced [3] the notion of epi-Lipschitz-like (ELL) sets that takes an intermediate place between the epi-Lipschitzian and CEL properties. He also proved in [3] that, for any closed subset $C \subset X$ of a normed space and its boundary point $c \in \operatorname{bd} C$, the Clarke normal cone is locally compact if C is CEL and tangentially regular at this point. The mentioned regularity property seems to be restrictive even in the finite dimensional setting. In particular, it never holds for the so-called Lipschitzian manifolds [38] which are locally homeomorphic to graphs of nonsmooth Lipschitz continuous vector functions; see [26, Section 3] for more details.

Further research showed that the regularity assumption can be avoided in infinite dimensions and the CEL property alone ensures the required local compactness if the Clarke normal cone is replaced with a different concept of generalized normals. The first result in this direction was obtained by Loewen [23] who proved, for any closed sets in reflexive spaces, that the CEL property implies the local compactness of the so-called limiting Fréchet normal cone [22] which is an infinite dimensional generalization of the normal cone introduced by Mordukhovich [24], [25]. An extension of this result to the more general case of weakly compactly generated Asplund spaces was obtained by Mordukhovich and Shao [30].

In the case of general Banach spaces a similar result was established by Jourani and Thibault [19] for the so-called G-normal cone of Ioffe [15] that is another infinite dimensional extension of Mordukhovich's construction. In [17], Jourani proved that the CEL property implies the local compactness of Ioffe's A-normal cone (which may be bigger than the G-cone) if the space is "weakly trustworthy." Recently Ioffe [16] established several characterizations of the CEL property in terms of normal cones satisfying certain requirements in corresponding Banach spaces. We refer the reader to [4], [10], [16], [17], [18], [19], [20], [21], [29], [28], [30] and their bibliographies for various applications of the CEL property to subdifferential calculus, metric regularity, Lipschitzian stability, necessary optimality conditions, and related aspects of nonlinear analysis and optimization in Banach spaces.

The primary goal of this paper is to provide intrinsic characterizations of the CEL property of closed convex sets in normed spaces. We are not familiar with any of such characterizations for either CEL or ELL convex sets even in particular infinite dimensional spaces. On the other hand, it has been known for a long time that a convex set is epi-Lipschitzian if and only if its interior is nonempty; see Rockafellar [37]. In this paper we prove that the CEL and ELL properties of closed convex sets agree in any normed spaces. Our main Theorem 2.5 in Section 2 contains also eight other characterizations of the CEL property one of which requires the additional Baire structure of the normed space in question. In particular, we show that a closed convex subset of a normed space is CEL if and only if its relative interior is nonempty and its span is a closed subspace of finite codimension.

In Section 3 we study supporting properties of CEL sets. The main characterization theorem of Section 2 allows us to establish that any boundary point of a CEL closed convex set in an arbitrary normed space is a proper support point of the set. In the case of Banach spaces we give a variational proof of this result and discuss its nonconvex generalizations. Section 4 concerns with characterizations of the CEL property for closed convex functions that are derived from the corresponding set characterizations applied to epigraphs. The final Section 5 contains some applications of the obtained results to constrained optimization via a generalized Slater interiority condition. Throughout the paper we use standard notation and terminology.

2. Characterizations of compactly epi-Lipschitzian Convex Sets

We give our main characterization theorem and proceed to prove it.

2.1. The main results: formulations and discussions

In the following, X denotes a normed linear space, and $\|.\|$ its norm. First we recall the definition of CEL and ELL sets [3, Definition 3.1]. We use the terminology compactly epi-Lipschitzian while in the literature it is sometimes referred as compactly epi-Lipschitz.

Definition 2.1. (i) A set C in X is *compactly epi-Lipschitzian (CEL)* if for all x in C, there are N_x a neighborhood of x, U a neighborhood of the origin, a positive ϵ and K a convex compact set such that

$$0 < \lambda < \epsilon \Longrightarrow C \cap N_x + \lambda U \subset C + \lambda K.$$

(ii) A subset C of X is *epi-Lipschitz-like (ELL)* if for all x in C there are N_x a neighborhood of x, Ω a convex set with polar set Ω^0 weakly$*$ locally compact, and a positive ϵ such that

$$0 < \lambda < \epsilon \Longrightarrow C \cap N_x + \lambda\Omega \subset C.$$

To characterize CEL convex sets, we recall what is a precompact set and a Baire space.

Definition 2.2. (i) A set P is *precompact* or *totally bounded* if for any open set U, there is a finite set F with $P \subset F + U$.

(ii) A set Σ is a *polytope* if it is the convex hull of a finite number of points.

(iii) A normed linear space X is a *Baire space* or *of the second category* if any countable covering of X with closed sets A_n, contains a set A_{n_0} with nonempty interior.

Remark 2.3. The Baire category theorem tells us that if X is a complete metric space, every nonempty open subset of X is of the second category [39, 41]. In particular, any complete metric space is a Baire space (see [41] for a reference on Baire spaces). Hence Baire spaces include Banach spaces, but there are examples of Baire spaces which are not complete as the following example shows [41, Exercise 3-1-4] (see also [1]) .

Example 2.4. Take X an infinite dimensional Banach space. It contains Y a nonclosed subspace with countable non-finite codimension. Name $(e_i)_i$ the sequence of linearly independent vectors such that $\operatorname{span}(Y \cup \cup_{i=1}^{\infty}\{e_i\}) = X$. Define $X_N := \operatorname{span}\{Y, e_1, \dots, e_N\}$. Then $X = \cup X_N$ and there is $\bar{N}$ with $X_{\bar{N}}$ of second category. Since $X_{\bar{N}}$ is not closed, it is a non-Banach Baire space.

Now we can state our main theorem.

Theorem 2.5. *Let C denote a closed convex set in a normed linear space X. The following are equivalent:*

(i) *The set C is CEL.*

(ii) *There is a convex compact set K with* $0 \in \operatorname{int}(C + K)$.

(iii) *There is a precompact set P with* $0 \in \operatorname{int}(C + P)$.

(iv) *There is a convex polytope* Σ *with* $0 \in \operatorname{int}(C+\Sigma)$.

(v) *There is a finite dimensional space* E *with* $\operatorname{int}(C+E) \neq \emptyset$.

(vi) *There is a point* x *in* C *such that the polar set of* $C-x$, $(C-x)^0$ *is weak-star locally compact.*

(vii) *The subspace spanned by* C, $\operatorname{span} C$, *is a finite-codimensional closed subspace and the relative interior* $\operatorname{ri} C := \operatorname{int}_{\operatorname{span} C} C$ *is nonempty.*

(viii) *The set* C *is ELL.*

(ix) *The projection* $Q : X \to M$ *on the subspace* $M := \operatorname{span} C$ *is linear continuous with its null space* $\mathcal{N}(Q)$ *finite dimensional and* $\operatorname{int}(Q(C))$ *nonempty.*

When X *is a Baire normed space, these are also equivalent to:*

(x) *There is a continuous linear open map* $Q : X \to Y$ *having a finite dimensional nullspace such that* $\operatorname{int}(Q(C))$ *is nonempty.*

In fact, our proof shows that the theorem still holds when Baire normed space is replaced by convex Baire spaces (see [40] for a reference on convex Baire spaces). That notion is slightly more general since there are examples of convex Baire spaces which are not Baire normed space.

This theorem calls for several kind of remarks.

Remark 2.6. First are some straightforward remarks:

- Clearly compact epi-Lipschitzness is invariant under translation.
  Hence we may always assume $0 \in C$ to simplify some of our arguments.
- The intersection of CEL convex sets need not be CEL. Indeed, take C_1 a pointed cone with interior in X an infinite dimensional subspace. Let C_2 be $-C_1$. Then C_1 and C_2 are CEL (since they have nonempty interior) but $C_1 \cap C_2 = \{0\}$ is not CEL (otherwise 0 would have a compact neighborhood, so X would be finite dimensional). For example, take the positive cone in c or in l_∞.
- Compact epi-Lipschitzness is clearly an infinite dimensional notion. Indeed, any set in a finite dimensional normed linear space is CEL (take K equals to the unit ball).

Remark 2.7. Next all three parts of Properties (vii) are needed:

- We do need $\operatorname{span} C$ to be closed. Take Φ a discontinuous linear function on an infinite dimensional Banach space and consider $C := \Phi^{-1}(0)$ an hyperplane. Then there is e in X such that $C + \mathbb{R}e = X$, so $\operatorname{span} C$ is finite codimensional. However, we have $\operatorname{int}(C + [-1,1]e)$ is empty.
  Indeed, suppose there is an open set U contained in $C + [-1,1]e$. Then $\Phi(U) \subset \Phi(C) + [-1,1]\Phi(e) = [-1,1]\Phi(e)$. So Φ is bounded which contradicts the fact Φ is not continuous. Consequently, there is no compact convex set K such that $\operatorname{int}(C+K) \neq \emptyset$, which means C is not CEL.
- Assuming $\operatorname{ri} C$ is nonempty and $\operatorname{span} C$ closed does not imply C is CEL. For example take $C := \mathbb{R}e$ in an infinite dimensional Banach space. Then $\operatorname{ri} C \neq \emptyset$ but C is not finite codimensional hence not CEL.
- We do need $\operatorname{ri} C \neq \emptyset$. Consider $X := l_2(\mathbb{N})$ the space of sequences with the norm $\|\cdot\|_2$, and $C := l_2^+$ the positive cone. Then $\operatorname{span} C = X$ is finite codimensional closed but $\operatorname{ri} C = \emptyset$.

- Property (vii) may be rewritten as: "there is a finite codimensional closed subspace M such that $C \subset M$ and $\mathrm{int}_M(C)$ is nonempty". This seems a weaker statement since Property (vii) merely states that it holds for $M = \mathrm{span}\, C$. However note that for any set S in a normed space X, $\mathrm{int}\,(S)$ nonempty yields $\mathrm{span}\, S = X$; consequently, the set M in the rephrased statement must be $\mathrm{span}\, C$. That fact follows from arguing by contradiction: take any s in $\mathrm{int}\,(S)$ and assume there is x not in $\mathrm{span}\, S$. Then build $x_n := x/n + s$. For n large enough, x_n is in $\mathrm{int}\,(S)$, so $x = n(x_n - s)$ is in $\mathrm{span}\, S$.

Remark 2.8. The convexity is very important in our result. Indeed, take $X := l_\infty(\mathbb{N})$ the bounded sequences in the supremum norm, define $f(x) := \liminf_{n\to\infty}|x_n|$, and consider $C := \{x \in X \ : \ f(x) \le 0\}$. Then C is CEL but neither ELL nor convex [3, Example 4.1].

Similarly, $\{(x, r) : \liminf x_k \le r \text{ in } l_2(\mathbb{R})\}$ is nonconvex CEL but not ELL.

Remark 2.9. In Property (iv) not only can we find a polytope Σ satisfying $\mathrm{int}\,(C + \Sigma) \neq \emptyset$, but also can we take $\Sigma = \sum_{k=1}^{N}[-1, 1]e_k$ where $e_1, \ldots, e_N$ are linearly independent vectors. This fact will be used several time to simplify our proofs.

2.2. Proof of the main theorem: general relations

The proof of Theorem 2.5 is split into several lemmas. We first show how all parts fit together.

Proof. First the relations (i)⇔(ii)⇒(iii)⇒(iv)⇔(v) are implied by Proposition 2.10 and Lemma 2.11. Next Lemma 2.14 gives (iv)⇒(x)⇒(ii) and Lemma 2.12, 2.15, and 2.16 give (iv)⇔(vii)⇔(ix); so all properties (i)–(v), (vii), (ix), and (x) are equivalent.

Now use Lemma 2.17 to get (vii) and (iv) imply (viii) and [3, Proposition 3.1 (a)] to obtain (viii)⇒(i).

Finally, [3, Lemma 2.1] gives (v)⇒(vi)⇒(ii). □

Let us start with the proof of our first characterization: (i)⇔ (ii). Note that it holds without any closedness assumption, so any convex set containing a CEL set is CEL (properties (i)–(vi) are clearly preserved under inclusion). In particular, if C is convex CEL, its closure $\bar{C}$ is CEL.

Proposition 2.10. *Let C be a convex subset of X. Then C is CEL if and only if there is K a convex compact set such that $0 \in \mathrm{int}\,(C + K)$*

Proof. Apply the notation of Definition 2.1. If C is CEL, take x in C and $\lambda := \epsilon/2$. Since

$$x + \lambda U \subset C \cap N_x + \lambda U \subset C + \lambda K,$$

we obtain $\lambda U \subset C + (\lambda K - x)$. Hence $0 \in \mathrm{int}\,(C + (\lambda K - x))$, which proves the necessary condition.

Fix $\bar{x} \in C$. Let V be a neighborhood of zero with $V \subset C + K$ and let U be a convex neighborhood of zero with $U+U\subset V$. Then for any $\lambda\in(0,1)$ one has

$$\begin{aligned} C\cap(\bar{x}+U)+\lambda U &\subset (1-\lambda)C+\lambda(\bar{x}+U)+\lambda U \\ &\subset (1-\lambda)C+\lambda(U+U)+\lambda\bar{x} \\ &\subset (1-\lambda)C+\lambda C+\lambda K+\lambda\bar{x} \\ &\subset C+\lambda(K+xb). \end{aligned}$$

This completes the proof because $K+\bar{x}$ is compact. $\square$

Lemma 2.11. *In Theorem* 2.5*, the following relations hold:*
(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) $\Leftrightarrow$(v).

Proof. Clearly, (ii)$\Rightarrow$(iii) holds (a compact set is precompact) as well as (iv)$\Rightarrow$(v) (span Σ is finite dimensional) and (v)$\Rightarrow$(iv)(because for any basis $e_1,\dots,e_N$ of E, $\Sigma=\sum_{i=1}^N[-1,1]e_i$ is a convex polytope spanning E and so satisfying $\operatorname{int}(C+\Sigma)\neq\emptyset$).

So the only remaining implication to prove is (iii)$\Rightarrow$(iv). Let U be a nonempty bounded open set with $U\subset C+P$. Using precompactness, there is a finite set F such that $P\subset F+U/2$. Define $\Sigma:=\operatorname{co}F$. We have

$$U\subset C+F+\frac{U}{2}.$$

By induction, we deduce

$$\begin{aligned} \frac{U}{2} &\subset \frac{C}{2}+\frac{\Sigma}{2}+\frac{U}{4}\subset\frac{C}{2}+\frac{C}{4}+\frac{\Sigma}{2}+\frac{\Sigma}{4}+\frac{U}{8}, \\ &\subset \cdots\subset(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})C+(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})\Sigma+\frac{U}{2^{n+1}}. \end{aligned}$$

Consequently,

$$\frac{U}{2}\subset(1-\frac{1}{2^n})(C+\Sigma)+\frac{U}{2^{n+1}}.$$

Taking the limit when n goes to infinity, we find $U/2\subset\overline{C+\Sigma}$. Since C is closed and Σ compact, we obtain $\operatorname{int}(C+\Sigma)\neq\emptyset$. $\square$

Lemma 2.12. *In Theorem* 2.5*, the following relation holds:* (vii)$\Leftrightarrow$(ix).

Proof. The set $M:=\operatorname{span}C$ is a finite codimensional closed subspace. We can write $M+N=X$ with $M\cap N=\{0\}$ and the null space $N:=\mathcal{N}(Q)$ finite dimensional. The result follows since X and $M\times N$ are isomorphic. $\square$

Remark 2.13. Note that we only need the projection to be continuous with finite dimensional null space and $\operatorname{int}(Q(C))$ nonempty. Indeed, Q is open is always true for a projection (at least in a normed space) and all projections are linear.

Next we show the particular part involving Baire spaces.

Lemma 2.14. *In Theorem* 2.5*, the following relations hold:*
(iv)$\Rightarrow$(x)$\Rightarrow$(ii).

Proof. Assume Property (iv) holds. Call $N := \operatorname{span}\Sigma$ the finite dimensional space spanned by Σ. Let $Y = X/N$ be the quotient space, and define $Qx := x + N$. Then N, the nullspace of Q, is finite dimensional. In addition, $Q(C) = C + N$ contains $C + \Sigma$. Hence it has nonempty interior.

Note that [39, p. 60, Proposition 3] shows that Q is open. So to end the proof, we show that Property (x) implies Property (ii).

Assume (x) holds and X is a Baire space. Name $N := \mathcal{N}(Q)$ and $B_r := \{x \in X : \|x\| \leq r\}$ the closed ball of radius r in X. The set $C_r := C + N \cap B_r$ is closed (since it is the sum of a closed set and a compact set). The set $Q(C + N) = Q(C)$ has nonempty interior in Y, i.e. , there is an open set V such that $V \subset Q(C + N)$. We deduce that $Q^{-1}(V) \subset Q^{-1}(Q(C + N))$. The continuity of Q implies that $Q^{-1}(V)$ is open and its linearity implies $Q^{-1}(Q(C + N)) = C + N$. Indeed take $x \in Q^{-1}(Q(C + N))$, there is $c \in C$ such that $Q(x) = Q(c)$. Hence $x = (x - c) + c \in N + C$; the reverse inclusion is obvious. Consequently $C + N$ has nonempty interior. So there is an open set U contained in $C + N = \cup_{r=1}^{\infty} C_r$. We deduce that there is r_0 with $\operatorname{int}(C_{r_0})$ nonempty. Therefore Property (ii) holds with $K = N \cap B_{r_0}$. □

Next we prove the part involving finite codimensional spaces.

2.3. Proof of the main theorem: finite codimension property

Lemma 2.15. *The following relation holds in Theorem* 2.5*:* (iv)⇒(vii)*.*

Proof. Without loss of generality, we can assume $0 \in C$. There are $\Sigma_N := \sum_{k=1}^{N} [-1, 1]e_k$ with $e_1, \ldots, e_N$ linearly independent and U an open neighborhood of 0 with $U \subset C + \Sigma_N$.

Step 1: The subspace $\operatorname{span} C$ is finite codimensional and closed.

Take $e \notin \operatorname{span} C$ and assume $C + [-1, 1]e$ has interior. We are going to prove $\operatorname{span} C$ is closed. For contradiction we suppose the open unit ball $B[0; 1]$ is contained in $C + [-1, 1]e$. Take $r > \|e\|$. If $\operatorname{span} C$ is not closed, it is dense. Since $e \notin \operatorname{span} C$ there is a sequence $x_n \in \operatorname{span} C$ with $x_n \to e \in B[0; r]$. Eventually $x_n \in B[0; r] \subset rC + [-r, r]e$ and so $x_n \in rC$ for n large enough. Using the closedness of C, we obtain the contradiction: $e \in rC \subset \operatorname{span} C$.

Now build $C_0 := C$, $C_k := C_{k-1} + [-1, 1]e_k$. All C_k have Property (vi) (since $C \subset C_k$ and that property is preserved under inclusion). Hence the previous argument shows C_{k-1} is closed in $\operatorname{span} C_k$ for $k = 1, \ldots, N$. Since $\operatorname{span} C_N = X$ (see Remark 2.7), all $\operatorname{span} C_k$ are closed. In particular, $\operatorname{span} C_0 = \operatorname{span} C$ is closed.

Since $C_N = C + \Sigma_N$ has nonempty interior, $\operatorname{span} C_N = X$ is finite codimensional. Moreover $\operatorname{span}(C_{k-1} + [-1, 1]e_k) = \operatorname{span} C_k$ yields $\operatorname{codim} C_{k-1} \leq \operatorname{codim} C_k < \infty$. In particular, $\operatorname{span} C_0 = \operatorname{span} C$ is finite codimensional which ends Step 1.

Step 2: The relative interior of C, $\operatorname{ri} C := \operatorname{int}_{\operatorname{span} C} C$ is nonempty.

To prove the claim we use the two properties: $0 \leq \alpha \leq \beta \Rightarrow \alpha C \subset \beta C$ (since $\alpha/\beta C + (1 - \alpha/\beta)0 \subset C$), and $\alpha, \beta \geq 0 \Rightarrow (\alpha + \beta)C = \alpha C + \beta C$ (another use of convexity).

Note $M := \operatorname{span} C = \operatorname{cone} C - \operatorname{cone} C$, $E_1 := \sum_{e_i \in M} [-1, 1]e_i$, and $E_2 := \sum_{e_i \notin M} [-1, 1]e_i$.

There is an open set $U \subset C + E_1 + E_2$ which satisfies

$$0 \in U \cap M \subset (C + E_1 + E_2) \cap M \subset C + E_1.$$

For $e_i \in M = \operatorname{cone} C - \operatorname{cone} C$, there are $\alpha_i^1, \alpha_i^2 \geq 0$ and $c_i^1, c_i^2 \in C$ such that $e_i = \alpha_i^1 c_i^1 - \alpha_i^2 c_i^2$. So any $u \in U \cap M \subset C + E_1$ can be written with $\lambda_i \in [-1, 1]$ as

$$\begin{aligned} u &= c + \sum_{i \in I} \lambda_i(\alpha_i^1 c_i^1 - \alpha_i^2 c_i^2), \\ &= c + \sum_{i \in I^+} (\lambda_i \alpha_i^1 c_i^1 + (1 - \lambda_i)\alpha_i^2 c_i^2 + \alpha_i^1 c_i^1) + \\ &\quad \sum_{i \in I^-} (-\lambda_i \alpha_i^2 c_i^2 + (1 + \lambda_i)\alpha_i^1 c_i^1 + \alpha_i^2 c_i^2) - \bar{x}, \end{aligned}$$

where $I^+ := \{i \in I \ : \ \lambda_i \in [0, 1]\}$, $I^- := \{i \in I \ : \ \lambda_i \in [-1, 0]\}$, $I := \{i \ : \ e_i \in M\}$, $c \in C$ and $\bar{x} := \sum_{i \in I} (\alpha_i^1 c_i^1 + \alpha_i^2 c_i^2)$. So

$$U \cap \operatorname{span} C \subset [1 + 2\sum_i (\alpha_i^1 + \alpha_i^2)]C - \bar{x}$$

which implies $\operatorname{ri} C$ is not empty. □

Lemma 2.16. *The following relation holds in Theorem* 2.5*:* (vii)⇒(iv).

Proof. Without loss of generality, we can assume $0 \in \operatorname{int}_{\operatorname{span} C} C$. Call $X_0 := \operatorname{span} C$ and $X_1 := X_0 + \mathbb{R} e_1$ with $X = X_0 + \operatorname{span}\{e_1, \ldots, e_N\}$ with $\|e_i\| = 1$. There is a ball B_0 with radius r_0 centered at 0 with $B_0 \cap X_0 \subset C$. We claim that there is another ball $B_1 \subset B_0$ with $B_1 \cap X_1 \subset C + [-1, 1]e_1$. Indeed, take y in $B_0 \cap X_1$ and not in X_0, then $y = \hat{c} + \alpha e_1$ for some $\hat{c}$ in C. Consider the linear functional $u : X_1 \to \mathbb{R}$ defined for all $x = c + \beta e_1$ by $u(x) = \beta$. Since $X_0 = \mathcal{N}(u)$ is closed, u is continuous [39, p. 382, Proposition 4]. Therefore $u(B_0)$ is bounded and hence there is r_1, $0 < r_1 < r_0$ such that for the ball B_1 centered at 0 and with radius r_1 one has $u(B_1) \subset [-1, 1]$, and for $y \in B_1 \cap X_1$ not in X_0, $y \in C + [-1, 1]e_1$. Moreover $B_1 \cap X_0 \subset B_0 \cap X_0 \subset C$. All in all, $B_1 \cap X_1 \subset C + [-1, 1]e_1$.

Now consider $X_2 := X_1 + \mathbb{R} e_2$. Applying the same argument gives the existence of a ball B_2 with $B_2 \cap X_2 \subset C + [-1, 1]e_1 + [-1, 1]e_2$. Consequently there is B_N in $X_N = X$ with $B_N \subset C + \sum_{i=1}^N [-1, 1]e_i$ which means (iv). □

Lemma 2.17. *The following relations hold in Theorem* 2.5*:*
(vii) *and* (iv)⇒(viii).

Proof. We can always assume $0 \in \operatorname{ri} C$.

Take $x = 0 \in C$, N_x a convex neighborhood of x such that $N_x \cap C - N_x \cap C \subset C$ ($0 \in \operatorname{ri} C$), $\epsilon := 1$, $0 < \lambda < \epsilon$, and $\Omega := x - C \cap N_x$. Then Ω is convex and

$$C \cap N_x + \lambda\Omega \ = C \cap N_x - \lambda C \cap N_x \ \subset C \cap N_x - C \cap N_x \subset C$$

since $\lambda(C \cap N_x) \subset C \cap N_x$ because $C \cap N_x$ is a convex set containing zero.

Taking (i)⇒(vi) and the definition of ELL into account, it remains to be proved that Ω is CEL. But there is $K := \sum_{i=1}^{N}[-1,1]e_i$ compact convex and an open ball B centered at 0 with $B \subset C + K$ which yields $B \subset B \cap C + K \subset C + K$ (same argument as in the previous lemma, we heavily use the closedness of $\operatorname{span} C$). Hence $B \cap C$ is CEL, which ends the proof. □

2.4. Proof of the main theorem: quasi-relative interior

Using quasi-relative interiors [5], we give an alternative proof of (iv)⇒(vii) under the additional assumption that C has nonempty quasi-relative interior. Even if the assumption is stronger, it clarifies the relation with quasi-relative interiors. Note that the proof uses Theorem 3.2 whose second proof does not depend on Theorem 2.5.

Definition 2.18. A point x is in the *quasi-relative interior* of a convex set C if for all nonzero continuous linear functional λ:

$$\lambda(C - x) \geq 0 \Rightarrow \lambda(C - x) = 0.$$

The set of quasi-relative interior points is denoted by $\operatorname{qri} C$.

In other words, x is a quasi-relative interior point of C if and only if $T_C(x)$, the tangent cone to C at x, is a subspace. In $\mathbb{R}^n$, the quasi-relative interior of a convex set is its relative interior.

If X is a separable Banach space and C a closed convex set, Borwein and Lewis [5] proved that C has a nonempty quasi-relative interior. So the next proposition applies in separable Banach spaces.

Proposition 2.19. *Assume C is a CEL closed convex set with nonempty quasi-relative interior. Then there is a finite codimensional closed subspace M such that $C \subset M$ and $\operatorname{int}_M(C)$ is nonempty.*

Proof. If 0 does not belong to C, it belongs to $C - \bar{c}$ where $\bar{c} \in C$. Since $\operatorname{qri}(C - \bar{c}) = \operatorname{qri}(C) - \bar{c}$ and $C - \bar{c}$ is closed convex CEL, we apply the proposition to $C - \bar{c}$. So we can always assume $0 \in C$.

Proposition 2.10 and Lemma 2.11 give (v): there is a finite dimensional subspace Σ_0 and a nonempty open set U with $U \subset C + \Sigma_0$. We name $\hat{c}$ a point in $\operatorname{qri} C$.

If $\operatorname{int}(C)$ is nonempty, the whole space $M := X$ satisfies the theorem: $U \cap X \subset C \subset X$ and $\operatorname{codim} X = 0$. Otherwise $\operatorname{int}(C) = \emptyset$, so $\hat{c}$ belongs to $\operatorname{bd} C$, the boundary of C. Applying Theorem 3.2, we deduce that $\hat{c}$ is a support point of C. So there is a nonzero continuous linear functional $\lambda_0 \in X^*$ such that $\lambda_0(C - \hat{c}) \geq 0$. Since $\hat{c}$ is in $\operatorname{qri} C$ and $0 \in C$, we obtain $\lambda_0(C) = 0$, i.e. , C is in the nullspace $M_1 := \mathcal{N}(()\lambda_0)$ of λ_0. Naming $\Sigma_1 := \Sigma_0 \cap M_1$ gives

$$U \cap M_1 \subset C + \Sigma_1 \subset M_1.$$

Now either the proposition holds with $M = M_1$ or we can build inductively a sequence of finite codimensional closed subspaces M_k and a sequence of finite dimensional subspaces Σ_k with $\operatorname{codim} M_k = k$ and $\dim \Sigma_k < \dim \Sigma_{k-1}$ (in fact $\dim \Sigma_k + 1 = \dim \Sigma_{k-1}$) such that

$$U \cap M_k \subset C + \Sigma_k \subset M_k.$$

Indeed, that property holds for $k = 1$. Suppose it holds for k, then either $\mathrm{int}_{M_k}(C)$ is nonempty so the theorem holds, or $\mathrm{int}_{M_k}(C)$ is empty and $\hat{c}$ belongs to $\mathrm{bd}\, C \subset \mathrm{supp}\, C \subset M_k$. Hence there is a nonzero continuous linear functional $\lambda_k \in (M_k)^*$ such that $\lambda_k(C - \hat{c}) \geq 0$. We apply the Hahn-Banach theorem [39, p. 77, Theorem 1] to extend λ_k to all X. The same argument as above gives $\lambda_k(C) = 0$. We define $M_{k+1} := N(\lambda_k)$ and $\Sigma_{k+1} := \Sigma_k \cap M_{k+1}$ to obtain

$$U \cap M_{k+1} \subset C + \Sigma_{k+1} \subset M_{k+1}$$

with $\dim \Sigma_{k+1} < \dim \Sigma_k$. Consequently both sequences exist.

To conclude, note that $n := \dim \Sigma_0$ is finite implies $\dim \Sigma_n = 0$. So $\Sigma_n = \{0\}$ and $\mathrm{codim}\, M_n = n$. Consequently the proposition holds with $M = M_n$. □

Remark 2.20. The assumption "C has nonempty quasi-relative interior" (which is true if the space is Banach and separable) is needed in the proof of the proposition.

Because of this assumption, the proof above does not cover all cases since the quasi-relative interior is empty for sets like $S := l_p^+(\mathbb{R})$ with $1 \leq p < \infty$ (the positive cone in $l_p(\mathbb{R}) := \{s : \mathbb{R} \to \mathbb{R} \;:\; \sum_{r\in\mathbb{R}} |s(r)|^p < \infty\}$). Indeed recall that $\sum_{r\in\mathbb{R}} |s(r)|^p = \sup_{F \text{ finite}} \sum_{r\in\mathbb{R}\cap F} |s(r)|$ and that the sum being finite implies the support of s is countable in S. Take any $\bar{s}$ in $l_p^+(\mathbb{R})$. Since the support of $\bar{s}$ is countable, there is $\bar{r}$ in $\mathbb{R}$ with $\bar{s}(\bar{r}) = 0$. Define the linear functional $f(s) := s(\bar{r})$. Using $f(\bar{s}) = \bar{s}(\bar{r}) = 0$ and $f(S) = \{s(\bar{r}) \;:\; s \in S\} \geq 0$ yields $\langle f, S - \bar{s}\rangle \geq 0$. Since there is $\hat{s}$ with $f(\hat{s}) = 1$, f is not always 0 on S. Hence $\bar{s}$ is a support point of S and $T_S(\bar{s})$ is not a subspace. All in all, $\mathrm{qri}\, S$ is empty.

3. Supporting properties of compactly epi-Lipschitzian sets

Starting from our main characterization theorem, we study supporting properties of CEL sets. An alternate proof of the main theorem in that section is presented for CEL sets in Banach spaces. The extension to nonconvex sets is then discussed.

3.1. Convex sets in normed spaces

In addition to characterizing CEL closed convex sets, we show that they have "nice" boundaries. Indeed, our next theorem will show that every boundary point of such a set is also a support point. First we need a technical lemma.

Lemma 3.1. *Let X be a normed linear space, K be a convex compact set, and C be a closed convex set. If $\bar{c}$ is in $\mathrm{bd}\, C$, the boundary of C, then there is $\bar{k}$ in $\mathrm{bd}\, K$ such that $\bar{c} + \bar{k} \in \mathrm{bd}(C + K)$.*

Proof. Take $\bar{c}$ in $\mathrm{bd}\, C$. Suppose, to the contrary, that for all $\bar{k}$ in $\mathrm{bd}\, K$, $\bar{c} + \bar{k}$ does not belong to $\mathrm{bd}(C + K)$. Then it must belong to $\mathrm{int}\,(C + K)$. Since K has empty interior (X has infinite dimension) the inclusion holds for all $k \in K$. Since K is compact, there is a positive ϵ such that

$$\bar{c} + K + \epsilon B \subset C + K \tag{3.1}$$

Now applying the Rådström cancellation principle [33] gives $\bar{c} + \epsilon B \subset C$ which contradicts the fact that $\bar{c}$ belongs to $\mathrm{bd}\, C$. □

Here is the announced theorem. Recall that a proper support point means that C is supported by an hyperplane and C is not included in the hyperplane. We will need such a proper hyperplane in the proof of Lemma 5.1.

Theorem 3.2. *Suppose X is a normed space and C a CEL closed convex set. Then every boundary point of C is also a proper support point of C.*

Our proof relies on the two classical separation theorems: with nonempty interior and in finite dimensions. See the next remark for an alternate proof and links to the Bishop-Phelps theorem.

Proof. Take $\bar{c}$ in the boundary of C. First suppose $\operatorname{int}(C)$ is nonempty. Then we can separate C from $\{\bar{c}\}$ (with the Hahn–Banach theorem for example): there is a nonzero continuous linear functional l such that $l(\bar{c}) = \sup_{c\in C} l(c)$ which means that $\bar{c}$ is a support point of C.

Now assume $\operatorname{int}(C)$ is empty. Theorem 2.5 implies C has nonempty relative interior. So we can apply the above argument to obtain a nonzero continuous linear functional l, defined on $\operatorname{span} C$, with $l(\bar{c}) = \sup_{c\in C} l(c)$. Now $\operatorname{span} C$ being finite codimensional allows us to extend l to a continuous linear functional $\hat{l}$ defined on all X. For example define $\hat{l}(x) := l(\operatorname{Proj}_{\operatorname{span} C}(x))$, where $\operatorname{Proj}_{\operatorname{span} C}$ is the projection onto $\operatorname{span} C$. So $\bar{c}$ is a proper support point of C. $\square$

Note that every point in C is a support point of C. Indeed if $\bar{c} \in C \setminus \operatorname{ri} C$ the above proof shows $\bar{c}$ is a proper support point. Otherwise $\bar{c} \in \operatorname{ri} C$. Write $X = \operatorname{span} C \oplus \mathbb{R}e \oplus Y$. Fix $x = c + re + y$ and take $l(x) = r$. Then $l(C) = 0$ and hence $\bar{c}$ is a support point of C.

Remark 3.3. Theorem 3.2 can be obtained from Lemma 4 in [2] in the case of Banach spaces based on our characterizations of CEL sets.

Remark 3.4. Theorem 3.2 does not always hold for non-CEL closed convex sets. Indeed, Fonf [13] proved that in every incomplete normed space there is a closed bounded convex set C with no support point. Theorem 3.2 tells us that such a set C cannot be CEL.

A more striking case when $\operatorname{bd} C$ equals $\operatorname{supp} C$ (the set of support points of C) is provided by the following example in which $C(S)$ could be ℓ_∞.

Example 3.5. Let S be a compact Hausdorff space, and $F \subset S$ be a closed non G_δ set. Then $C := \{f \in C(S) \ : \ f \geq 0 \text{ and } f(x) = 0 \text{ for } x \in F\}$ verifies

$$\operatorname{bd} C = C = \operatorname{supp} C.$$

We end that subsection with a way to generate CEL closed convex sets. Indeed, a consequence of the next proposition is that in any normed linear space, projections of CEL closed convex sets are CEL: note that any closed convex set containing a CEL closed convex set is CEL. Note also that the closed convex hull of a CEL set is necessarily CEL by Proposition 2.10.

Proposition 3.6. *Assume C is CEL, $Q : X \to Y$ is an open linear continuous map, and X, Y are normed linear spaces. Then $Q(C)$ is CEL.*

Proof. There is a convex compact set K and an open set U with $U \subset C + K$. Thus, $Q(U) \subset Q(C) + Q(K)$. Now since Q is open, $Q(U)$ is open and since Q is continuous, $Q(K)$ is compact. Hence, we found a compact set $K' = Q(K)$ with $\operatorname{int}(Q(C) + K')$ nonempty, which means that $Q(C)$ is CEL. □

3.2. Variational arguments in Banach spaces

Let us present another proof of Theorem 3.2 in the case of Banach spaces X, i.e. , under an additional completeness assumption in the theorem. This proof is fully independent of Theorem 2.5: it is actually based on variational arguments and admits generalizations to nonconvex sets; see the next subsection. For the case of convex closed sets C under consideration we use the classical Bishop-Phelps theorem on the density of support points in the boundary of C; see, e.g., [32, Theorem 3.18].

Take any $\bar{c}$ in the boundary of C. According to the Bishop-Phelps theorem, we find sequences $\{c_n\} \subset X$ and $\{\xi_n\} \in X^*$ satisfying $c_n \to \bar{c}$ as $n \to \infty$, $\|\xi_n\| = 1$, and

$$\xi_n \in N(c_n; C) := \{\xi \in X^* |\ \langle \xi, c - c_n \rangle \leq 0\} \tag{3.2}$$

for all $n = 1, 2, \ldots$, where $N(\cdot; C)$ signifies the normal cone of convex analysis. Since C is CEL, we can apply Loewen's result in [23, Proposition 3.7] to conclude that there exist a compact set $S \subset X$, a neighborhood U of $\bar{c}$, and a number $\gamma > 0$ such that

$$N(c; C) \subset K_\gamma(S) := \{\xi \in X^* |\ \gamma \|\xi\| \leq \max_{s \in S} |\langle \xi, s \rangle|\} \tag{3.3}$$

for all $c \in C \cap U$. Note that the mentioned result of [23] covers the general case of nonconvex sets $C \subset X$ where $N(c; C)$ in (3.3) is replaced with the so-called *Fréchet normal cone* to C at $c \in C$ defined by

$$\widehat{N}(c; C) := \{\xi \in X^* |\ \limsup_{x \to c,\, x \in C} \frac{\langle \xi, x - c \rangle}{\|x - c\|} \leq 0\}. \tag{3.4}$$

It is well known that constructions (3.2) and (3.4) agree for convex sets.

Due to the weak* compactness of the unit ball in the dual space X^* we select a subnet $\xi_\nu \in N(c_\nu; C)$ in (3.2) which weakly* converges to some $\bar{\xi} \in X^*$. Passing to the limit in (3.2), we easily get that $\langle \bar{\xi}, c - \bar{c} \rangle \leq 0$.

It remains to prove that $\|\bar{\xi}\| \neq 0$. Assume on the contrary that $\bar{\xi} = 0$. Using the compactness of the set $S \subset X$ in (3.3), we conclude that $\langle \xi_\nu, s \rangle \to 0$ uniformly in S. Thus (3.3) implies that $\xi_\nu \to 0$ in the *norm* topology of X^*. But this is impossible due to $\|\xi_\nu\| = 1$ for all ν. The obtained contradiction completes the proof of Theorem 3.2 in Banach spaces.

3.3. Nonconvex generalizations

The above arguments can be extended to the nonconvex case using variational principles and appropriate concepts of normal cones in nonsmooth analysis. If X is an *Asplund space* (that is, a Banach space where every convex continuous function is generically Fréchet differentiable, in particular, any reflexive space; see [32]), then a proper analogue of the

Bishop-Phelps theorem is obtained by Mordukhovich and Shao [27] via the density of the set

$$c \in \text{bd } C \quad \text{with} \quad \widehat{N}(c; C) \neq \{0\} \tag{3.5}$$

involving the Fréchet normal cone (3.4). Moreover, the density of (3.5) for every closed set $C \subset X$ is shown to be a *characterization* of Asplund space; see [12]. Now using Loewen's result mentioned above, we conclude similarly to Subsection 3.2 that for any closed set C, CEL at $\bar{c}$, one has

$$N(\bar{c}; C) \neq \{0\} \quad \text{at every} \quad \bar{c} \in \text{bd } C \tag{3.6}$$

in terms of the *limiting normal cone*

$$N(\bar{c}; C) := \limsup_{c \to \bar{c},\ c \in C} \widehat{N}(c; C) \tag{3.7}$$

introduced in [22] as an extension of the finite dimensional construction of Mordukhovich [24]. In (3.7) "limsup" connotes the *sequential* Painlevé-Kuratowski upper limit of multifunctions with respect to the norm topology in X and the weak* topology in X^*. Note that we can use the sequential vs. topological (net) upper limit in (3.6) and (3.7) since a bounded set in X^* is weakly* sequentially compact for any Asplund space X; see [32].

When C is convex, the normal cone (3.7) reduces to the normal cone of convex analysis. Thus (3.6) can be viewed as an extention of our support point theorem (Theorem 3.2) to nonconvex CEL sets in Asplund spaces. Note that in this form it does not hold outside of Asplund spaces. In fact it was shown by Fabian and Mordukhovich [12] that in any non-Asplund space X there is an *epi-Lipschitzian* set $C \subset X$ for which (3.6) is violated at *every* boundary point. To cover the case of arbitrary Banach spaces, one needs to use a different normal cone for nonconvex sets.

An appropriate construction was introduced by Ioffe under the name of the (approximate) *G-normal cone* denoted by $N_G(\cdot; C)$, see [15]. This construction is another infinite dimensional extension of [24] being generally more complicated than (3.6). It agrees with (3.6) in certain most important situations but may be bigger (never smaller) than (3.6) even for epi-Lipschitzian sets in spaces with Fréchet smooth renorms; see [29, Section 9] for more details and further references.

Similarly to the arguments in [27] one can show that the density result (3.5) holds in terms of $N_G(\cdot; C)$ for any closed set C in a Banach space X. Then we need to pass to the limit and get an analogue of (3.6) in terms of the G-normal cone for any closed CEL set. To make it possible, we can use the result of Jourani and Thibault [19, Lemma 3] that ensures a local compactness property of type (3.3) for the G-normal cone under the CEL assumption on C at $\bar{c}$. This justifies the N_G-analogue of Theorem 3.2 for CEL sets in arbitrary Banach spaces.

Note that density (properness) results of type (3.5) where obtained by Borwein and Strojwas [7, 8] for various normal cones dual to some tangent cones in Banach spaces. However, for arbitrary CEL sets those normal cones may not possess a local compactness property of type (3.3), called "normal compactness" in [29], that is crucial for the limiting procedure. In particular, the Clarke normal cone is not locally compact unless C is ELL; see [3]. The normal compactness property with the Clarke normal cone is shown to be satisfied for ELL sets in [20]. Ioffe recently proved [16] that the CEL property of $C \subset X$ at $\bar{c}$ is actually *equivalent* to property (3.3), if $N(\cdot; C)$ is either the Fréchet normal cone on an Asplund space X or the G-normal cone on an arbitrary Banach space. It follows from Borwein [3, Example 4.1] that this result does not hold for the Clarke normal cone in $X = l_\infty$.

4. Characterizations of compactly epi-Lipschitzian Convex Functions

The analogous of our main theorem for CEL sets is presented for CEL closed convex functions. As usual the link between functions and sets is provided by the epigraph

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} \ : \ f(x) \leq r\}.$$

We say that a closed convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is CEL if its epigraph is CEL. As for sets, we can always assume $f(0) = 0$.

The next theorem characterizes CEL convex functions. The notation δ_K denotes the indicator function of the set K: $\delta_K(x) = 0$ if $x \in K$, $+\infty$ otherwise. We denote by f^* the Legendre–Fenchel conjugate of f (see [14, 35])

$$f^*(s) := \sup_{x \in X} [\langle s, x \rangle - f(x)],$$

and $f \Box g$ the inf-convolution of f and g

$$f \Box g(x) := \inf_{y \in X} [f(y) + g(x - y)].$$

The core denotes the algebraic interior of a set

$$\operatorname{core} C := \{x \in C \ : \ \forall d \in X, \exists T > 0 \ : \ |t| \leq T \Rightarrow x + td \in C\}.$$

Theorem 4.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper closed convex function. The following are equivalent:*

(i) *The function f is CEL.*
(ii) *There is a convex compact set K such that $f \Box \delta_K$ is continuous at 0.*
(iii) *There is a convex compact set K such that $f^* + \delta_K^*$ has bounded level sets.*
(iv) *There is a convex compact set K such that* $0 \in \operatorname{core}(\operatorname{dom} f \Box \delta_K)$.

To prove the theorem, we need several basic steps. First we recall the following well-known result. Its simple proof is included for the sake of self-containedness.

Lemma 4.2. *Take $\sigma : X \to \mathbb{R} \cup \{+\infty\}$.*

- *If* $0 \in \operatorname{int}(\operatorname{epi} \sigma)$*, then σ is bounded on some neighborhood of* 0.
- *If σ is finite and continuous at* 0*, then* $\operatorname{int}(\operatorname{epi} \sigma)$ *is nonempty.*

Proof. There is an open ball B of center 0 and radius r with $0 \in B \subset \operatorname{epi} \sigma$. If there is a sequence x_n converging to 0 with $\sigma(x_n) = \infty$, then for n large enough, $(x_n, \sigma(x_n)) \in B$, i.e. , $\|x_n\| + |\sigma(x_n)| < r$ which contradicts $\sigma(x_n) = \infty$. This proves the first part of the lemma.

Next, assume σ is (finite and) continuous at 0. Set $I := (\sigma(0)-1, \sigma(0)+1)$. Then

$$\begin{aligned}(0,2)+\sigma^{-1}(I)\times I &= \sigma^{-1}(I)\times(\sigma(0)+1,\sigma(0)+3)\\ &\subset \{(x,\alpha)\ :\ \sigma(x)<\alpha\}\subset \operatorname{epi}\sigma.\end{aligned}$$

Since $\sigma^{-1}(I)$ is open, $\operatorname{int}(\operatorname{epi}\sigma)$ is nonempty. □

Remark 4.3. Note that even if a function is continuous CEL convex it may not send bounded sets to bounded sets.

Indeed take $X = l_2$ and $f(x) = \sum_{n=1}^{\infty}|x_n|^{2n}$. Then f is convex and continuous (since it is lower semi-continuous and finite in a Banach space). Moreover f is CEL since its epigraph spans $l_2\times\mathbb{R}$ (which is clearly closed and of finite codimension), and has nonempty interior (by the previous lemma since f is continuous at 0). However, $f(2e_n) = 2^{2n}$ implies that if we denote B the unit ball, $f(2B)$ is unbounded.

So f is a CEL convex continuous function that does not send bounded sets to bounded sets.

Since the strict epigraph of the inf-convolution is the sum of the strict epigraphs (see Remark [14, IV.2.3.3]) we need to relate compact epi-Lipschitzness to strict epigraphs. Recall that

$$\operatorname{epi}_s f := \{(x,r)\in X\times\mathbb{R}\ :\ f(x)<r\}.$$

Lemma 4.4. *If f is a proper convex function,* $\operatorname{span}(\operatorname{epi} f) = \operatorname{span}(\operatorname{epi}_s f)$, *and* $\operatorname{ri}(\operatorname{epi} f) = \operatorname{ri}(\operatorname{epi}_s f)$. *In particular,* $\operatorname{epi} f$ *is CEL if and only if* $\operatorname{epi}_s f$ *is CEL.*

Proof. The second assertion clearly follows by using Theorem 2.5(vii).

Step 1: $\operatorname{span}(\operatorname{epi} f) = \operatorname{span}(\operatorname{epi}_s f)$.

Take y in $\operatorname{span}(\operatorname{epi} f)$. There are $\alpha_i \ge 0$, $x_i \in X$, and $r_i \in \mathbb{R}$ with $f(x_i) \le r_i$ for $i = 1,2$ and $y = \alpha_1(x_1,r_1) - \alpha_2(x_2,r_2)$. Since without loss of generality we can assume $(0,0)\in\operatorname{epi} f$, the points $(0,a)$ are in $\operatorname{span}(\operatorname{epi} f)$ for any $a\in\mathbb{R}$. (The convexity allows to write $\operatorname{span}(\operatorname{epi} f) = \operatorname{cone}(\operatorname{epi} f) - \operatorname{cone}(\operatorname{epi} f)$).

Either $\alpha_2 = 0$ and we can write

$$y = \alpha_1(x_1,r_1) = \alpha_1(x_1,r_1+1) - (0,\alpha_1)$$

so y belongs to $\operatorname{span}(\operatorname{epi}_s f)$. Or $\alpha_2 \neq 0$ and we write

$$y = \alpha_1(x_1,r_1) - \alpha_2(x_2,r_2) = \alpha_1(x_1,r_1+1) - \alpha_2(x_2, r_2 + \frac{\alpha_1}{\alpha_2}).$$

So y is again in $\operatorname{span}(\operatorname{epi}_s f)$. The reverse inclusion is obvious.

Step 2: $\operatorname{ri}(\operatorname{epi} f) = \operatorname{ri}(\operatorname{epi}_s f)$.

Take $y\in\operatorname{ri}(\operatorname{epi} f)$, and note $M := \operatorname{span}(\operatorname{epi} f) = \operatorname{span}(\operatorname{epi}_s f)$, and $B(y,\delta)$ the open ball of center y and radius δ. There is $\delta > 0$ such that $B(y,\delta)\cap M \subset \operatorname{epi} f$. Take $u := (x_u, r_u)\in B(y,\delta/2)\cap M$ and consider $u' := u-(0,\delta/3)$. Using the triangle inequality we deduce $u'\in B(y,\delta)$. Since u' is also in M, it is in $\operatorname{epi} f$. Hence u is in $\operatorname{epi}_s f$. We conclude that $B(y,\delta/2)\cap M\subset \operatorname{epi}_s f$. The reverse inclusion is clear. □

Now we prove Theorem 4.1.

Proof. Step 1: (ii) implies (i).

Assume (ii). Apply Lemma 4.2 and Lemma 4.4 to obtain

$$\emptyset \neq \operatorname{int}(\operatorname{epi}(f \Box \delta_K)) = \operatorname{int}(\operatorname{epi}_s(f \Box \delta_K)) = \operatorname{int}(\operatorname{epi}_s f + \operatorname{epi}_s \delta_K).$$

Note that $\operatorname{epi}_s \delta_K \subset K \times (0,\infty)$ and $\operatorname{epi}_s f + K \times (0,\infty) \subset \operatorname{epi}_s f + K \times \{0\}$ to get $\operatorname{int}(\operatorname{epi}_s f + K') \neq \emptyset$ with $K' := K \times \{0\}$ a compact convex set. Consequently $\operatorname{epi}_s f$, and so $\operatorname{epi} f$, is CEL.

Step 2: (i) implies (ii).

Conversely, assume (i). Then there is K' a compact convex set with $0 \in \operatorname{int}(\operatorname{epi}_s f + K')$. Define $K := \operatorname{Proj}_X K'$ and $\bar{t} := \min_{(x,t)\in K'} t$. Since both projections Proj_X and $\operatorname{Proj}_{\mathbb{R}}$ are continuous and K' is compact, K is compact convex, $\bar{t}$ is well-defined and we have

$$\operatorname{epi}_s f + K' \subset \operatorname{epi}_s f + K \times \{\bar{t}\}.$$

We obtain $0 \in \operatorname{int}(\operatorname{epi}_s \tilde{f} + K \times (0,\infty))$ where $\tilde{f} := f(.) - \bar{t}$. Applying again Remark [14, IV.2.3.3] we deduce $0 \in \operatorname{int}(\operatorname{epi}_s(\tilde{f} \Box \delta_K))$. Applying Lemma 4.2, the function $\tilde{f} \Box \delta_K$ is bounded on a neighborhood of 0.

To conclude we note that $\tilde{f} \Box \delta_K$ is a convex function, so it is continuous at 0 (for example see [14, Lemma IV.3.1.1] whose proof still holds in a normed linear space). Since $\tilde{f} \Box \delta_K = f \Box \delta_K - \bar{t}$, (ii) holds.

Step 3: (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

The key result we use is a theorem proved by Moreau [31] and by Rockafellar [34] which implies that a proper convex function on a normed linear space is strongly continuous at 0 if and only if its conjugate has bounded level sets.

More precisely, [34, Theorem 7A(a)] and [34, Corollary 4D] give (ii) $\Leftrightarrow$ (iii), while [34, Theorem 4C] gives (iii)$\Leftrightarrow$(iv). $\square$

5. Applications to constrained optimization

In order to apply the Fenchel duality theorem of [5], we first prove that the relative interior is equal to the quasi-relative interior for CEL closed convex sets.

Lemma 5.1. *Assume C is CEL, closed, and convex. Then* $\operatorname{ri} C = \operatorname{qri} C$.

Proof. Without loss of generality we can take $x = 0$ to prove both inclusions.

Step 1: We show that $\operatorname{ri} C \subset \operatorname{qri} C$.

Take $x = 0 \in \operatorname{ri} C$ and let us prove $\operatorname{cl}(\mathbb{P}C) = \operatorname{span} C$. Fix a neighborhood V of zero with $V \cap \operatorname{span} C \subset C$. Then for any $\lambda > 0$ one has

$$(\lambda V) \cap \operatorname{span} C = \lambda V \cap \lambda \operatorname{span} C \subset \lambda C.$$

So $(\mathbb{P}V) \cap \operatorname{span} C \subset \mathbb{P}C$ and hence, since $\mathbb{P}V = X$ one gets $(\mathbb{P}V) \cap \operatorname{span} C = \operatorname{span} C$. So $\operatorname{span} C \subset \mathbb{P}C$. As $\mathbb{P}C \subset \operatorname{span} C$, one has $\operatorname{span} C = \mathbb{P}C$. In particular $\operatorname{span} C = \operatorname{cl}(\mathbb{P}C)$ since $\operatorname{span} C$ is closed.

Step 2: We show that $x \notin \operatorname{ri} C \Rightarrow x \notin \operatorname{qri} C$.

If $x \notin$ ri and $x \in C$ then $x \in \operatorname{bd} C$. Applying Theorem 3.2, there is a nonzero continuous linear functional λ such that $\lambda(C \setminus \{x\} - x) > 0$. Since X is infinite dimensional and C is CEL, C is not reduced to $\{x\}$. So there is $x' \in C$ with $\lambda(x' - x) > 0$. Consequently we found a nonzero continuous linear functional λ such that $\lambda(C - x) \geq 0$ but $\lambda(C - x) \neq 0$. In other words, $x \notin \operatorname{qri} C$. □

To write our next theorem, we use the same notations as in [5]: X is a normed space, $g : X \to (-\infty, \infty]$ and $h : \mathbb{R}^n \to (-\infty, \infty]$ are convex proper; and $A : X \to \mathbb{R}^n$ is a continuous linear function.

Theorem 5.2. *If g is closed convex CEL and*

$$\begin{array}{lll} \textit{either} & A(\operatorname{ri}(\operatorname{dom} g)) \cap \operatorname{ri}(\operatorname{dom} h) \neq \emptyset & \\ \textit{or} & A(\operatorname{ri}(\operatorname{dom} g)) \cap \operatorname{dom} h \neq \emptyset & \textit{and } h \textit{ is polyhedral,} \end{array}$$

then

$$\inf\{g(x) + h(Ax) \; : \; x \in X\} = \max\{-g^*(A^T\lambda) - h^*(-\lambda) \; : \; \lambda \in \mathbb{R}^n\}.$$

Proof. If the function g is CEL convex, then its domain is also CEL by Proposition 3.6. So Lemma 5.1 gives $\operatorname{qri}(\operatorname{dom} g) = \operatorname{ri}(\operatorname{dom} g)$. Applying Fenchel duality theorem [5, Corollary 4.3] gives the same generalized Slater condition as in finite dimension: we need to find a point in $\operatorname{ri}(\operatorname{dom} g)$ with image by A in $\operatorname{ri}(\operatorname{dom} h)$. □

So the CEL property ensures sufficient amount of compactness to recover the finite dimensional results (see our discussions in Section 1 and in Subsections 3.2 and 3.3).

The subsequent sum rule and minimax theorem have been stated in [5]. In fact, Ye used our main characterization theorem (Theorem 2.5) to apply a sum rule in her proof of necessary conditions for optimal control of strongly monotone variational inequalities [42].

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