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PSEUDO-CATEGORIES

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(*communicated by Ross Street*)

Abstract

We provide a complete description of the category of pseudo-categories (including pseudo-functors, natural and pseudo-natural transformations and pseudo modifications). A pseudo-category is a non strict version of an internal category. It was called a weak category and weak double category in some earlier papers. When internal to Cat it is at the same time a generalization of a bicategory and a double category. The category of pseudo-categories is a kind of "tetracategory" and it turns out to be cartesian closed in a suitable sense.

1. Introduction

The notion of pseudo-category[1] considered in this paper is closely related and essentially is a special case of several higher categorical structures studied for example by Grandis and Paré [8], Leinster [3], Street [11],[12], among several others. We have arrived to the present definition of pseudo-category (which some authors would probably call a pseudo double category) while describing internal bicategories in Ab [5]. We even found it easier, for our particular purposes, to work with pseudo-categories than to work with bicategories. Defining a pseudo-category we begin with a 2-category, take the definition of an internal category there, and replace the equalities in the associativity and identity axioms by the existence of suitable isomorphisms which then have to satisfy some coherence conditions. That is, let **C** be a 2-category, a pseudo-category in (internal to) **C** is a system

$$(C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)$$

where C_0, C_1 are objects of **C**,

$$d, c : C_1 \longrightarrow C_0\ ,\ e : C_0 \longrightarrow C_1\ ,\ m : C_1 \times_{C_0} C_1 \longrightarrow C_1$$

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[1] In the previous work [6] the word "weak" was used with the same meaning. We claim that "pseudo" is more apropriate because it is the intermediate term between precategory and internal category. Also it agrees with the notion of pseudo-functor, already well established.

are morphisms of $\mathbf{C}$, with $C_1 \times_{C_0} C_1$ the object in the pullback diagram

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ \pi_1 \downarrow & & \downarrow c \\ C_1 & \xrightarrow{d} & C_0 \end{array} \quad ;$$

$$\alpha : m\,(1_{C_1} \times_{C_0} m) \longrightarrow m\,(m \times_{C_0} 1_{C_1}) \ ,$$
$$\lambda : m\,\langle ec, 1_{C_1}\rangle \longrightarrow 1_{C_1} \ , \ \rho : m\,\langle 1_{C_1}, ed\rangle \longrightarrow 1_{C_1},$$

are 2-cells of $\mathbf{C}$ (which are isomorphisms), the following conditions are satisfied

$$de = 1_{c_0} = ce, \tag{1.1}$$

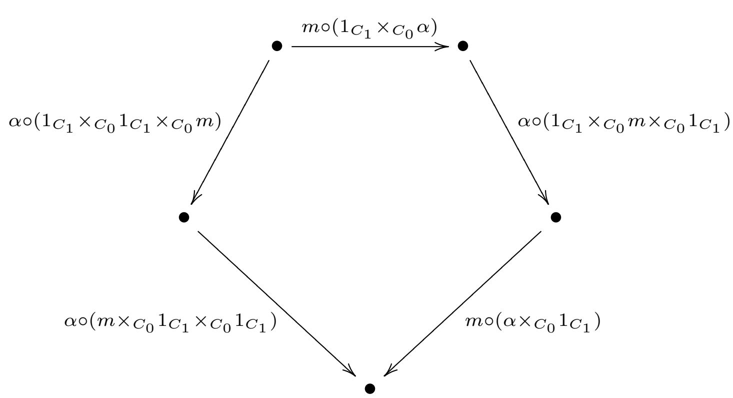
$$dm = d\pi_2 \ , \ cm = c\pi_1, \tag{1.2}$$

$$\begin{aligned} d \circ \lambda = 1_d = d \circ \rho, \\ c \circ \lambda = 1_c = c \circ \rho, \end{aligned} \tag{1.3}$$

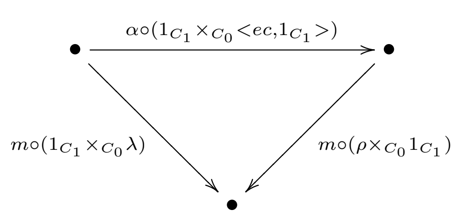
$$d \circ \alpha = 1_{d\pi_3} \ , \ c \circ \alpha = 1_{c\pi_1}, \tag{1.4}$$

$$\lambda \circ e = \rho \circ e, \tag{1.5}$$

and the following diagrams commute



(1.6)



(1.7)

Examples:

1. When **C**=Set with the discrete 2-category structure (only identity 2-cells) one obtains the definition of an ordinary category since α, λ, ρ are all identities;

2. When **C**=Set with the codiscrete 2-category structure (exactly one 2-cell for each pair of morphisms) one obtain the definition of a precategory since α, λ, ρ always exist and the coherence conditions are trivially satisfied;
(This result applies equally to any category)

3. When **C**=Grp considered as a 2-category: every group is a (one object) category and the inclusion functor

$$\mathrm{Grp} \longrightarrow \mathrm{Cat}$$

induces a 2-category structure in Grp, where a 2-cell

$$\tau : f \longrightarrow g \ \ , \ (f, g : A \longrightarrow B \text{ group homomorphisms})$$

is an element $\tau \in B$, such that for every $x \in A$,

$$g(x) = \tau f(x) \tau^{-1}.$$

With this setting, a pseudo-category in Grp is described (see [7]) by a group homomorphism

$$\partial : X \longrightarrow B,$$

an arbitrary element

$$\delta \in \ker \partial$$

and an action of B in X (denoted by $b \cdot x$ for $b \in B$ and $x \in X$) satisfying

$$\partial(b \cdot x) = b\partial(x) b^{-1}$$
$$\partial(x) \cdot x' = x + x' - x$$

for every $b \in B, \ x, x' \in X$. Note that the difference to a crossed module (description of an internal category in Grp) is that in a crossed module the element $\delta = 1$.
The pseudo-category so obtained is as follows: objects are the elements of B, arrows are pairs $(x, b) : b \longrightarrow \partial x + b$ and the composition of $(x', \partial x + b) : \partial x + b \longrightarrow \partial x' + \partial x + b$ with $(x, b) : b \longrightarrow \partial x + b$ is the pair $(x' + x - \delta + b \cdot \delta, b) : b \longrightarrow \partial x' + \partial x + b$. The isomorphism between $(0, \partial x + b) \circ (x, b) = (x, b) \circ (0, b)$ and (x, b) is the element $(\delta, 0) \in X \rtimes B$. Associativity is satisfied, since $(x'', \partial x' + \partial x + b) \circ ((x', \partial x + b) \circ (x, b)) = ((x'', \partial x' + \partial x + b) \circ (x', \partial x + b)) \circ (x, b)$.

4. When **C**=Mor(Ab) the 2-category of morphisms of abelian groups, the above definition gives a structure which is completely determined by a commutative

square

$$\begin{array}{ccc} A_1 & \xrightarrow{\partial} & A_0 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k_0} \\ B_1 & \xrightarrow{\partial'} & B_0 \end{array}$$

together with three morphisms

$$\lambda, \rho : A_0 \longrightarrow A_1,$$
$$\eta : B_0 \longrightarrow A_1,$$

satisfying conditions

$$k_1\lambda = 0 = k_1\rho,$$
$$k_1\eta = 0,$$

and it may be viewed as a structure with objects, vertical arrows, horizontal arrows and squares, in the following way (see [6], p. 409, for more details)

$$\begin{array}{ccc} b & \xrightarrow{(b,x)} & b + k_0(x) \\ \binom{b}{d}\downarrow & \begin{pmatrix} b & x \\ d & y \end{pmatrix} & \downarrow\binom{b+k_0(x)}{d+k_1(y)} \\ b + \partial'(d) & \xrightarrow[(b+\partial'(d),x+\partial(y))]{} & * \end{array},$$

where $*$ stands for $b + \partial'(d) + k_0(x + \partial(y)) = b + k_0(x) + \partial'(d + k_1(y))$.

5. When **C**=Top (with homotopy classes as 2-cells) we find the following particular example. Let X be a space and consider the following diagram

$$X^I \times_X X^I \xrightarrow{m} X^I \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} X$$

where X^I is equipped with the compact open topology and $X^I \times_X X^I$ with the product topology (I is the unit interval), with

$$X^I \times_X X^I = \{\langle g, f\rangle \mid f(0) = g(1)\}$$

and d, e, c, m defined as follows

$$d(f) = f(0)$$
$$c(f) = f(1)$$
$$e_x(t) = x$$
$$m(f,g) = \begin{cases} g(2t), t < \frac{1}{2} \\ f(2t-1), t \geqslant \frac{1}{2} \end{cases}$$

with $f, g : I \longrightarrow X$ (continuous maps) and $x \in X$. The homotopies α, λ, ρ are the usual ones.

6. When $\mathbf{C}$=Cat the objects C_0 and C_1 are (small) categories, and the morphisms d, c, e, m are functors. We denote the objects of C_0 by the first capital letters in the alphabet (possible with primes) $A, A', B, B', ...$ and the morphisms by first small letters in the alphabet $a : A \longrightarrow A', b : B \longrightarrow B', ...$. We will denote the objects of C_1 by small letters as $f, f', g, g', ...$ and the morphisms by small greek letters as $\varphi : f \longrightarrow f', \gamma : g \longrightarrow g', ...$. We will also consider that the functors d and c are defined as follows

$$\begin{array}{ccc} C_1 & & C_0 \\ & d \nearrow & a : A \longrightarrow A' \\ \varphi : f \longrightarrow f' & & \\ & c \searrow & b : B \longrightarrow B' \end{array}$$

hence, the objects of C_1 are arrows $f : A \longrightarrow B, f' : A' \longrightarrow B'$, that we will always represent using inplace notation as $A \xrightarrow{\;f\;} B$, $A' \xrightarrow{\;f'\;} B'$ to distinguish from the morphisms of C_0, and thus the morphisms of C_1 are of the form

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle a}\downarrow & \Downarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{\quad f' \quad} & B' \end{array} \; .$$

The functor e sends $a : A \longrightarrow A'$ to

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ {\scriptstyle a}\downarrow & \Downarrow{\scriptstyle id_a} & \downarrow{\scriptstyle a} \\ A' & \xrightarrow{\quad id_{A'} \quad} & A' \end{array} \; ,$$

while the functor m sends $\langle \gamma, \varphi \rangle$ to $\gamma \otimes \varphi$ as displayed in the diagram below

$$\begin{array}{ccccc} A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad g \quad} & C \\ {\scriptstyle a}\downarrow & \Downarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle b} & \Downarrow{\scriptstyle \gamma} & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{\quad f' \quad} & B' & \xrightarrow{\quad g' \quad} & C' \end{array} \longmapsto \begin{array}{ccc} A & \xrightarrow{\quad g\otimes f \quad} & C \\ {\scriptstyle a}\downarrow & \Downarrow{\scriptstyle \gamma\otimes\varphi} & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{\quad g'\otimes f' \quad} & C' \end{array} \; .$$

Each component of α is of the form

$$\begin{array}{ccc} A & \xrightarrow{\; h\otimes(g\otimes f) \;} & D \\ {\scriptstyle 1_A}\downarrow & \Downarrow{\scriptstyle \alpha_{h,g,f}} & \downarrow{\scriptstyle 1_D} \\ A & \xrightarrow{\; (h\otimes g)\otimes f \;} & D \end{array} \; ,$$

while the components of λ and ρ are given by

$$\begin{array}{ccc} A & \xrightarrow{id_B \otimes f} & B \\ {\scriptstyle 1_A}\downarrow & \Downarrow \lambda_f & \downarrow {\scriptstyle 1_B} \\ A & \xrightarrow{f} & B \end{array} \quad , \quad \begin{array}{ccc} A & \xrightarrow{f \otimes id_A} & B \\ {\scriptstyle 1_A}\downarrow & \Downarrow \rho_f & \downarrow {\scriptstyle 1_B} \\ A & \xrightarrow{f} & B \end{array} \quad .$$

Thus, a description of pseudo-category in Cat is as follows.

A pseudo-category in Cat is a structure with
- objects: $A, A', A'', B, B', ...$
- morphisms: $a : A \longrightarrow A', a' : A' \longrightarrow A'', b : B \longrightarrow B', ...$
- *pseudo-morphisms*: $A \xrightarrow{f} B$, $A' \xrightarrow{f'} B'$, $B \xrightarrow{g} C$, ...
- and cells:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & \Downarrow \varphi & \downarrow {\scriptstyle b} \\ A' & \xrightarrow{f'} & B' \end{array} \quad , \quad \begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle a'}\downarrow & \Downarrow \varphi' & \downarrow {\scriptstyle b'} \\ A'' & \xrightarrow{f''} & B'' \end{array} \quad , \quad \begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle b}\downarrow & \Downarrow \gamma & \downarrow {\scriptstyle c} \\ B' & \xrightarrow{g'} & C' \end{array} \quad , ...$$

where objects and morphisms form a category

$$a''\left(a'a\right) = \left(a''a'\right)a,$$
$$1_{A'}a = a1_A;$$

pseudo-morphisms and cells also form a category

$$\varphi''\left(\varphi'\varphi\right) = \left(\varphi''\varphi'\right)\varphi,$$
$$1_{f'}\varphi = \varphi 1_f,$$

with 1_f being the cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\downarrow & \Downarrow 1_f & \downarrow {\scriptstyle 1_B} \\ A & \xrightarrow{f} & B \end{array} \quad ;$$

for each pair of *pseudo-composable* cells γ, φ, there is a pseudo-composition $\gamma \otimes \varphi$

$$\begin{array}{ccc} A & \xrightarrow{g \otimes f} & C \\ {\scriptstyle a}\downarrow & \Downarrow \gamma\otimes\varphi & \downarrow {\scriptstyle c} \\ A' & \xrightarrow{g' \otimes f'} & C' \end{array} \quad ;$$

satisfying

$$\left(\gamma'\gamma\right) \otimes \left(\varphi'\varphi\right) = \left(\gamma' \otimes \varphi'\right)\left(\gamma \otimes \varphi\right), \tag{1.8}$$
$$1_{g\otimes f} = 1_g \otimes 1_f;$$

for each morphism $a : A \longrightarrow A'$, there is a pseudo-identity id_a

$$\begin{array}{ccc} A & \xrightarrow{\quad id_A \quad} & A \\ {\scriptstyle a}\downarrow & \Downarrow {\scriptstyle id_a} & \downarrow{\scriptstyle a} \\ A' & \xrightarrow{\quad id_{A'} \quad} & A' \end{array}$$

satisfying

$$\begin{aligned} id_{1_A} &= 1_{id_A} \\ id_{a'a} &= id_{a'} id_a; \end{aligned}$$

there is a special cell $\alpha_{h,g,f}$ for each triple of composable pseudo-morphisms h, g, f

$$\begin{array}{ccc} A & \xrightarrow{\ h\otimes(g\otimes f)\ } & D \\ {\scriptstyle 1_A}\downarrow & \Downarrow {\scriptstyle \alpha_{h,g,f}} & \downarrow{\scriptstyle 1_D} \\ A & \xrightarrow{\ (h\otimes g)\otimes f\ } & D \end{array} \quad ,$$

natural in each component, i.e., the following diagram of cells

$$\begin{array}{ccc} h \otimes (g \otimes f) & \xrightarrow{\alpha_{h,g,f}} & (h \otimes g) \otimes f \\ {\scriptstyle \eta\otimes(\gamma\otimes\varphi)}\downarrow & & \downarrow{\scriptstyle (\eta\otimes\gamma)\otimes\varphi} \\ h' \otimes (g' \otimes f') & \xrightarrow{\alpha_{h',g',f'}} & (h' \otimes g') \otimes f' \end{array}$$

commutes for every triple of pseudo-composable cells φ, γ, η

$$\begin{array}{ccccccc} A & \xrightarrow{\quad f \quad} & B & \xrightarrow{\quad g \quad} & C & \xrightarrow{\quad h \quad} & D \\ {\scriptstyle a}\downarrow & \Downarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle b} & \Downarrow{\scriptstyle \gamma} & \downarrow{\scriptstyle c} & \Downarrow{\scriptstyle \eta} & \downarrow{\scriptstyle d} \\ A' & \xrightarrow{\quad f' \quad} & B' & \xrightarrow{\quad g' \quad} & C' & \xrightarrow{\quad h' \quad} & D' \end{array} \quad ;$$

to each pseudo-morphism $f : A \longrightarrow B$ there are two special cells

$$\begin{array}{ccc} A & \xrightarrow{\ id_B\otimes f\ } & B \\ {\scriptstyle 1_A}\downarrow & \Downarrow{\scriptstyle \lambda_f} & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \quad , \qquad \begin{array}{ccc} A & \xrightarrow{\ f\otimes id_A\ } & B \\ {\scriptstyle 1_A}\downarrow & \Downarrow{\scriptstyle \rho_f} & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \quad ,$$

natural in f, that is, to every cell φ as above, the following diagrams of cells commute

$$\begin{array}{ccc} id_B \otimes f & \xrightarrow{\lambda_f} & f \\ {\scriptstyle id_{1_B}\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ id_{B'} \otimes f' & \xrightarrow{\lambda_{f'}} & f' \end{array} \quad , \qquad \begin{array}{ccc} f \otimes id_A & \xrightarrow{\rho_f} & f \\ {\scriptstyle \varphi\otimes id_{1_A}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ f' \otimes id_{B'} & \xrightarrow{\lambda_{f'}} & f' \end{array} \quad .$$

And furthermore, the following conditions are satisfied whenever the compositions are defined

$$\begin{array}{ccccc} & f\otimes(g\otimes(h\otimes k)) & \xrightarrow{f\otimes\alpha_{g,h,k}} & f\otimes((g\otimes h)\otimes k) & \\ {}^{\alpha_{f,g,h\otimes k}}\swarrow & & & & \searrow{}^{\alpha_{f,g\otimes h,k}} \\ (f\otimes g)\otimes(h\otimes k) & & & & (f\otimes(g\otimes h))\otimes k \\ & {}_{\alpha_{f\otimes g,h,k}}\searrow & & \swarrow{}_{\alpha_{f,g,h}\otimes k} & \\ & & ((f\otimes g)\otimes h)\otimes k & & \end{array}$$

$$\begin{array}{ccc} f\otimes(1\otimes g) & \xrightarrow{\alpha_{f,1,g}} & (f\otimes 1)\otimes g \\ {}_{f\otimes\lambda_g}\searrow & & \swarrow{}_{\rho_f\otimes g} \\ & f\otimes g & \end{array} .$$

Examples of pseudo-categories internal to Cat include the usual bicategories of Spans, Bimodules, homotopies, ... where in each case it is also allowed to consider the natural morphisms between the objects in order to obtain a vertical categorical structure. For example in the case of spans we would have sets as objects, maps as morphisms, spans $A \longleftarrow S \longrightarrow B$ as pseudo-morphisms and the cells being triples (h, k, l) with the following two squares commutative

$$\begin{array}{ccccc} A & \longleftarrow & S & \longrightarrow & B \\ {\scriptstyle h}\downarrow & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l} \\ A' & \longleftarrow & S' & \longrightarrow & B \end{array} .$$

A pseudo-category in Cat has the following structures: a category (with objects and morphisms); a category (with pseudo-morphisms and cells); a bicategory (considering only the morphisms that are identities); a double category (if all the special cells are identity cells).

Other examples as Cat (with modules as pseudo-morphisms) may be found in [3] or [8].

The present description of pseudo double category (internal pseudo-category in Cat) is the same given by Leinster [3] and differs from the one considered by Grandis and Paré [8] in the sense that they also have

$$id_A = id_A \otimes id_A.$$

In the following sections we will provide a complete description of pseudo-functors, natural and pseudo-natural transformations and pseudo-modifications. We prove that all the compositions are well defined (except for the horizontal composition of pseudo-natural transformations which is only defined up to an isomorphism). In the end we show that the category of pseudo-categories (internal to some ambient 2-category $\mathbf{C}$) is Cartesian closed up to isomorphism. We will give all the definitions in terms of the internal structure to some ambient 2-category and also explain what is obtained in the case where the ambient 2-category is Cat. While doing some proofs we will make use of Yoneda embedding and consider the diagrams in Cat rather than in the abstract ambient 2-category.

We will also freely use known definitions and results from [4],[1],[2] and [10].

2. Pseudo-Functors

Let $\mathbf{C}$ be a 2-category and suppose

$$C = (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)\,, \qquad (2.1)$$
$$C' = (C_0', C_1', d', c', e', m', \alpha', \lambda', \rho')$$

are two pseudo-categories in $\mathbf{C}$.

A pseudo-functor $F : C \longrightarrow C'$ is a system

$$F = (F_0, F_1, \mu, \varepsilon)$$

where $F_0 : C_0 \longrightarrow C_0', F_1 : C_1 \longrightarrow C_1'$ are morphisms of $\mathbf{C}$,

$$\mu : F_1 m \longrightarrow m' \left(F_1 \times_{F_0} F_1\right)\ ,\ \varepsilon : F_1 e \longrightarrow e' F_0,$$

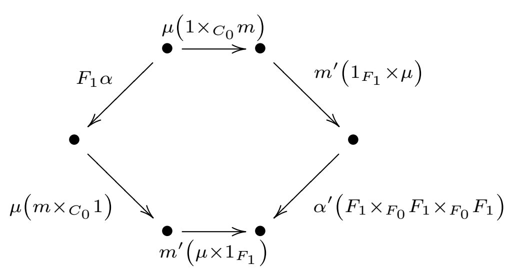
are 2-cells of $\mathbf{C}$ (that are isomorphisms[2]), the following conditions are satisfied

$$d' F_1 = F_0 d, \qquad (2.2)$$
$$c' F_1 = F_0 c,$$

$$d' \circ \mu = 1_{F_0 d \pi_2}, \qquad (2.3)$$
$$c' \circ \mu = 1_{F_0 c \pi_1},$$

$$d' \circ \varepsilon = 1_{F_0}, \qquad (2.4)$$
$$c' \circ \varepsilon = 1_{F_0},$$

and the following diagrams commute



, (2.5)

[2]Some authors (example Grandis and Paré in [8, 9]) consider the notion of pseudo - which corresponds to the present one - but also consider the notions of lax and colax where the 2-cells may not be isomorphisms.

$$\begin{array}{ccc} \bullet & \xrightarrow{F_1\rho} & \bullet \\ {\scriptstyle \mu\langle 1,ed\rangle}\downarrow & & \uparrow{\scriptstyle \rho' F_1} \\ \bullet & \xrightarrow{m'(1_{F_1}\times\varepsilon)} & \bullet \end{array} \quad , \tag{2.6}$$

$$\begin{array}{ccc} \bullet & \xrightarrow{F_1\lambda} & \bullet \\ {\scriptstyle \mu\langle ec,1\rangle}\downarrow & & \uparrow{\scriptstyle \lambda' F_1} \\ \bullet & \xrightarrow{m'(\varepsilon\times 1_{F_1})} & \bullet \end{array} \quad .$$

Consider the particular case of **C**=Cat. Let

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle a}\downarrow & \Downarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{\quad f' \quad} & B' \end{array}$$

be a cell in the pseudo-category C. A pseudo-functor $F : C \longrightarrow C'$, consists of four maps (sending objects to objects, morphisms to morphisms, pseudo-morphisms to pseudo-morphisms and cells to cells - that we will denote only by F to keep notation simple)

$$\begin{array}{ccc} FA & \xrightarrow{\quad Ff \quad} & FB \\ {\scriptstyle Fa}\downarrow & \Downarrow{\scriptstyle F\varphi} & \downarrow{\scriptstyle Fb} \\ FA' & \xrightarrow{\quad Ff' \quad} & FB' \end{array} \quad ;$$

a special cell $\mu_{f,g}$

$$\begin{array}{ccc} FA & \xrightarrow{\quad F(g\otimes f) \quad} & FC \\ {\scriptstyle 1}\downarrow & \Downarrow{\scriptstyle \mu_{f,g}} & \downarrow{\scriptstyle 1} \\ FA & \xrightarrow{\quad Fg\otimes Ff \quad} & FC \end{array}$$

to each pair of composable pseudo-morphisms f, g; a special cell ε_A

$$\begin{array}{ccc} FA & \xrightarrow{\quad F(id_A) \quad} & FA \\ {\scriptstyle 1}\downarrow & \Downarrow{\scriptstyle \epsilon_A} & \downarrow{\scriptstyle 1} \\ FA & \xrightarrow{\quad id_{FA} \quad} & FA \end{array}$$

to each object A, and satisfying the commutativity of the following diagrams

$$\begin{array}{ccc}
F\left(f\otimes\left(g\otimes h\right)\right) & \xrightarrow{F(\alpha_{f,g,h})} & F\left(\left(f\otimes g\right)\otimes h\right) \\
\Big\downarrow{\scriptstyle \mu_{f,g\otimes h}} & & \Big\downarrow{\scriptstyle \mu_{f\otimes g,h}} \\
F\left(f\right)\otimes F\left(g\otimes h\right) & & F\left(f\otimes g\right)\otimes F\left(h\right) \\
\Big\downarrow{\scriptstyle F(f)\otimes\mu_{g,h}} & & \Big\downarrow{\scriptstyle \mu_{f,g}\otimes F(h)} \\
F\left(f\right)\otimes\left(F\left(g\right)\otimes F\left(h\right)\right) & \xrightarrow[\alpha'_{Ff,Fg,Fh}]{} & \left(F\left(f\right)\otimes F\left(g\right)\right)\otimes F\left(h\right)
\end{array},$$

$$\begin{array}{ccc}
F\left(f\otimes id_A\right) & \xrightarrow{F(\rho_f)} & F\left(f\right) \\
{\scriptstyle \mu_{f,id_A}}\Big\downarrow & & \Big\uparrow{\scriptstyle \rho'_{Ff}} \\
F\left(f\right)\otimes F\left(id_A\right) & \xrightarrow{F(f)\otimes\varepsilon_A} & F\left(f\right)\otimes id_{F(A)}
\end{array},$$

$$\begin{array}{ccc}
F\left(id_B\otimes f\right) & \xrightarrow{F(\lambda_f)} & F\left(f\right) \\
{\scriptstyle \mu_{id_B,f}}\Big\downarrow & & \Big\uparrow{\scriptstyle \lambda'_{Ff}} \\
F\left(id_B\right)\otimes F\left(f\right) & \xrightarrow{\varepsilon_B\otimes F(f)} & id_{F(B)}\otimes F\left(f\right)
\end{array},$$

whenever the pseudo-compositions are defined.

Return to the general case.

Let $F : C \longrightarrow C'$ and $G : C' \longrightarrow C''$ be pseudo-functors in a 2-category $\mathbf{C}$. Consider C and C' as in (2.1) and let

$$\begin{aligned}
C'' &= \left(C''_0, C''_1, d'', c'', e'', m'', \alpha'', \lambda'', \rho''\right), \\
F &= \left(F_0, F_1, \mu^F, \varepsilon^F\right), \\
G &= \left(G_0, G_1, \mu^G, \varepsilon^G\right).
\end{aligned}$$

The composition of the pseudo-functors F and G is defined by the formula

$$GF = \left(G_0F_0, G_1F_1, \left(\mu^G\circ\left(F_1\times_{F_0}F_1\right)\right)\cdot\left(G_1\circ\mu^F\right), \left(\varepsilon^G\circ F_0\right)\cdot\left(G_1\circ\varepsilon^F\right)\right) \quad (2.7)$$

where $\circ$ represents the horizontal composition in $\mathbf{C}$ and $\cdot$ represents the vertical composition, as displayed in the diagram below

$$\begin{array}{ccccc} C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 & \xleftarrow{e} & C_0 \\ {\scriptstyle F_1\times_{F_0}F_1}\downarrow & \mu^F \Downarrow & {\scriptstyle F_1}\downarrow & \varepsilon^F \Downarrow & \downarrow{\scriptstyle F_0} \\ C'_1 \times_{C'_0} C'_1 & \xrightarrow{m'} & C'_1 & \xleftarrow{e'} & C'_0 \\ {\scriptstyle G_1\times_{G_0}G_1}\downarrow & \mu^G \Downarrow & {\scriptstyle G_1}\downarrow & \varepsilon^G \Downarrow & \downarrow{\scriptstyle G_0} \\ C''_1 \times_{C''_0} C''_1 & \xrightarrow{m''} & C''_1 & \xleftarrow{e''} & C''_0 \end{array} .$$

Proposition 1. *The above formula to compose pseudo-functors is well defined.*

Proof. Consider the system

$$GF = \left(G_0F_0, G_1F_1, \mu^{GF}, \varepsilon^{GF}\right)$$

with $\mu^{GF}, \varepsilon^{GF}$ as in (2.7). We will show that GF is a pseudo-functor from the pseudo-category C to the pseudo-category C''.

It is clear that $G_0F_0 : C_0 \longrightarrow C''_0, G_1F_1 : C_1 \longrightarrow C''_1$, are morphisms of the ambient 2-category $\mathbf{C}$ and $\mu^{GF} : G_1F_1m \longrightarrow m''\left(G_1F_1 \times_{G_0F_0} G_1F_1\right),\ \varepsilon^{GF} : G_1F_1e \longrightarrow e''G_0F_0$ are 2-cells of $\mathbf{C}$ and they are isomorphisms.

Conditions (2.2) are satisfied and

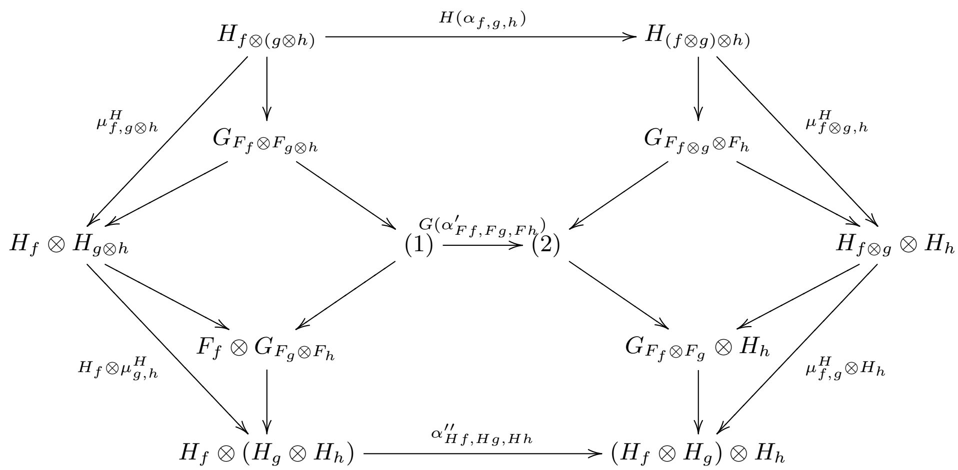
$$\begin{aligned} d''\mu^{GF} &= d\left(\left(\mu^G \circ (F_1 \times_{F_0} F_1)\right) \cdot \left(G_1 \circ \mu^F\right)\right) \\ &= \left(d \circ \mu^G \circ (F_1 \times_{F_0} F_1)\right) \cdot \left(d \circ G_1 \circ \mu^F\right) \\ &= \left(1_{G_0d'\pi'_2} \circ (F_1 \times_{F_0} F_1)\right) \cdot \left(G_0 \circ d' \circ \mu^F\right) \\ &= \left(1_{G_0d'\pi'_2\left(F_1 \times_{F_0} F_1\right)}\right) \cdot \left(G_0 \circ 1_{F_0d\pi_2}\right) \\ &= 1_{G_0d'F_1\pi_2} \cdot 1_{G_0F_0d\pi_2} \\ &= 1_{G_0F_0d\pi_2}, \end{aligned}$$

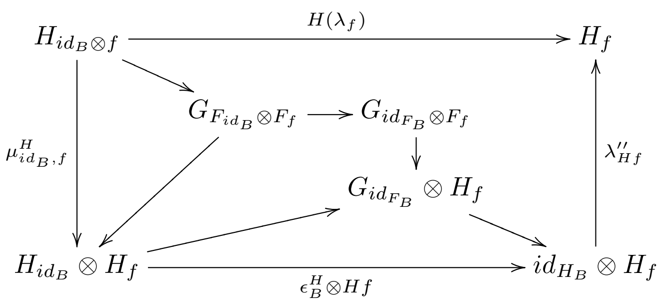
as well $c''\mu^{GF} = 1_{G_0F_0c\pi_1}$, hence (2.3) holds. Also

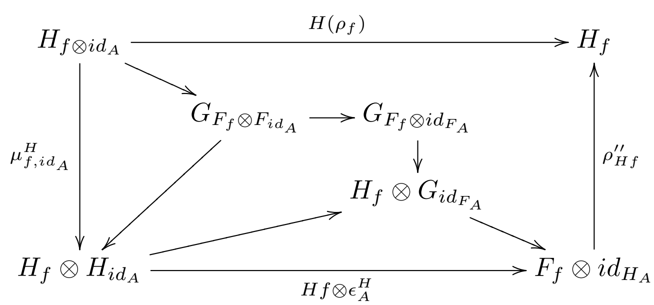
$$\begin{aligned} d''\varepsilon^{GF} &= d''\left(\left(\varepsilon^G \circ F_0\right) \cdot \left(G_1 \circ \varepsilon^F\right)\right) \\ &= \left(d'' \circ \varepsilon^G \circ F_0\right) \cdot \left(d''G_1 \circ \varepsilon^F\right) \\ &= \left(1_{G_0} \circ F_0\right) \cdot \left(G_0d' \circ \varepsilon^F\right) \\ &= 1_{G_0F_0} \cdot \left(G_0 \circ 1_{F_0}\right) \\ &= 1_{G_0F_0} \cdot 1_{G_0F_0} = 1_{G_0F_0}, \end{aligned}$$

and similarly $c''\varepsilon^{GF} = 1_{G_0F_0}$, so conditions (2.4) are satisfied.

Commutativity of diagrams (2.5), (2.6) follows from Yoneda Lemma and the commutativity of the following diagrams







where $(1) = G_{F_f\otimes(F_g\otimes F_h)}$ and $(2) = G_{(F_f\otimes F_g)\otimes F_h}$. We also use the abbreviations $H = GF$ and F_f or Ff instead of $F(f)$ to save space in the diagram. □

Composition of pseudo-functors is associative and there is an identity pseudo-functor for every pseudo-category, namely the pseudo-functor

$$1_C = (1_{C_0}, 1_{C_1}, 1_m, 1_e)$$

for the pseudo-category

$$C = (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho).$$

Given a 2-category $\mathbf{C}$, we define the category $\mathrm{PsCat}(\mathbf{C})$ consisting of all pseudo-categories and pseudo-functors internal to $\mathbf{C}$.

3. Natural and pseudo-natural transformations

Let $\mathbf{C}$ be a 2-category and suppose

$$\begin{aligned} C &= (C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)\,, \\ C' &= (C'_0, C'_1, d', c', e', m', \alpha', \lambda', \rho') \end{aligned} \tag{3.1}$$

are pseudo-categories in $\mathbf{C}$ and

$$\begin{aligned} F &= \left(F_0, F_1, \mu^F, \varepsilon^F\right), \\ G &= \left(G_0, G_1, \mu^G, \varepsilon^G\right) \end{aligned} \tag{3.2}$$

are pseudo-functors from C to C'.

A **natural transformation** $\theta : F \longrightarrow G$ is a pair $\theta = (\theta_0, \theta_1)$ of 2-cells of $\mathbf{C}$

$$\begin{aligned} \theta_0 &: F_0 \longrightarrow G_0 \\ \theta_1 &: F_1 \longrightarrow G_1 \end{aligned}$$

satisfying

$$\begin{aligned} d' \circ \theta_1 &= \theta_0 \circ d \\ c' \circ \theta_1 &= \theta_0 \circ c \end{aligned}$$

and the commutativity of the following diagrams of 2-cells

$$\begin{array}{ccc} \bullet & \xrightarrow{\theta_1 \circ m} & \bullet \\ {\scriptstyle \mu^F}\big\downarrow & & \big\downarrow{\scriptstyle \mu^G} \\ \bullet & \xrightarrow{m' \circ \left(\theta_1 \times_{\theta_0} \theta_1\right)} & \bullet \end{array}$$

$$\begin{array}{ccc} \bullet & \xrightarrow{\theta_1 \circ e} & \bullet \\ {\scriptstyle \varepsilon^F}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon^G} \\ \bullet & \xrightarrow{e' \circ \theta_0} & \bullet \end{array} .$$

A **pseudo-natural transformation** $T : F \longrightarrow G$ is a pair

$$T = (t, \tau)$$

where $t : C_0 \longrightarrow C'_1$ is a morphism of $\mathbf{C}$,

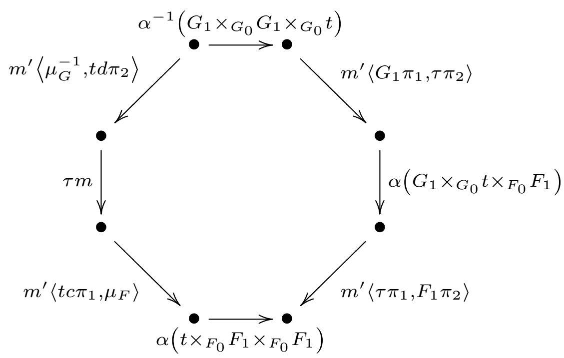
$$\tau : m' \langle G_1, td \rangle \longrightarrow m' \langle tc, F_1 \rangle$$

is a 2-cell (that is an isomorphism); the following conditions are satisfied

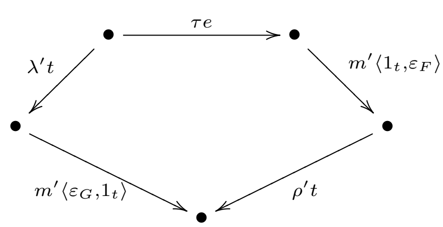
$$\begin{aligned} d't &= F_0 \\ c't &= G_0 \end{aligned} \tag{3.3}$$

$$\begin{aligned} d' \circ \tau &= 1_{d'F_1} \\ c' \circ \tau &= 1_{c'G_1} \end{aligned} \tag{3.4}$$

and the following diagrams of 2-cells are commutative[3]



, (3.5)



. (3.6)

In the case **C**=Cat: let W, W' be two pseudo-categories in Cat, and $F, G : W \longrightarrow W'$ two pseudo-functors. Given a cell

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle a}\downarrow & \Downarrow \varphi & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{\quad f' \quad} & B' \end{array}$$

in W, we will write

$$\begin{array}{ccc} FA & \xrightarrow{\quad Ff \quad} & FB \\ {\scriptstyle Fa}\downarrow & \Downarrow F\varphi & \downarrow{\scriptstyle Fb} \\ FA' & \xrightarrow{\quad Ff' \quad} & FB' \end{array} \quad \text{and} \quad \begin{array}{ccc} GA & \xrightarrow{\quad Gf \quad} & GB \\ {\scriptstyle Ga}\downarrow & \Downarrow G\varphi & \downarrow{\scriptstyle Gb} \\ GA' & \xrightarrow{\quad Gf' \quad} & GB' \end{array}$$

---

[3] $G_1 \times_{G_0} t \times_{F_0} F_1 : C_1 \times_{C_0} C_0 \times_{C_0} C_1 \longrightarrow C_1 \times_{C_0} C_1 \times_{C_0} C_1$
$t \times_{F_0} F_1 \times_{F_0} F_1 : C_0 \times_{C_0} C_1 \times_{C_0} C_1 \longrightarrow C_1 \times_{C_0} C_1 \times_{C_0} C_1$
$G_1 \times_{G_0} G_1 \times_{G_0} t : C_1 \times_{C_0} C_1 \times_{C_0} C_0 \longrightarrow C_1 \times_{C_0} C_1 \times_{C_0} C_1$

for the image of φ under F and G.

The description of natural and pseudo-natural transformations in this particular case is as follows:

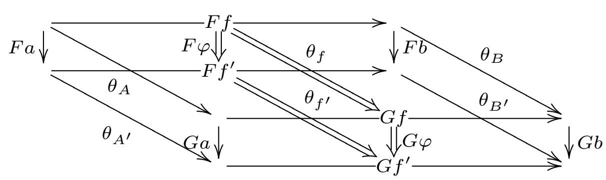
- While a natural transformation $\theta : F \longrightarrow G$ is a family of cells

$$\begin{array}{ccc} FA & \xrightarrow{\quad Ff \quad} & FB \\ {\scriptstyle \theta_A}\downarrow & \Downarrow \theta_f & \downarrow{\scriptstyle \theta_B} \\ GA & \xrightarrow[\quad]{\quad Gf \quad} & GB \end{array} \quad ,$$

one for each pseudo-morphism f in W, such that for every cell φ in W, the square

$$\begin{array}{ccc} Ff & \xrightarrow{\theta_f} & Gf \\ {\scriptstyle F\varphi}\downarrow & & \downarrow{\scriptstyle G\varphi} \\ Ff' & \xrightarrow{\theta_{f'}} & Gf' \end{array}$$

is commutative as displayed in the picture below



;

and furthermore, given two composable pseudo-morphisms g, f and an object A in W, the following squares are commutative

$$\begin{array}{ccc} F(g \otimes f) & \xrightarrow{\mu^F_{g,f}} & Fg \otimes Ff \\ {\scriptstyle \theta_{g\otimes f}}\downarrow & & \downarrow{\scriptstyle \theta_g \otimes \theta_f} \\ G(g \otimes f) & \xrightarrow{\mu^G_{g,f}} & Gg \otimes Gf \end{array}$$

$$\begin{array}{ccc} F(id_A) & \xrightarrow{\varepsilon^F_A} & id_{FA} \\ {\scriptstyle \theta_{id_A}}\downarrow & & \downarrow{\scriptstyle id_{\theta_A}} \\ G(id_A) & \xrightarrow[\varepsilon^G_A]{} & id_{GA} \end{array} \quad .$$

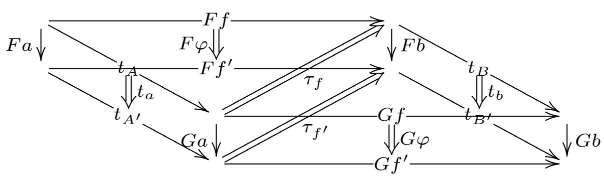
- Rather a pseudo-natural transformation $T : F \longrightarrow G$ consists of two families of cells

$$\begin{array}{ccc} FA & \xrightarrow{\quad t_A \quad} & GA \\ {\scriptstyle Fa}\downarrow & \Downarrow {\scriptstyle t_a} & \downarrow{\scriptstyle Ga} \\ FA' & \xrightarrow{\quad t_{A'} \quad} & GA' \end{array}$$

and

$$\begin{array}{ccc} FA & \xrightarrow{\quad Gf \otimes t_A \quad} & GB \\ {\scriptstyle 1}\downarrow & \Downarrow {\scriptstyle \tau_f} & \downarrow{\scriptstyle 1} \\ FA & \xrightarrow{\quad t_B \otimes Ff \quad} & GB \end{array}$$

with a a morphism and f a pseudo-morphism of W, as displayed in the following picture



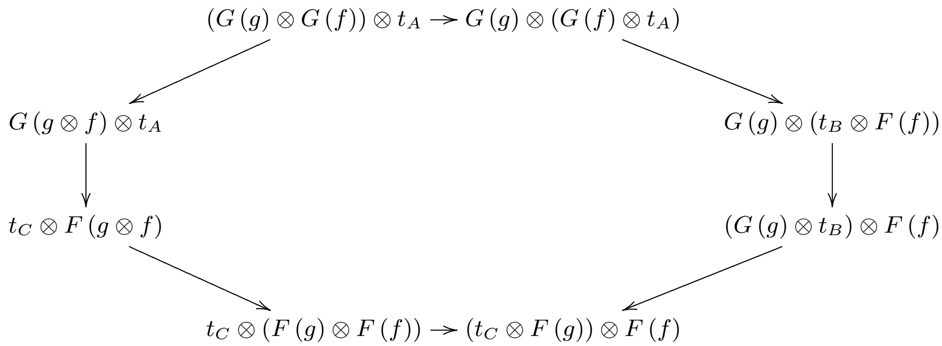
such that (t is a functor)

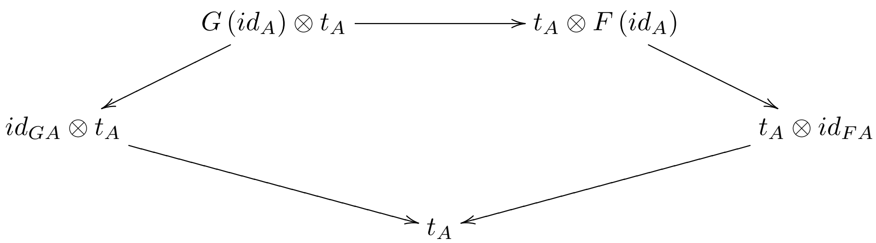
$$t_{a'a} = t_a t_{a'}$$
$$t_{1_A} = 1_{t_A}$$

(τ is natural)

$$\begin{array}{ccc} Gf \otimes t_A & \xrightarrow{\ \tau_f\ } & t_B \otimes Ff \\ {\scriptstyle G\varphi\otimes t_a}\downarrow & & \downarrow{\scriptstyle t_b \otimes F\varphi} \\ Gf' \otimes t_{A'} & \xrightarrow{\ \tau_{f'}\ } & t_{B'} \otimes Ff' \end{array},$$

and for every two composable pseudo-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, the following diagrams of cells in W' are commutative

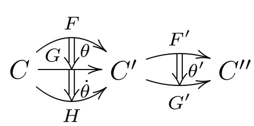




.

Return to the general case.

Let $\mathbf{C}$ be a 2-category and suppose C, C', C'' are pseudo-categories in $\mathbf{C}$ and $F, G, H : C \longrightarrow C'$, $F', G' : C' \longrightarrow C''$ are pseudo-functors. Natural transformations $\theta, \theta', \dot{\theta}$



may be composed horizontally with $\theta' \circ \theta = (\theta'_0, \theta'_1) \circ (\theta_0, \theta_1) = (\theta'_0 \circ \theta_0, \theta'_1 \circ \theta_1)$ obtained from the horizontal composition of 2-cells of $\mathbf{C}$, and vertically with $\dot{\theta} \cdot \theta = \left(\dot{\theta}_0, \dot{\theta}_1\right) \cdot (\theta_0, \theta_1) = \left(\dot{\theta}_0 \cdot \theta_0, \dot{\theta}_1 \cdot \theta_1\right)$ obtained from the vertical composition of 2-cells of $\mathbf{C}$. Clearly both compositions are well defined, are associative, have identities and satisfy the middle interchange law. This fact may be stated as in the following theorem.

Theorem 1. *Let* $\boldsymbol{C}$ *be a 2-category. The category PsCat(*$\boldsymbol{C}$*) (with pseudo-categories, pseudo-functors and natural transformations) is a 2-category.*

Composition of pseudo-natural transformations is much more delicate.

Again let $\mathbf{C}$ be a 2-category and suppose C, C' are pseudo-categories in $\mathbf{C}$, $F, G, H : C \longrightarrow C'$ are pseudo-functors (as above) and consider the pseudo-natural transformations

$$F \xrightarrow{T} G \xrightarrow{S} H$$

with

$$T = (t, \tau), \quad S = (s, \sigma).$$

Vertical composition of pseudo-natural transformations S and T is defined as

$$S \otimes T = (m' \langle s, t\rangle, \sigma \otimes \tau) \tag{3.7}$$

where

$$\sigma \otimes \tau = \alpha \langle sc, tc, F_1\rangle \cdot m' \langle 1_{sc}, \tau\rangle \cdot \alpha^{-1} \langle sc, G_1, td\rangle \cdot m' \langle \sigma, 1_{td}\rangle \cdot \alpha \langle H_1, sd, td\rangle . \tag{3.8}$$

The above formula in the case **C**=Cat is expressed as follows

$$(s \otimes t)_a = s_a \otimes t_a, \qquad \begin{array}{ccccc} FA & \xrightarrow{t_A} & GA & \xrightarrow{s_A} & HA \\ {\scriptstyle Fa}\downarrow & \Downarrow t_a & \downarrow{\scriptstyle Ga} & \Downarrow s_a & \downarrow{\scriptstyle Ha} \\ FA' & \xrightarrow{t_{A'}} & GA' & \xrightarrow{s_{A'}} & HA' \end{array} ;$$

and

$$(\sigma \otimes \tau)_f = \alpha (s_B \otimes \tau_f) \alpha^{-1} (\sigma_f \otimes t_A) \alpha,$$

as displayed in the following picture

$$\begin{array}{ccc} Hf \otimes (s_A \otimes t_A) & \xrightarrow{(\sigma\tau)_f} & (s_B \otimes t_B) \otimes Ff \\ {\scriptstyle \alpha}\downarrow & & \uparrow{\scriptstyle \alpha} \\ (Hf \otimes s_A) \otimes t_A & & s_B \otimes (t_B \otimes Ff) \\ {\scriptstyle \sigma_f \otimes t_A}\downarrow & & \uparrow{\scriptstyle s_B \otimes \tau_f} \\ (s_B \otimes Gf) \otimes t_A & \xrightarrow[\alpha^{-1}]{} & s_B \otimes (Gf \otimes t_A) \end{array} .$$

Return to the general case.

Theorem 2. *The vertical composition of pseudo-natural transformations is well defined.*

Proof. Consider C, C' as in (3.1), F, G as in (3.2), $H = (H_0, H_1, \mu_H, \varepsilon_H)$ and S, T as above. Clearly $(st) = m' \langle s, t\rangle : C_0 \longrightarrow C'_1$ is a morphism of **C** and $\sigma\tau : m' \langle H_1, (st)\, d\rangle \longrightarrow m' \langle (st)\, c, F_1\rangle$ is a 2-cell of **C** that is an isomorphism (is defined as a composition of isomorphisms).

Conditions (3.3) and (3.4) are satisfied

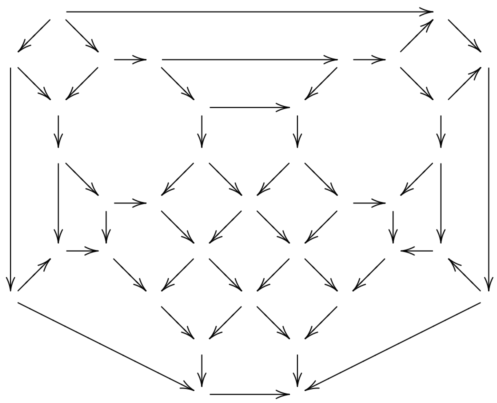
$$\begin{aligned} d'm' \langle s, t\rangle &= d'\pi'_2 \langle s, t\rangle \\ &= d't \\ &= F_0, \end{aligned}$$

also $c'm'\langle s,t\rangle = c's = H_0$, and

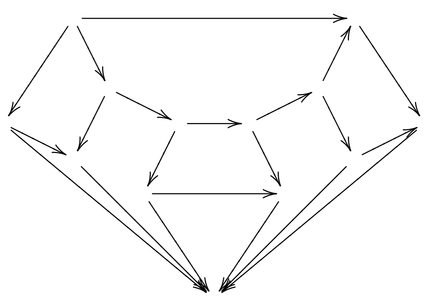
$$\begin{aligned} d'\circ(\sigma\otimes\tau) &= d'\circ\left(\alpha\langle sc,tc,F_1\rangle\cdot m'\langle 1_{sc},\tau\rangle\cdot\alpha^{-1}\langle sc,G_1,td\rangle\cdot m'\langle\sigma,1_{td}\rangle\cdot\alpha\langle H_1,sd,td\rangle\right)\\ &= \left(d'\circ\alpha\langle sc,tc,F_1\rangle\right)\cdot\left(d'\circ m'\langle 1_{sc},\tau\rangle\right)\cdot\\ &\qquad\left(d'\circ\alpha^{-1}\langle sc,G_1,td\rangle\right)\cdot\left(d'\circ m'\langle\sigma,1_{td}\rangle\right)\cdot\left(d'\circ\alpha\langle H_1,sd,td\rangle\right)\\ &= 1_{d'F_1}\cdot 1_{d'F1}\cdot 1_{d'td}\cdot 1_{d'td}\cdot 1_{d'td}\\ &= 1_{d'F_1}\cdot 1_{d'td} = 1_{d'F_1}\cdot 1_{F_0d} = 1_{d'F_1}\cdot 1_{d'F_1} = 1_{d'F_1}\end{aligned}$$

with similar computations for $c'\circ(\sigma\otimes\tau) = 1_{c'H_1}$.

Commutativity of diagrams (3.5) and (3.6) is obtained using Yoneda Lemma, writing the respective diagrams and adding all the possible arrows to fill them in order to obtain the following *mask* and *diamond*



(mask)



(diamond)

in which squares commute by naturality, hexagons commute by definition of $(\sigma\otimes\tau)$, octagons commute because S, T are pseudo-natural transformations, pentagons in the diamond commute by the same reason and all the other pentagons and triangles commute by coherence. □

The horizontal composition of pseudo-natural transformations is only defined up to an isomorphism and it will be considered at the end of this paper.

In the next section we define square pseudo-modification (simply called pseudo-modification) and show that given two pseudo-categories C, C', we obtain a pseudo-category by considering the pseudo-functors as objects, natural transformations as morphisms, pseudo-natural transformations as pseudo-morphisms and pseudo-modifications as cells. So, in particular, we will show that the vertical composition of pseudo-natural transformations is associative and has identities up to isomorphism. We also show that PsCat is Cartesian closed up to isomorphism, that is, instead of an isomorphism of categories PsCat$(A \times B, C)\cong$PsCat$(A$,PsCAT$(B, C))$ we get an equivalence of categories PsCat$(A \times B, C)\sim$PsCat$(A$,PsCAT$(B, C))$.

4. Pseudo-modifications

Let $\mathbf{C}$ be a 2-category. Suppose C, C' are pseudo-categories in $\mathbf{C}$, $F, G, H, K : C \longrightarrow C'$ are pseudo-functors, $T = (t, \tau) : F \longrightarrow G$, $T' = (t', \tau') : H \longrightarrow K$ are pseudo-natural transformations and $\theta = (\theta_0, \theta_1) : F \longrightarrow H, \theta' = (\theta'_0, \theta'_1) : G \longrightarrow K$ are two natural transformations.

A **pseudo-modification** Φ (that will be represented as)

$$\begin{array}{ccc} F & \xrightarrow{\quad T \quad} & G \\ {\scriptstyle\theta}\downarrow & \Downarrow{\scriptstyle\Phi} & \downarrow{\scriptstyle\theta'} \\ H & \xrightarrow[\quad T' \quad]{} & K \end{array}$$

is a 2-cell of $\mathbf{C}$

$$\Phi : t \longrightarrow t'$$

satisfying

$$\begin{aligned} d' \circ \Phi &= \theta_0 \\ c' \circ \Phi &= \theta'_0 \end{aligned} \tag{4.1}$$

and the commutativity of the square

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad \tau \quad} & \bullet \\ {\scriptstyle m'\langle\theta'_1, \Phi\circ d\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle m'\langle\Phi\circ c, \theta_1\rangle} \\ \bullet & \xrightarrow[\quad \tau' \quad]{} & \bullet \end{array} . \tag{4.2}$$

Consider the case where $\mathbf{C}$=Cat. Suppose W, W' are two pseudo-categories in Cat, $F, G, H, K : W \longrightarrow W'$ are pseudo-functors, $T : F \longrightarrow G, T' : H \longrightarrow K$ are pseudo-natural transformations and $\theta : F \longrightarrow G, \theta' : H \longrightarrow K$ are natural transformations.

A pseudo-modification Φ

$$\begin{array}{ccc} F & \xrightarrow{\quad T \quad} & G \\ {\scriptstyle\theta}\downarrow & \Downarrow{\scriptstyle\Phi} & \downarrow{\scriptstyle\theta'} \\ H & \xrightarrow[\quad T' \quad]{} & K \end{array}$$

is a family of cells

$$\begin{array}{ccc} FA & \xrightarrow{\quad t_A \quad} & GA \\ {\scriptstyle \theta_A}\downarrow & \Downarrow {\scriptstyle \Phi_A} & \downarrow{\scriptstyle \theta'_A} \\ HA & \xrightarrow{\quad t'_A \quad} & KA \end{array}$$

of W', for each object A in W, where the square

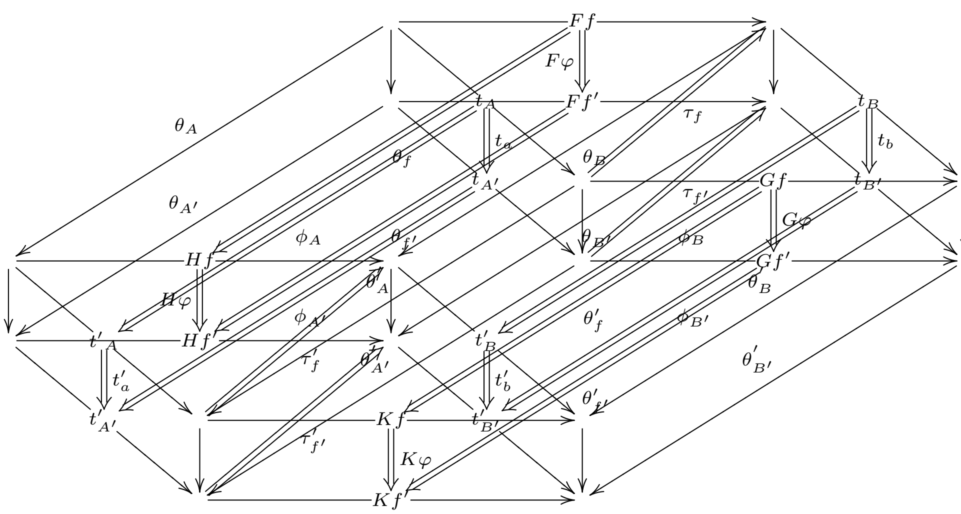
$$\begin{array}{ccc} t_A & \xrightarrow{\Phi_A} & t'_A \\ {\scriptstyle t_a}\downarrow & & \downarrow{\scriptstyle t'_a} \\ t_{A'} & \xrightarrow{\Phi_{A'}} & t'_{A'} \end{array}, \qquad (4.3)$$

commutes for every morphism $a : A \longrightarrow A'$ in W (naturality of Φ) and the square

$$\begin{array}{ccc} Gf \otimes t_A & \xrightarrow{\tau_f} & t_B \otimes Ff \\ {\scriptstyle \theta'_f \otimes \Phi_A}\downarrow & & \downarrow{\scriptstyle \Phi_B \otimes \theta_f} \\ Kf \otimes t'_A & \xrightarrow[\tau'_f]{} & t'_B \otimes Hf \end{array}, \qquad (4.4)$$

commutes for every pseudo-morphism $f : A \longrightarrow B$ in W.

Both squares (4.3) and (4.4) may be displayed together with full information, for a φ in W, as follows



(4.5)

Return to the general case.

Let **C** be a 2-category and consider C, C' two pseudo-categories in **C** as in (3.1). Suppose T, T', T'' are pseudo-natural transformations between pseudo-functors from C to C': we define for

$$T \xrightarrow{\Phi} T' \xrightarrow{\Phi'} T''$$

a composition $\Phi'\Phi$ as the composition of 2-cells in **C**, and clearly it is well defined, is associative and has identities. Now for $\theta, \theta', \theta''$ natural transformations between pseudo-functors from C to C', we define for

$$\theta \xrightarrow{\Phi} \theta' \xrightarrow{\Psi} \theta''$$

a pseudo-composition $\Psi \otimes \Phi = m' \langle \Psi, \Phi \rangle$.

Proposition 2. *Let **C** be a 2-category and suppose Ψ, Φ are pseudo-modifications*

$$\begin{array}{ccccc} F & \xrightarrow{T} & G & \xrightarrow{S} & H \\ {\scriptstyle\theta}\downarrow & \Downarrow{\scriptstyle\Phi} & \downarrow{\scriptstyle\theta'} & \Downarrow{\scriptstyle\Psi} & \downarrow{\scriptstyle\theta''} \\ F' & \xrightarrow{T'} & G' & \xrightarrow{S'} & H' \end{array}$$

with F, G, H, F', G', H' pseudo-functors from C to C' (pseudo-categories as in (3.1)), S, T, S', T' pseudo-natural transformations and $\theta, \theta', \theta''$ natural transformations as considered above.
The formula

$$\Psi \otimes \Phi = m' \langle \Psi, \Phi \rangle$$

for pseudo-composition of pseudo-modifications is well defined.

Proof. Recall that the composition of pseudo-modifications is given by

$$S \otimes T = (m' \langle s, t \rangle, (\sigma \otimes \tau))$$

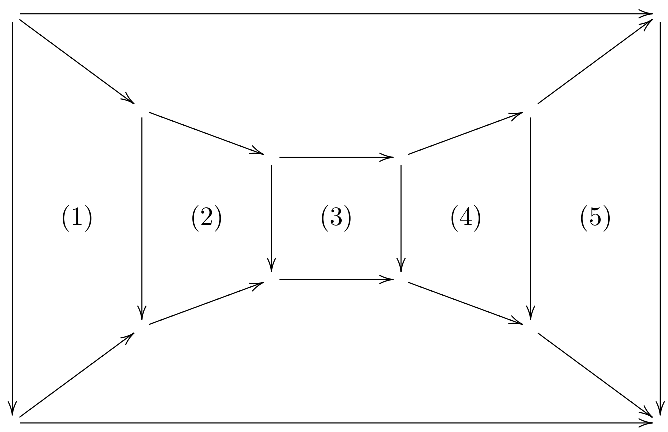
with $(\sigma \otimes \tau)$ given as in (3.8), hence

$$m' \langle \Psi, \Phi \rangle : m' \langle s, t \rangle \longrightarrow m' \langle s', t' \rangle$$

is a 2-cell of **C** as required.
Conditions (4.1) are satisfied,

$$\begin{aligned} d'm' \circ \langle \Psi, \Phi \rangle &= d'\pi_2' \circ \langle \Psi, \Phi \rangle = d' \circ \Phi = \theta_0 \\ c'm \circ \langle \Psi, \Phi \rangle &= c'\pi_1' \langle \Psi, \Phi \rangle = c' \circ \Psi = \theta_0''. \end{aligned}$$

To prove commutativity of square (4.2) we use Yoneda Lemma and the following diagram, obtained by adapting (4.2) to the present case and filling its interior



where hexagons commute by definition of $(\sigma \otimes \tau)$ and $(\sigma' \otimes \tau')$, squares $(1), (3), (5)$ commute by naturality of α' while squares $(2), (4)$ commute because Ψ, Φ are pseudo-modifications (satisfy (4.4)) together with the fact that pseudo-composition (in C') satisfies the middle interchange law (1.8). □

Composition of pseudo-natural transformations is not associative, however there is a special pseudo-modification for each triple of composable pseudo-natural transformations.

Proposition 3. *Let $\boldsymbol{C}$ be 2-category and suppose $F, G, H, K : C \longrightarrow C'$ are pseudo-functors in $\boldsymbol{C}$ and that $S = (s, \sigma), T = (t, \tau), U = (u, \upsilon)$ are pseudo-natural transformations as follows*

$$F \xrightarrow{S} G \xrightarrow{T} H \xrightarrow{U} K.$$

The 2-cell $\alpha'_{U,T,S} = \alpha' \langle u, t, s \rangle$ is a pseudo-modification

$$\begin{array}{ccc} F & \xrightarrow{U \otimes (T \otimes S)} & K \\ {\scriptstyle 1}\downarrow & \Downarrow \alpha'_{U,T,S} & \downarrow{\scriptstyle 1} \\ F & \xrightarrow{(U \otimes T) \otimes S} & K \end{array} ,$$

and it is natural in S, T, U, in the sense that the square

$$\begin{array}{ccc} U \otimes (T \otimes S) & \xrightarrow{\alpha' \langle u,t,s \rangle} & (U \otimes T) \otimes S \\ {\scriptstyle \varphi \otimes (\gamma \otimes \delta)}\downarrow & & \downarrow{\scriptstyle (\varphi \otimes \gamma) \otimes \delta} \\ U' \otimes (T' \otimes S') & \xrightarrow{\alpha' \langle u',t',s' \rangle} & (U' \otimes T') \otimes S' \end{array}$$

commutes for every pseudo-modification $\varphi : U \longrightarrow U', \gamma : T \longrightarrow T', \delta : S \longrightarrow S'$.

Proof. The 2-cell $\alpha' \langle u,t,s\rangle$ is obtained from

$$C_0 \xrightarrow{\langle u,t,s\rangle} C'_1 \times_{C'_0} C'_1 \times_{C'_0} C'_1 \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \alpha'}} C_1,$$

and

$$\begin{aligned} U \otimes (T \otimes S) &= (m\,(1 \times m)\,\langle u,t,s\rangle\,, (\upsilon \otimes (\tau \otimes \sigma))) \\ (U \otimes T) \otimes S &= (m\,(m \times 1)\,\langle u,t,s\rangle\,, ((\upsilon \otimes \tau) \otimes \sigma))\,, \end{aligned}$$

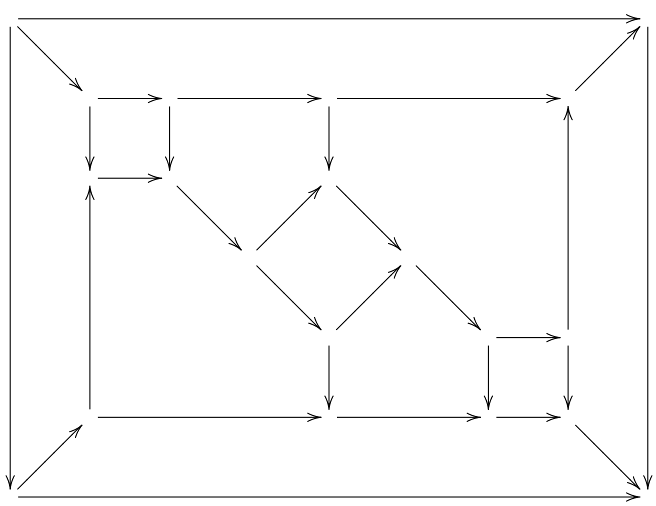
hence

$$\alpha' \langle u,t,s\rangle : m\,(1 \times m)\,\langle u,t,s\rangle \longrightarrow m\,(m \times 1)\,\langle u,t,s\rangle$$

is a 2-cell of $\mathbf{C}$.
Conditions (4.1) are satisfied

$$\begin{aligned} d' \circ \alpha' \circ \langle u,t,s\rangle &= 1_{d'\pi'_3}\,\langle u,t,s\rangle = 1_{d's} = 1_{F_0} \\ c' \circ \alpha' \circ \langle u,t,s\rangle &= 1_{c'\pi'_1}\,\langle u,t,s\rangle = 1_{c'u} = 1_{K_0}. \end{aligned}$$

Commutativity of (4.2) follows from Yoneda Lemma and the commutativity of the following diagram



where hexagons commute because S, T, U are pseudo-natural transformations, squares commute by naturality and pentagons by coherence.

To prove naturality we observe that

$$\begin{aligned} ((\varphi \otimes \gamma) \otimes \delta) \cdot (\alpha' \langle u,t,s\rangle) &= (m' \langle m \langle \varphi,\gamma\rangle\,, \delta\rangle) \cdot (\alpha' \langle u,t,s\rangle) \\ &= (m'\,(m' \times 1)\,\langle \varphi,\gamma,\delta\rangle) \cdot (\alpha' \langle u,t,s\rangle) \\ &= \left(1_{m'(m'\times 1)} \cdot \alpha'\right) \circ \left(\langle \varphi,\gamma,\delta\rangle \cdot 1_{\langle u,t,s\rangle}\right) \\ &= \alpha' \circ \langle \varphi,\gamma,\delta\rangle \end{aligned}$$

and

$$\begin{aligned}(\alpha' \langle u', t', s' \rangle) \cdot (\varphi \otimes (\gamma \otimes \delta)) &= (\alpha' \langle u', t', s' \rangle) \cdot (m' \langle \varphi, m' \langle \gamma, \delta \rangle \rangle) \\ &= (\alpha' \langle u', t', s' \rangle) \cdot (m' (1 \times m') \langle \varphi, \gamma, \delta \rangle) \\ &= \left(\alpha' \cdot 1_{m'(1 \times m')}\right) \circ \left(1_{\langle u', t', s' \rangle} \langle \varphi, \gamma, \delta \rangle\right) \\ &= \alpha' \circ \langle \varphi, \gamma, \delta \rangle .\end{aligned}$$

□

For every pseudo-functor there is a pseudo-identity pseudo-natural transformation and a pseudo-identity pseudo-modification.

Proposition 4. *Consider a pseudo-functor $F = (F_0, F_1, \mu_F, \varepsilon_F) : C \longrightarrow C'$ in a 2-category $\boldsymbol{C}$ (with C, C' pseudo-categories in $\boldsymbol{C}$ as in (3.1)). The pair $\left(e'F_0, \left(\lambda'^{-1}\rho'\right) \circ F_1\right)$ is a pseudo-natural transformation in PsCat($\boldsymbol{C}$)*

$$id_F = \left(e'F_0, \lambda'^{-1}\rho' F_1\right) : F \longrightarrow F,$$

and the 2-cell $1_{e'F_0} : e'F_0 \longrightarrow e'F_0$ is a pseudo-modification in PsCat($\boldsymbol{C}$)

$$\begin{array}{ccc} F & \xrightarrow{\quad id_F \quad} & F \\ {\scriptstyle 1}\big\downarrow & \Downarrow 1_{id_F} & \big\downarrow{\scriptstyle 1} \\ F & \xrightarrow{\quad id_F \quad} & F \end{array} .$$

Proof. Clearly $e'F_0 : C_0 \longrightarrow C_1'$ is a morphism of $\mathbf{C}$, and

$$\lambda'^{-1}\rho' F_1 : m \langle F_1, e'd'F_1 \rangle \longrightarrow m' \langle e'c'F_1, F_1 \rangle$$

is a 2-cell (that is an isomorphism) of $\mathbf{C}$.

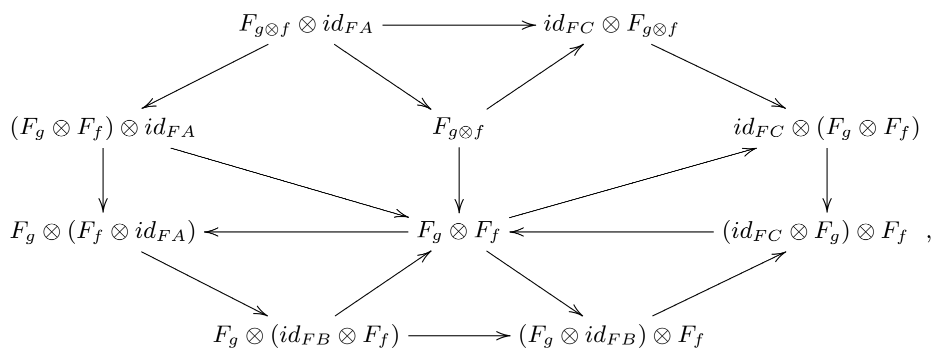
Conditions (3.3) and (3.4) are satisfied,

$$\begin{aligned} d'e'F_0 &= F_0 \\ c'e'F_0 &= F_0 \end{aligned}$$

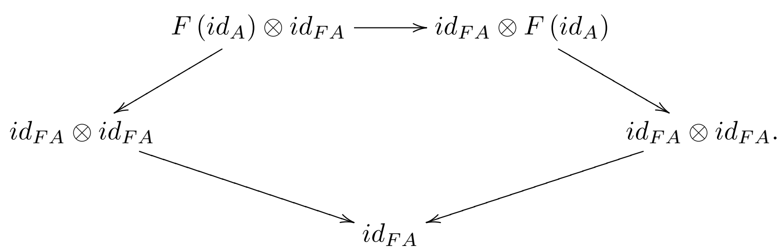
$$\begin{aligned} d' \circ \left(\lambda'^{-1}\rho' F_1\right) &= d' \circ \left(\lambda'^{-1}\rho'\right) \circ F_1 \\ &= \left(d'\lambda'^{-1} F_1\right) \cdot \left(d'\rho' F_1\right) \\ &= \left(1_{d'F_1}\right) \cdot \left(1_{d'F_1}\right) \\ &= \left(1_{d'F_1}\right), \end{aligned}$$

and similarly for $c' \circ \left(\lambda'^{-1}\rho' F_1\right) = 1_{c'F_1}$.

Commutativity of (3.5) is obtained using Yoneda Lemma and the commutativity of the diagram



while (3.6) follows in a similar way as observed in the diagram



This proves that id_F is a pseudo-natural transformation. To prove $1_{id_F}=1_{e'F_0}$ is a pseudo-modification we note that

$$1_{e'F_0}:e'F_0\longrightarrow e'F_0$$

is a 2-cell of **C**,

$$\begin{aligned} d'\circ 1_{e'F_0} &= 1_{d'e'F_0}=1_{F_0},\\ c'\circ 1_{e'F_0} &= 1_{c'e'F_0}=1_{F_0}. \end{aligned}$$

To prove commutativity of square (4.2) we use Yoneda Lemma and the commutativity of the following square

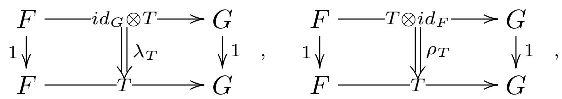
$$\begin{array}{ccc} Ff\otimes id_{FA} & \xrightarrow{\lambda'^{-1}_{Ff}\rho'_{Ff}} & id_{FB}\otimes Ff \\ {\scriptstyle 1_{Ff}\otimes 1_{id_{FA}}}\downarrow & & \downarrow{\scriptstyle 1_{id_{FB}}\otimes 1_{Ff}} \\ Ff\otimes id_{FA} & \xrightarrow{\lambda'^{-1}_{Ff}\rho'_{Ff}} & id_{FB}\otimes Ff \end{array}.$$

□

Proposition 5. *Let **C** be a 2-category and suppose $F,G:C\longrightarrow C'$ are pseudo-functors in **C**.*
For every pseudo-natural transformation

$$T=(t,\tau):F\longrightarrow G$$

there are two special pseudo-modifications

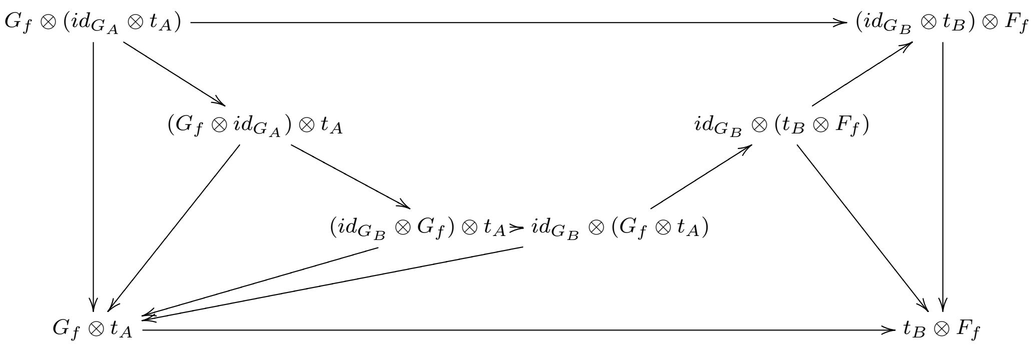


with $\lambda_T = \lambda' \circ t, \rho_T = \rho' \circ t$ *both natural in* T*.*

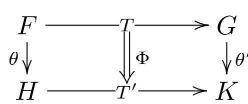
Proof. It is clear that $\lambda' \circ t : m' \langle t, e' F_0 \rangle \longrightarrow t$ is a 2-cell of $\mathbf{C}$, and

$$d' \circ \lambda' \circ t = 1_{d't} = 1_{F_0}$$
$$c' \circ \lambda' \circ t = 1_{c't} = 1_{G_0}.$$

The commutativity of square (4.2) is obtained from the commutativity of diagram



In order to prove naturality of λ_T consider a internal pseudo-modification



as defined in (4.1); then, on the one hand we have

$$\begin{aligned}\Phi \cdot (\lambda' \circ t) &= \left(1_{C_1'} \circ \Phi\right) \cdot (\lambda' \circ 1_t) \\ &= \left(1_{C_1'} \cdot \lambda'\right) \circ (\Phi \cdot 1_t) \\ &= \lambda' \circ \Phi\end{aligned}$$

and on the other hand we have

$$\begin{aligned}(\lambda' \circ t') \cdot (m' \langle e'\theta_0', \Phi \rangle) &= (\lambda' \circ t') \cdot \left(m' \left\langle e'c', 1_{C_1'} \right\rangle \circ \Phi\right) \\ &= \left(\lambda' \cdot 1_{m' \left\langle e'c', 1_{C_1'} \right\rangle}\right) \circ (1_{t'} \cdot \Phi) \\ &= \lambda' \circ \Phi.\end{aligned}$$

The proof on rho is similar. □

The three last propositions lead us to the following theorem.

Theorem 3. *Let* $\boldsymbol{C}$ *be a 2-category, and consider* C, C' *two pseudo-categories in* $\boldsymbol{C}$*. The data:*

- *objects: pseudo-functors from* C *to* C'*;*
- *morphisms: natural transformations (between pseudo-functors from* C *to* C'*);*
- *pseudo-morphisms: pseudo-natural transformations (between pseudo-functors from* C *to* C'*);*
- *cells: pseudo-modifications (between such natural and pseudo-natural transformations);*

form a pseudo-category (in Cat).

Proof. Natural transformations and pseudo-functors form a category: theorem 1. pseudo-modifications and pseudo-natural transformations also form a category: the composition is associative and has identities (that inherit the structure of 2-cells of the ambient 2-category).

For every pseudo-natural transformation $T = (t, \tau) : F \longrightarrow G$, the identity pseudo-modification is $1_T = 1_t$

$$\begin{array}{ccc} F & \xrightarrow{\quad T \quad} & G \\ {\scriptstyle 1}\downarrow & \Downarrow 1_T & \downarrow{\scriptstyle 1} \\ F & \xrightarrow{\quad T \quad} & G \end{array} .$$

For each pair of pseudo-composable pseudo-modifications Φ, Ψ, there is a (well defined - proposition 2) pseudo-composition $\Phi \otimes \Psi = m' \langle \Phi, \Psi \rangle$ satisfying (1.8)

$$(\Phi\Phi') \otimes (\Psi\Psi') = m' \langle \Phi\Phi', \Psi\Psi' \rangle$$

$$\begin{aligned} (\Phi \otimes \Psi)(\Phi' \otimes \Psi') &= (m' \langle \Phi, \Psi \rangle)(m' \langle \Phi', \Psi' \rangle) \\ &= (1_{m'} 1_{m'}) \circ (\langle \Phi, \Psi \rangle \langle \Phi', \Psi' \rangle) \\ &= m' \langle \Phi\Phi', \Psi\Psi' \rangle ; \end{aligned}$$

and $1_{T \otimes S} = 1_T \otimes 1_S$,

$$1_{m\langle t,s \rangle} = 1_m \circ 1_{\langle t,s \rangle} = 1_m \circ \langle 1_t, 1_s \rangle = m \langle 1_t, 1_s \rangle .$$

For each natural transformation $\theta : F \longrightarrow G$ there is a pseudo-modification

$$\begin{array}{ccc} F & \xrightarrow{\quad id_F \quad} & F \\ {\scriptstyle \theta}\downarrow & \Downarrow id_\theta & \downarrow{\scriptstyle \theta} \\ G & \xrightarrow{\quad id_G \quad} & G \end{array}$$

with $id_\theta = e'\theta_0$, satisfying

$$id_{1_F} = e' 1_{F_0} = 1_{e' F_0} = 1_{id_F} ,$$

$$id_{\theta'\theta} = e' \circ (\theta'_0 \theta_0) = (e' \circ \theta'_0)(e' \circ \theta_0) = id_{\theta'} id_\theta .$$

By Proposition 3 there is a special pseudo-modification $\alpha_{T,U,S} = \alpha\langle T, U, S\rangle$ for each triple of composable pseudo-natural transformations T, U, S, natural in each component and satisfying the pentagon coherence condition.

By Proposition 5 there are two special pseudo-modifications λ_T, ρ_T to each pseudo-natural transformation $T : F \longrightarrow G$, natural in T and satisfying the triangle coherence condition. □

5. Conclusion and final remarks

The mathematical object PsCat that we have just defined has the following structure:

- objects: $A, B, C, ...$
- morphisms: $f : A \longrightarrow B, ...$
- 2-cells: $\theta : f \longrightarrow g, ...(f, g : A \longrightarrow B)$
- pseudo-cells: $f \xrightarrow{T} g$, ...
- tetra cells: $$\begin{array}{ccc} f & \xrightarrow{\quad T \quad} & g \\ \theta\big\downarrow & \Downarrow \Phi & \big\downarrow\theta' \\ f' & \xrightarrow{\quad T' \quad} & g' \end{array} \quad , ...$$

where objects, morphisms and 2-cells form a 2-category and for each pair of objects A, B, the morphisms, 2-cells, pseudo-cells and tetra cells from A to B form a pseudo-category.

Two questions arise at this moment:

- What is happening from $\mathrm{PsCat}(B, C)\times\mathrm{PsCat}(A, B)$ to $\mathrm{PsCat}(A, C)$?

- What is the relation between $\mathrm{PsCat}(A \times B, C)$ and $\mathrm{PsCat}(A, \mathrm{PsCAT}(B, C))$?

The answer to the second question is easy to find out. If starting with a pseudo-functor in $\mathrm{PsCat}(A \times B, C)$, say

$$h : A \times B \longrightarrow C,$$

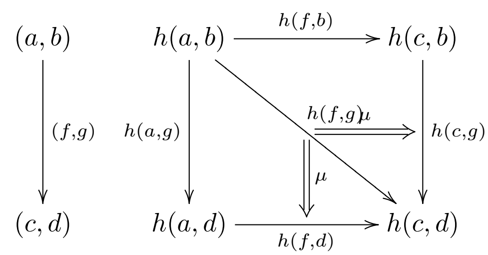
by going to $\mathrm{PsCat}\left(A, C^B\right)$ and coming back we will obtain either

$$h\,(c, g) \otimes h\,(f, b)$$

or

$$h\,(f, d) \otimes h\,(a, g)$$

instead of $h(f,g)$ as displayed in the diagram below



And since they are all isomorphic via μ and τ we have that the relation is an equivalence of categories.

A similar phenomena happens when trying to define horizontal composition of pseudo-natural transformations (while trying to answer the first question): there are two equally good ways to define a horizontal composition and they differ by an isomorphism.

Let $\mathbf{C}$ be a 2-category and C, C', C''pseudo-categories in $\mathbf{C}$, consider S, T pseudo-natural transformations as in

$$C \underset{G}{\overset{F}{\rightrightarrows}} \!\!\downarrow T\; C' \underset{G'}{\overset{F'}{\rightrightarrows}} \!\!\downarrow S\; C''$$

there are two possibilities to define horizontal composition

$$S \circ_{w1} T = m'' \langle sG_0, F_1t' \rangle$$

and

$$S \circ_{w2} T = m'' \langle G_1't, sF_0 \rangle$$

as displayed in the following picture

$$\begin{array}{ccc} C_1 & \xleftarrow{\;e\;} & C_0 \\ F_1 \big\downarrow G_1 & \swarrow t & F_0 \big\downarrow G_0 \\ C_1' & \xleftarrow{\;e'\;} & C_0' \\ F_1' \big\downarrow G_1' & \swarrow s & F_0' \big\downarrow G_0' \\ C_1'' & \xleftarrow{\;e''\;} & C_0'' \end{array} .$$

Hence we have two isomorphic functors from $\mathrm{PsCat}(B,C) \times \mathrm{PsCat}(A,B)$ to $\mathrm{PsCat}(A,C)$ both defining a horizontal composition.

We note that this behaviour, of composition beeing defined up to isomorphism, also occurs while trying to compose homotopies. So one can expect further relations between the theory of pseudo-categories and homotopy theory to be investigated.

For instance the category Top itself may be viewed as a structure with objects (spaces), morphisms (continuous mappings), 2-cells (homotopy classes of homotopies), pseudo-cells (simple homotopies) and tetra cells (homotopies between homotopies).

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