

INEQUALITIES FOR J -CONTRACTIONS INVOLVING THE α -POWER MEAN

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Abstract: A selfadjoint involutive matrix J endows \mathbb{C}^n with an indefinite inner product $[\cdot, \cdot]$ given by $[x, y] := \langle Jx, y \rangle$, $x, y \in \mathbb{C}^n$. We present some inequalities of indefinite type involving the α -power mean and the chaotic order. These results are in the vein of those obtained by E. Kamei [6, 7].



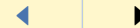
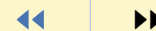
Inequalities for J -contractions

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1. Introduction

For a selfadjoint involution matrix J , that is, $J = J^*$ and $J^2 = I$, we consider \mathbb{C}^n with the indefinite Krein space structure endowed by the indefinite inner product $[x, y] := y^* Jx$, $x, y \in \mathbb{C}^n$. Let M_n denote the algebra of $n \times n$ complex matrices. The J -adjoint matrix $A^\#$ of $A \in M_n$ is defined by

$$[Ax, y] = [x, A^\# y], \quad x, y \in \mathbb{C}^n,$$

or equivalently, $A^\# = JA^*J$. A matrix $A \in M_n$ is said to be J -selfadjoint if $A^\# = A$, that is, if JA is selfadjoint. For a pair of J -selfadjoint matrices A, B , the J -order relation $A \geq^J B$ means that $[Ax, x] \geq [Bx, x]$, $x \in \mathbb{C}^n$, where this order relation means that the selfadjoint matrix $JA - JB$ is positive semidefinite. If A, B have positive eigenvalues, $\text{Log}(A) \geq^J \text{Log}(B)$ is called the J -chaotic order, where $\text{Log}(t)$ denotes the principal branch of the logarithm function. The J -chaotic order is weaker than the usual J -order relation $A \geq^J B$ [11, Corollary 2].

A matrix $A \in M_n$ is called a J -contraction if $I \geq^J A^\#A$. If A is J -selfadjoint and $I \geq^J A$, then all the eigenvalues of A are real. Furthermore, if A is a J -contraction, by a theorem of Potapov-Ginzburg [2, Chapter 2, Section 4], all the eigenvalues of the product $A^\#A$ are nonnegative.

Sano [11, Corollary 2] obtained the indefinite version of the Löwner-Heinz inequality of indefinite type, namely for A, B J -selfadjoint matrices with nonnegative eigenvalues such that $I \geq^J A \geq^J B$, then $I \geq^J A^\alpha \geq^J B^\alpha$, for any $0 \leq \alpha \leq 1$. The Löwner-Heinz inequality has a famous extension which is the Furuta inequality. An indefinite version of this inequality was established by Sano [10, Theorem 3.4] and Bebiano *et al.* [3, Theorem 2.1] in the following form: Let A, B be J -selfadjoint matrices with nonnegative eigenvalues μ and $\mu I \geq^J A \geq^J B$ (or $A \geq^J B \geq^J \mu I$) for

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some $\mu > 0$. For each $r \geq 0$,

$$(1.1) \quad \left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

and

$$(1.2) \quad \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq^J \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

hold for all $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

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2. Inequalities for α -Power Mean

For J -selfadjoint matrices A, B with positive eigenvalues, $A \geq^J B$ and $0 \leq \alpha \leq 1$, the α -power mean of A and B is defined by

$$A\sharp_{\alpha}B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\alpha} A^{\frac{1}{2}}.$$

Since $I \geq^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ (or $I \leq^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$) the J -selfadjoint power $\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\alpha}$ is well defined.

The essential part of the Furuta inequality of indefinite type can be reformulated in terms of α -power means as follows. If A, B are J -selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$, then for all $p \geq 1$ and $r \geq 0$

$$(2.1) \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq^J A$$

and

$$(2.2) \quad B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \geq^J B.$$

The indefinite version of Kamei's satellite theorem for the Furuta inequality [7] was established in [4] as follows: If A, B are J -selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$, then

$$(2.3) \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq^J B \leq^J A \leq^J B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$$

for all $p \geq 1$ and $r \geq 0$.

Remark 1. Note that by (2.3) and using the fact that $X\sharp AX \geq^J X\sharp BX$ for all $X \in M_n$ if and only if $A \geq^J B$, we have $A^{1+r} \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}}$ and $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} \geq^J B^{1+r}$.



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B^{1+r} . Applying the Löwner-Heinz inequality of indefinite type, with $\alpha = \frac{1}{1+r}$, we obtain

$$A \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1}{p+r}} \quad \text{and} \quad \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^{\frac{1}{p+r}} \geq^J B$$

for all $p \geq 1$ and $r \geq 0$.

In [4], the following extension of Kamei's satellite theorem of the Furuta inequality was shown.

Lemma 2.1. *Let A, B be J -selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$A^{-r} \#_{\frac{t+r}{p+r}} B^p \leq^J B^t \quad \text{and} \quad A^t \leq^J B^{-r} \#_{\frac{t+r}{p+r}} A^p,$$

for $r \geq 0$ and $0 \leq t \leq p$.

Theorem 2.2. *Let A, B be J -selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$(2.4) \quad A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq^J \left(A^{-r} \#_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}} \\ \leq^J B \leq^J A \leq^J \left(B^{-r} \#_{\frac{t+r}{p+r}} A^p \right)^{\frac{1}{t}} \leq^J B^{-r} \#_{\frac{1+r}{p+r}} A^p,$$

for $r \geq 0$ and $1 \leq t \leq p$.

Proof. Without loss of generality, we may consider $\mu = 1$, otherwise we can replace A and B by $\frac{1}{\mu}A$ and $\frac{1}{\mu}B$. Let $1 \leq t \leq p$. Applying the Löwner-Heinz inequality of indefinite type in Lemma 2.1 with $\alpha = \frac{1}{t}$, we get

$$\left(A^{-r} \#_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}} \leq^J B \leq^J A \leq^J \left(B^{-r} \#_{\frac{t+r}{p+r}} A^p \right)^{\frac{1}{t}}.$$



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Let $A_1 = A$ and $B_1 = \left(A^{-r} \#_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}}$. Note that

$$(2.5) \quad A^{-r} \#_{\frac{1+r}{p+r}} B^p = A^{-r} \#_{\frac{1+r}{t+r}} \left(A^{-r} \#_{\frac{t+r}{p+r}} B^p \right) = A_1^{-r} \#_{\frac{1+r}{t+r}} B_1^t.$$

Since $\mu I \geq^J A_1 \geq^J B_1$, applying Lemma 2.1 to A_1 and B_1 , with $t = 1$ and $p = t$, we obtain

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq^J B_1 = \left(A^{-r} \#_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}}.$$

The remaining inequality in (2.4) can be obtained in an analogous way using the second inequality in Lemma 2.1, with $t = 1$ and $p = t$. \square

Theorem 2.3. *Let A, B be J -selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then*

$$\left(A^{-r} \#_{\frac{t_1+r}{p+r}} B^p \right)^{\frac{1}{t_1}} \leq^J \left(A^{-r} \#_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{1}{t_2}} \quad \text{and} \quad \left(B^{-r} \#_{\frac{t_1+r}{p+r}} A^p \right)^{\frac{1}{t_1}} \geq^J \left(B^{-r} \#_{\frac{t_2+r}{p+r}} A^p \right)^{\frac{1}{t_2}}$$

for $r \geq 0$ and $1 \leq t_2 \leq t_1 \leq p$.

Proof. Without loss of generality, we may consider $\mu = 1$, otherwise we can replace A and B by $\frac{1}{\mu}A$ and $\frac{1}{\mu}B$. Let $A_1 = A$ and $B_1 = \left(A^{-r} \#_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{1}{t_2}}$. By Lemma 2.1 and the Löwner Heinz inequality of indefinite type with $\alpha = \frac{1}{t_2}$, we have $B_1 \leq^J B \leq^J A_1 \leq^J I$. Applying Lemma 2.1 to A_1 and B_1 , with $p = t_2$, we obtain

$$(2.6) \quad A_1^{-r} \#_{\frac{t_1+r}{t_2+r}} B_1^{t_2} \leq^J B_1^{t_1} = \left(A^{-r} \#_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{t_1}{t_2}}.$$

On the other hand,

$$(2.7) \quad A_1^{-r} \#_{\frac{t_1+r}{p+r}} B^p = A_1^{-r} \#_{\frac{t_1+r}{t_2+r}} \left[\left(A^{-r} \#_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{1}{t_2}} \right]^{t_2} = A_1^{-r} \#_{\frac{t_1+r}{t_2+r}} B_1^{t_2}.$$



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By (2.6) and (2.7),

$$A^{-r} \#_{\frac{t_1+r}{p+r}} B^p \leq^J \left(A^{-r} \#_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{t_1}{t_2}}.$$

Using the Löwner-Heinz inequality of indefinite type with $\alpha = \frac{1}{t_1}$, we have

$$\left(A^{-r} \#_{\frac{t_1+r}{p+r}} B^p \right)^{\frac{1}{t_1}} \leq^J \left(A^{-r} \#_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{1}{t_2}}.$$

The remaining inequality can be obtained analogously. □

Theorem 2.4. Let A, B be J -selfadjoint matrices with nonnegative eigenvalues and $\mu I \geq^J A \geq^J B$ for some $\mu > 0$. Then

$$A^{-r} \#_{\frac{t+r}{p+r}} B^p \leq^J \left(A^{-r} \#_{\frac{1+r}{p+r}} B^p \right)^t \leq^J B^t \leq^J A^t \leq^J \left(B^{-r} \#_{\frac{1+r}{p+r}} A^p \right)^t \leq^J B^{-r} \#_{\frac{t+r}{p+r}} A^p$$

for $0 \leq t \leq 1 \leq p$ and $r \geq 0$.

Proof. By the indefinite version of Kamei's satellite theorem for the Furuta inequality and since $0 \leq t \leq 1$, we can apply the Löwner-Heinz inequality of indefinite type with $\alpha = t$, to get

$$\left(A^{-r} \#_{\frac{1+r}{p+r}} B^p \right)^t \leq^J B^t \leq^J A^t \leq^J \left(B^{-r} \#_{\frac{1+r}{p+r}} A^p \right)^t.$$

Note that

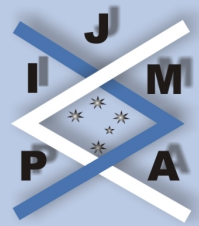
$$A^{-r} \#_{\frac{t+r}{p+r}} B^p = (A^t)^{-\frac{r}{t}} \#_{\frac{t+r}{1+r}} \left[\left(A^{-r} \#_{\frac{1+r}{p+r}} B^p \right)^t \right]^{\frac{1}{t}}.$$

Since $\mu I \geq^J A^t$, for all $t > 0$ [10] and $A^t \geq^J \left(A^{-r} \#_{\frac{1+r}{p+r}} B^p \right)^t$, applying the indefinite version of Kamei's satellite theorem for the Furuta inequality with A and B

replaced by A^t and $\left(A^{-r} \#_{\frac{1+r}{p+r}} B^p\right)^t$, respectively, and with r replaced by r/t and p replaced by $1/t$, we have

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq^J \left(A^{-r} \#_{\frac{1+r}{p+r}} B^p\right)^t.$$

The remaining inequality can be obtained analogously. □



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3. Inequalities Involving the J -Chaotic Order

The following theorem is the indefinite version of the *Chaotic Furuta inequality*, a result previously stated in the context of Hilbert spaces by Fujii, Furuta and Kamei [5].

Theorem 3.1. *Let A, B be J -selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then the following statements are mutually equivalent:*

- (i) $\text{Log}(A) \geq^J \text{Log}(B)$;
- (ii) $(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \leq^J A^r$, for all $p \geq 0$ and $r \geq 0$;
- (iii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq^J B^r$, for all $p \geq 0$ and $r \geq 0$.

Under the chaotic order $\text{Log}(A) \geq^J \text{Log}(B)$, we can obtain the satellite theorem of the Furuta inequality. To prove this result, we need the following lemmas.

Lemma 3.2 ([10]). *If A, B are J -selfadjoint matrices with positive eigenvalues and $A \geq^J B$, then $B^{-1} \geq^J A^{-1}$.*

Lemma 3.3 ([10]). *Let A, B be J -selfadjoint matrices with positive eigenvalues and $I \geq^J A$, $I \geq^J B$. Then*

$$(ABA)^\lambda = AB^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^2 B^{\frac{1}{2}} \right)^{\lambda-1} B^{\frac{1}{2}} A, \quad \lambda \in \mathbb{R}.$$

Theorem 3.4 (Satellite theorem of the chaotic Furuta inequality). *Let A, B be J -selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. If $\text{Log}(A) \geq^J \text{Log}(B)$ then*

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq^J B \quad \text{and} \quad B^{-r} \#_{\frac{1+r}{p+r}} A^p \geq^J A$$



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for all $p \geq 1$ and $r \geq 0$.

Proof. Let $\text{Log}(A) \geq^J \text{Log}(B)$. Interchanging the roles of r and p in Theorem 3.1 from the equivalence between (i) and (iii), we obtain

$$(3.1) \quad \left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{\frac{p}{p+r}} \geq^J B^p,$$

for all $p \geq 0$ and $r \geq 0$. From Lemma 3.3, we get

$$A^{-\frac{r}{2}} \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} A^{-\frac{r}{2}} = B^{\frac{p}{2}} \left[\left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{-\frac{p}{p+r}} \right]^{\frac{p-1}{p}} B^{\frac{p}{2}}.$$

Hence, applying Lemma 3.2 to (3.1), noting that $0 \leq (p-1)/p \leq 1$ and using the Löwner-Heinz inequality of indefinite type, we have

$$A^{-\frac{r}{2}} \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1+r}{p+r}} A^{-\frac{r}{2}} \leq^J B^{\frac{p}{2}} B^{1-p} B^{\frac{p}{2}} = B.$$

The result now follows easily. The remaining inequality can be analogously obtained. \square

As a generalization of Theorem 3.4, we can obtain the next characterization of the chaotic order.

Theorem 3.5. *Let A, B be J -selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A$, $\mu I \geq^J B$ for some $\mu > 0$. Then the following statements are equivalent:*

- (i) $\text{Log}(A) \geq^J \text{Log}(B)$;
- (ii) $A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq^J B^t$, for $r \geq 0$ and $0 \leq t \leq p$;
- (iii) $B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \geq^J A^t$, for $r \geq 0$ and $0 \leq t \leq p$;



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$$(iv) A^{-r} \sharp_{\frac{-t+r}{p+r}} B^p \leq^J A^{-t}, \text{ for } r \geq 0 \text{ and } 0 \leq t \leq r;$$

$$(v) B^{-r} \sharp_{\frac{-t+r}{p+r}} A^p \geq^J B^{-t}, \text{ for } r \geq 0 \text{ and } 0 \leq \delta \leq r.$$

Proof. We first prove the equivalence between (i) and (iv). By Theorem 3.1, $\text{Log}(A) \geq^J \text{Log}(B)$ is equivalent to $(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \leq^J A^r$, for all $p \geq 0$ and $r \geq 0$. Henceforth, since $0 \leq t \leq r$ applying the Löwner-Heinz inequality of indefinite type, we easily obtain

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{-t+r}{p+r}} = \left[(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \right]^{\frac{-t+r}{r}} \leq^J A^{r-t}.$$

Analogously, using the equivalence between (i) and (iii) in Theorem 3.1, we easily obtain that (i) is equivalent to (v).

(ii) \Leftrightarrow (v) Suppose that (ii) holds. By Lemma 3.3 and using the fact that $X^\# A X \geq^J X^\# B X$ for all $X \in M_n$ if and only if $A \geq^J B$, we have

$$\begin{aligned} A^{\frac{r}{2}} B^t A^{\frac{r}{2}} &\geq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{t+r}{p+r}} \\ &= A^{\frac{r}{2}} B^{\frac{p}{2}} \left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{\frac{t-p}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}}. \end{aligned}$$

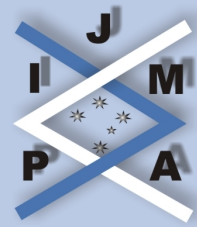
It easily follows by Lemma 3.2, that

$$B^{p-t} \leq^J \left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{\frac{-t+p}{p+r}},$$

for $r \geq 0$ and $0 \leq t \leq p$. Replacing p by r , we obtain (v). □

In an analogous way, we can prove that (v) \Leftrightarrow (iii).

Remark 2. Consider two J -selfadjoint matrices A, B with positive eigenvalues and $\mu I \geq^J A, \mu I \geq^J B$ for some $\mu > 0$. Let $1 \leq t \leq p$. Applying the Löwner



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Heinz inequality of indefinite type in Theorem 3.5 (ii) with $\alpha = \frac{1}{t}$, we obtain that $\text{Log}(A) \geq^J \text{Log}(B)$ if and only if

$$\left(A^{-r} \sharp_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}} \leq^J B.$$

Consider $A_1 = A$ and $B_1 = \left(A^{-r} \sharp_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}}$. Following analogous steps to the proof of Theorem 2.2 we have

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p = A_1^{-r} \sharp_{\frac{1+r}{t+r}} B_1^t.$$

Since $B_1 \leq^J B \leq^J \mu I$ and $A_1 \leq^J \mu I$, applying Theorem 3.5 (ii) to A_1 and B_1 , with $t = 1$ and $p = t$, we obtain $\text{Log}(A_1) \geq^J \text{Log}(B_1)$ if and only if

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq^J \left(A^{-r} \sharp_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}}.$$

Note that $\text{Log}(A_1) \geq^J \text{Log}(B_1)$ is equivalent to $\text{Log}(A) \geq^J \text{Log}(B)$, when $r \rightarrow 0^+$. In this way we can easily obtain Corollary 3.6, Corollary 3.8 and Corollary 3.8 from Theorem 3.5:

Corollary 3.6. *Let A, B be J -selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A, \mu I \geq^J B$ for some $\mu > 0$. Then $\text{Log}(A) \geq^J \text{Log}(B)$ if and only if*

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq^J \left(A^{-r} \sharp_{\frac{t+r}{p+r}} B^p \right)^{\frac{1}{t}} \leq^J B \quad \text{and} \quad A \leq^J \left(B^{-r} \sharp_{\frac{t+r}{p+r}} A^p \right)^{\frac{1}{t}} \leq^J B^{-r} \sharp_{\frac{1+r}{p+r}} A^p,$$

for $r \geq 0$ and $1 \leq t \leq p$.

Corollary 3.7. *Let A, B be J -selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A, \mu I \geq^J B$ for some $\mu > 0$. Then $\text{Log}(A) \geq^J \text{Log}(B)$ if and only if*

$$\left(A^{-r} \sharp_{\frac{t_1+r}{p+r}} B^p \right)^{\frac{1}{t_1}} \leq^J \left(A^{-r} \sharp_{\frac{t_2+r}{p+r}} B^p \right)^{\frac{1}{t_2}} \quad \text{and} \quad \left(B^{-r} \sharp_{\frac{t_1+r}{p+r}} A^p \right)^{\frac{1}{t_1}} \geq^J \left(B^{-r} \sharp_{\frac{t_2+r}{p+r}} A^p \right)^{\frac{1}{t_2}}$$



for $r \geq 0$ and $1 \leq t_2 \leq t_1 \leq p$.

Corollary 3.8. Let A, B be J -selfadjoint matrices with positive eigenvalues and $\mu I \geq^J A, \mu I \geq^J B$ for some $\mu > 0$. Then $\text{Log}(A) \geq^J \text{Log}(B)$ if and only if

$$A^{-r} \#_{\frac{t+r}{p+r}} B^p \leq^J \left(A^{-r} \#_{\frac{1+r}{p+r}} B^p \right)^t \leq^J B^t \quad \text{and} \quad A^t \leq^J \left(B^{-r} \#_{\frac{1+r}{p+r}} A^p \right)^t \leq^J B^{-r} \#_{\frac{t+r}{p+r}} A^p,$$

for $r \geq 0$ and $0 \leq t \leq 1 \leq p$.

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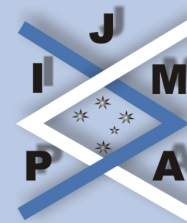
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References

- [1] T. ANDO, Löwner inequality of indefinite type, *Linear Algebra Appl.*, **385** (2004), 73–84.
- [2] T.Ya. AZIZOV AND I.S. IOKHVIDOV, *Linear Operators in Spaces with an Indefinite Metric*, Nauka, Moscow, 1986, English Translation: Wiley, New York, 1989.
- [3] N. BEBIANO, R. LEMOS, J. da PROVIDÊNCIA AND G. SOARES, Further developments of Furuta inequality of indefinite type, preprint.
- [4] N. BEBIANO, R. LEMOS, J. da PROVIDÊNCIA AND G. SOARES, Operator inequalities for J -contractions, preprint.
- [5] M. FUJII, T. FURUTA AND E. KAMEI, Furuta's inequality and its application to Ando's theorem, *Linear Algebra Appl.*, **179** (1993), 161–169.
- [6] E. KAMEI, Chaotic order and Furuta inequality, *Scientiae Mathematicae Japonicae*, **53**(2) (2001), 289–293.
- [7] E. KAMEI, A satellite to Furuta's inequality, *Math. Japon.*, **33** (1988), 883–886.
- [8] E. KAMEI, Parametrization of the Furuta inequality, *Math. Japon.*, **49** (1999), 65–71.
- [9] M. FUJII, J.-F. JIANG AND E. KAMEI, Characterization of chaotic order and its application to Furuta inequality, *Proc. Amer. Math. Soc.*, **125** (1997), 3655–3658.
- [10] T. SANO, Furuta inequality of indefinite type, *Math. Inequal. Appl.*, **10** (2007), 381–387.



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- [11] T. SANO, On chaotic order of indefinite type, *J. Inequal. Pure Appl. Math.*, **8**(3) (2007), Art. 62. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=890>]



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