Journal of Integer Sequences, Vol. 13 (2010),
Article 10.2.3

Some Extremal Postage Stamp Bases

Michael F. Challis
Hempnall House
Lundy Green Hempnall
Norwich NR15 2NU
United Kingdom
mike@hempnallhouse.co.uk

John P. Robinson
Center for Bioinformatics and Computational Biology
University of Iowa
Iowa City, IA 52242
USA
john-robinson@uiowa.edu

Abstract

A set of k positive integers is a postage stamp basis for n if every positive integer up to n can be expressed as the sum of no more than h values from the set. An extremal basis is one for which n is as large as possible. For the case $h = k = 8$, the unique extremal basis is $A = \{1, 8, 13, 58, 169, 295, 831, 1036\}$, with $n = 3485$. Several other new extremal bases are presented, along with corrections to a previous article.

1 Introduction

The global postage-stamp problem consists of determining, for given positive integers h and k, a set of k positive integers

$$A_k = \{a_1 = 1 < a_2 < \cdots < a_k\}$$

such that

(a) sums of h or fewer of these a_j can realize the numbers $1, 2, \ldots, n$, and

(b) the value of n is as large as possible.

The h-range for a particular set A_k is denoted by $n_h(A_k)$ and the extremal value by $n_h(k)$. Mossige [3] presented efficient search algorithms for determining $n_h(k)$, which Shallit [8] has shown to be NP hard in k. Various techniques (e.g., Challis [2]) have been used to reduce the effort to compute $n_h(k)$, and a further improvement is described below.

The most recent results are presented in an appendix. Corrections and improvements to Robinson [6] are also given.

2 Tree representation

A set A_k is said to be admissible (strictly, h-admissible) if $n_h(A_k) \geq a_k$; clearly inadmissible sets are of no interest when searching for the extremal value $n_h(k)$.

The natural tree representation for all admissible postage stamp sets associates $a_1 = 1$ with the root. The root branches out to h admissible nodes at the second level for the a_2 values $2, 3, \ldots, h+1$. At level three and beyond, the number of successor states for a particular state can vary. For any given admissible denomination vector A there is a path.

For the diagonal case $h = k$, all possible admissible sets were examined for the cases $h = 6, 7, 8$ and about 100 million random admissible sets for $h = 9$. Table 1 summarizes these experiments. The average fan-out is over all admissible sets. Note that the average fan-out increases down the tree and for the same level grows slowly with h. The number of admissible sets for $h = 9$ is an estimate based on the experimental average values.

	Admissible	Average fan-out by level							
h	cases	1	2	3	4	5	6	7	8
6	5.2×10^6	6	11.8	23.9	43.2	71.6			
7	5.6×10^9	7	15.0	32.5	64.0	114	234		
8	2.5×10^{13}	8	18.5	42.8	90.8	174	307	525	
9[1]	3.7×10^{17}	9	22.3	54.7	125	256	472	860	1502

Table 1: Cases and fan-out

3 Algorithm for case $h = k = 8$

The number of admissible postage stamp sets grows rapidly for the diagonal case (Table 1). Level $k - 1$ has the largest average fan-out; e.g., over 500 for $h = 8$. We extend the method of Challis [2] wherein a set A can be rejected if it can be shown that a particular needed total x cannot be represented by A. Whereas Challis uses a software cache to speed up his *generate*(x) procedure, we use look-up tables. We next outline this speed-up for $h = 8$.

[1] Estimates.

For each preamble of $a_1 = 1$ to a_7, eight intermediate look-up tables are calculated. Each table lists all totals of a_1 to a_7 using *len* denominations or less, for $1 \leq len \leq 8$. These calculations are amortized over all the admissible values of a_8. Next, the target total x is checked, in one look-up step in the table $len = 8$, to determine if some combination of a_1 to a_7 totals to x. If not, then the table $len = 7$ is checked for the total $x - a_8$; if it exists, this yields a combination totaling x using one copy of x_8. If not, another a_8 is subtracted, and the $len = 6$ table is checked for a combination using two copies of x_8. Continue down to $len = 1$, finally checking whether $8a_8 = x$. When a combination for x is found, the scan down the eight tables is exited and the entire process is repeated with another target x. If no combination equal to x is found in these nine steps, A with a_8 is rejected, and another value for a_8 is tested.

The set of target values for x are the totals needed to equal the current best A. If we generate all these totals, then $n_8(A)$ is computed, and the target set is updated. This process yielded a net speed-up factor of more than 20 over the unoptimized procedure for $h = 8$.

There are more than 10^{13} admissible sets in the $(8, 8)$ case. The unique extremal basis $A = \{1, 8, 13, 58, 169, 295, 831, 1036\}$ with $n = 3485$ was determined in 2002 by the first author and independently in 2009 by the second using the program described above. Challis [2] describes two different algorithms (the *K-program* and *H-program*) and the 2002 result was obtained using the *K-program*, whereas the 2009 result was obtained using an independent algorithm based on the *H-program*.

4 New extremal sets

Many other new results have been obtained by Challis, and a complete table of extremal bases known to him at the time of writing (September 2009) is included as Appendix A to this paper.

These results have been obtained using the algorithms described in Challis [2], but with one further improvement made to the H-program which enables a number of candidate sets A_k to be rejected as a group.

Suppose that we have already determined a reasonably good lower bound T for $n_h(k)$. In [2] we show that the following value $X < T$ is a "difficult" target for A_k: that is, in almost all cases (provided $h \gg k$) there is no generation of X as the sum of no more than h values from A_k:

$$X = (C_k - 1)a_k + (C_{k-1} - 1)a_{k-1} + \ldots + (C_2 - 1)a_2 + (a_2 - 1) \tag{1}$$

where

$$T = C_k a_k + R_k \text{ for } 0 \leq R_k < a_k,$$

and

$$a_i = C_{i-1}a_{i-1} + R_{i-1} \text{ for } 0 \leq R_{i-1} < a_{i-1},\ 3 \leq i < k.$$

We now suppose that we can find a value x such that

$$x \leq X \text{ and } x > (C_k - 2)a_k + (h - C_k + 2)a_{k-1},$$

where the second condition above means that if x is to have a valid generation as the sum of no more than h values from A_k, then it will require at least $(C_k - 1)$ values a_k. Of course there is no guarantee that we can find such values for given T and A_k, but in practice it turns out often to be the case.

Suppose further that x cannot be generated by A_k at all. We now show how under certain conditions it is possible to derive a value x' which cannot be generated by A'_k where $a'_k = a_k - r$, $r > 0$, $a'_i = a_i$, $1 \leq i < k$. As a_k decreases, so C_k may increase and C_{k-1} may decrease; other values C_i will remain unchanged. One extra condition that we require is that C_{k-1} also remains unchanged. So we have:

$$a'_k = a_k - r,\ C'_k = C_k + j,\ r > 0,\ j \geq 0$$

with

$$C'_{(k-1)} = C_{k-1} \tag{2}$$

Now consider

$$x' = x + j(a_k - r) - r(C_k - 1)$$

and suppose further (our second condition) that x', in analogy with x, satisfies

$$x' > (C'_k - 2)a'_k + (h - C'_k + 2)a_{k-1}. \tag{3}$$

We show that A'_k cannot generate x' by contradiction. Suppose that such a generation exists. Then it must include exactly $(C'_k - 1)$ values a'_k and so will be of the form:

$$x' = (C'_k - 1)a'_k + f_{k-1}a_{k-1} + \ldots + f_2a_2 + f_1$$

with

$$(C'_k - 1) + f_{k-1} + \ldots + f_1 \leq h$$

Substituting for x', C'_k and a'_k we have

$$x + j(a_k - r) - r(C_k - 1) = (C_k + j - 1)(a_k - r) + f_{k-1}a_{k-1} + \ldots + f_2a_2 + f_1$$

or

$$x = (C_k - 1)a_k + f_{k-1}a_{k-1} + \ldots + f_2a_2 + f_1 \text{ with } (C'_k - 1) + f_{k-1} + \ldots + f_1 \leq h.$$

But since $C'_k \leq C_k$ this means that x can be generated by A_k, which is contrary to hypothesis.

We now know that x' cannot be generated by A'_k, but in order to reject A'_k we must also show that x' is less than T. If we write

$$x = (C_k - 1)a_k + F,$$

we have

$$x' = (C_k - 1)a_k + F + j(a_k - r) - r(C_k - 1) = (C'_k - 1)a'_k + F.$$

From (1) we know that $F \leq (C_{k-1} - 1)a_{k-1} + \ldots + (C_2 - 1)a_2 + (a_2 - 1)$, and so

$$x' \leq (C'_k - 1)a'_k + (C_{k-1} - 1)a_{k-1} + \ldots + (C_2 - 1)a_2 + (a_2 - 1) = X'$$

where X' is the difficult target for A'_k, so $x' < T$.

Once we have discovered a suitable value x we can reject all sets A'_k for $r = 0, 1, \ldots$ without further ado so long as the two conditions (2) and (3) are met. It is not difficult to calculate when these conditions will fail, and as the average length of the "chain" of rejected sets is quite substantial for large h, the extra cost of the requisite book-keeping code is more than offset by the savings made. This is illustrated in the following table:

k	h	Average chain length	Speed increase
4	150	1088	7%
5	35	395	11%
6	14	124	9%

5 Corrections to "Some extremal postage stamp 2-bases"

Of minor importance are the following corrections to Table 1 of the original article [6]:

Stride s	Original value	Correct value
13	1,654	1,658
17	246,196	246,169
21	69,076,273	69,075,740

A counting bug was found in the original program which was most serious for $s = 21$.

The type of 2-bases investigated by Robinson [6] were also investigated by Mossige [5], who first describes preambles $\mathrm{PA}(s, a_s)$ and then describes a property *extensibility*. Consider the following family of bases derived from $\mathrm{PA}(s, a_s)$:

$$A_{p+s} = \{1, a_2, \ldots, a_s = b_0, a_s + s = b_1, a_s + 2s = b_2, \ldots, a_s + ps = b_p\}$$

We say $\mathrm{PA}(s, a_s)$ is *extensible* if A_{p+s} is admissible for all p.

Next we say that an extensible preamble $\mathrm{PA}(s, a_s)$ is *symmetricizable* if the family of the symmetric bases A_k derived from it in the obvious way (see Robinson [6]) are admissible for all $k \geq k_0$ for some k_0. It is straightforward to show that $\mathrm{PA}(s, a_s)$ is extensible if A_{p+s} is admissible for some p such that $b_{p-1} \geq 2b_0$, and that it is symmetricizable if the derived symmetric basis is admissible for some p such that $b_p \geq 2b_0$. These facts were used by Challis when coding his algorithm to determine optimal preambles $\mathrm{PA}(s, a_s)$.

What is perhaps surprising is that an extensible preamble is not necessarily symmetricizable, and this is the reason for the two errors in Table 2 of the original article: the preambles for $s = 16$ and $s = 24$, although extensible, are not symmetricizable. As an example, consider the 2-basis for $k = 52$ derived from $\mathrm{PA}(16, 61)$; it is easy to show that the value 415 cannot be generated by this basis.

Let us call a preamble that is extensible but not symmetricizable *anomalous*; then there are no anomalous preambles for $s < 14$. Even afterwards, there are very few, although the proportion increases slightly as s increases. For $s = 21$, just 1.3% of extensible preambles are anomalous.

Here are corrected versions of Table 2 and Table 3 from Robinson [6]:

Table 2: Most efficient PAs for $s = 11, \ldots, 26$

s	A
11	$\{1, 3, 4, 7, 8, 9, 16, 17, 21, 24, 35\}$
13	$\{1, 2, 5, 7, 10, 11, 19, 21, 22, 25, 29, 30, 43\}$
15	$\{1, 2, 5, 6, 8, 9, 13, 19, 22, 27, 29, 33, 40, 41, 56\}$
16	$\{1, 2, 3, 7, 8, 9, 12, 15, 22, 26, 30, 36, 37, 43, 45, 61\}$
17	$\{1, 2, 5, 6, 7, 12, 13, 16, 26, 28, 31, 37, 38, 42, 44, 49, 66\}$
19	$\{1, 2, 3, 6, 9, 11, 12, 15, 16, 27, 32, 37, 45, 48, 52, 55, 61, 62, 80\}$
20	$\{1, 2, 4, 5, 11, 13, 14, 19, 29, 35, 37, 43, 46, 47, 50, 52, 56, 58, 68, 88\}$
21	$\{1, 2, 3, 6, 10, 14, 17, 19, 26, 29, 36, 41, 49, 51, 54, 55, 58, 60, 67, 74, 95\}$
22	$\{1, 3, 5, 7, 8, 12, 14, 18, 26, 32, 33, 42, 43, 50, 60, 63, 68, 79, 81, 83, 97, 105\}$
24	$\{1, 3, 5, 6, 13, 15, 16, 18, 22, 38, 41, 44, 47, 52, 55, 58, 59, 60, 74, 80, 81, 91, 93, 117\}$
26	$\{1, 3, 4, 6, 7, 14, 16, 19, 20, 28, 36, 38, 39, 48, 49, 60, 61, 70, 76, 77, 89, 93, 95, 99, 109, 135\}$

The changes are an alternative PA(16, 61) that is symmetricizable, a new PA(17, 66) that gives rise to an optimal basis for $k = 57$, a replacement for PA(24, 118), and the addition of PA(26, 135).

Table 3: Best $n_2(A_k)$ for the symmetric bases

s	a_s	k-range	$n_2(A_k)$
11	35	30-40	$22k - 344$
13	43	40-43	$26k - 504$
15	56	43-52	$30k - 676$
16	61	52-56	$32k - 780$
17	66	56-58	$34k - 892$
19	80	58-62	$38k - 1124$
20	88	62-67	$40k - 1248$
22	105	67-80	$44k - 1516$
24	117	80-82	$48k - 1836$
26	135	82-?	$52k - 2164$

Interestingly, Mossige missed PA(16, 61) and noted that for $9 \le s \le 19$ all of the best bases were derived from preambles with odd s, so although he calculated as far as $s = 23$ he omitted $s = 20$ and $s = 22$.

Recently (see section 7), an exhaustive search for optimal 2-bases for $k = 21$ found the following to be extremal:

k	$n(2,k)$	A
21	164	$\{1,3,4,6,10,13,15,21,29,37,45,53,61,69,73,75,78,79,82,84,88\}$
		$\{1,3,4,6,10,13,15,21,29,37,45,53,61,69,73,75,78,79,80,82,84\}$
		$\{1,3,4,6,10,13,15,21,29,37,45,53,61,67,69,72,76,78,79,81,82\}$
		$\{1,3,4,5,8,14,20,26,32,38,44,50,56,62,68,74,77,78,79,81,82\}$

This is the first example for $k > 14$ where there are non-symmetric extremal solutions in addition to the expected symmetric bases.

6 Discussion

It was found that $n(8) = 3485$ is 40 larger than the next best $(8,8)$ set with $n_8(A) = 3445$. The improved rejection factor technique presented in this note, when applied to the $h = 9$ case, appears to be about 50. Our estimate of more than 10^{17} admissible cases indicates that this approach is insufficient for $h = 9$ with the current computer technology. In the random experiments it was found that $n(9) \geq 9338$, which is more than twice the Fibonacci lower bound of 4180 [1].

7 Appendix A: Table of extremal ranges $n(h,k)$ with corresponding h-bases A_k

Note that the smaller values in the following table were previously published by the first author [2].

h = 2

k	$n(2,k)$	a_i						
3	8	1	3	4				
4	12	1	3	5	6			
5	16	1	3	5	7	8		
6	20	1	2	5	8	9	10	
6	20	1	3	4	8	9	11	
6	20	1	3	4	9	11	16	
6	20	1	3	5	6	13	14	
6	20	1	3	5	7	9	10	
7	26	1	2	5	8	11	12	13
7	26	1	3	4	9	10	12	13
7	26	1	3	5	7	8	17	18

h = 2 (continued)

	k	$n(2,k)$	a_i																					
	8	32	1	2	5	8	11	14	15	16														
	8	32	1	3	5	7	9	10	21	22														
	9	40	1	3	4	9	11	16	17	19	20													
	10	46	1	2	3	7	11	15	19	21	22	24												
	10	46	1	2	5	7	11	15	19	21	22	24												
	11	54	1	2	3	7	11	15	19	23	25	26	28											
	11	54	1	2	5	7	11	15	19	23	25	26	28											
	11	54	1	3	4	9	11	16	18	23	24	26	27											
	11	54	1	3	5	6	13	14	21	22	24	26	27											
	12	64	1	3	4	9	11	16	21	23	28	29	31	32										
	13	72	1	3	4	9	11	16	20	25	27	32	33	35	36									
	14	80	1	2	5	8	11	14	17	20	23	24	25	51	53	55								
∞	14	80	1	3	4	5	8	14	20	26	32	35	36	37	39	40								
	14	80	1	3	4	9	10	15	16	21	22	24	25	51	53	55								
	15	92	1	3	4	5	8	14	20	26	32	38	41	42	43	45	46							
	16	104	1	3	4	5	8	14	20	26	32	38	44	47	48	49	51	52						
	17	116	1	3	4	5	8	14	20	26	32	38	44	50	53	54	55	57	58					
	18	128	1	3	4	5	8	14	20	26	32	38	44	50	56	59	60	61	63	64				
	19	140	1	3	4	5	8	14	20	26	32	38	44	50	56	62	65	66	67	69	70			
	20	152	1	3	4	5	8	14	20	26	32	38	44	50	56	62	68	71	72	73	75	76		
	21	164	1	3	4	6	10	13	15	21	29	37	45	53	61	69	73	75	78	79	82	84	88	
	21	164	1	3	4	6	10	13	15	21	29	37	45	53	61	69	73	75	78	79	80	82	84	
	21	164	1	3	4	6	10	13	15	21	29	37	45	53	61	67	69	72	76	78	79	81	82	
	21	164	1	3	4	5	8	14	20	26	32	38	44	50	56	62	68	74	77	78	79	81	82	
	22	180	1	3	4	6	10	13	15	21	29	37	45	53	61	69	77	81	83	86	87	90	92	96
	22	180	1	3	4	6	10	13	15	21	29	37	45	53	61	69	77	81	83	86	87	88	90	92
	22	180	1	3	4	6	10	13	15	21	29	37	45	53	61	69	75	77	80	84	86	87	89	90

h = 3

k	$n(3,k)$	a_i													
3	15	1	4	5											
4	24	1	4	7	8										
5	36	1	4	6	14	15									
6	52	1	3	7	9	19	24								
6	52	1	4	6	14	17	29								
7	70	1	4	5	15	18	27	34							
8	93	1	3	6	10	24	26	39	41						
9	121	1	3	8	9	14	32	36	51	53					
10	154	1	2	6	8	19	28	40	43	91	103				
11	186	1	2	3	8	11	26	38	56	69	85	89			
11	186	1	4	6	13	16	27	44	49	73	81	91			
12	225	1	3	8	13	15	16	49	53	84	88	108	114		
13	271	1	4	6	14	16	20	39	56	79	100	113	122	131	
14	323	1	2	4	9	15	27	38	43	46	97	107	127	147	157

h = 4

k	$n(4,k)$	a_i									
3	26	1	5	8							
4	44	1	3	11	18						
5	70	1	3	11	15	32					
6	108	1	4	9	16	38	49				
6	108	1	5	8	27	29	44				
7	162	1	4	9	24	35	49	51			
7	162	1	4	10	15	37	50	71			
7	162	1	5	8	25	31	52	71			
8	228	1	3	8	19	33	39	92	102		
9	310	1	4	10	11	28	33	78	118	143	
9	310	1	5	7	22	31	36	83	117	133	
10	422	1	4	9	24	26	42	104	115	174	185
11	550	1	4	9	20	34	52	62	137	149	229	242

h = 5

k	$n(5,k)$	a_i								
3	35	1	6	7						
4	71	1	4	12	21					
4	71	1	5	12	28					
5	126	1	4	9	31	51				
6	211	1	4	13	24	56	61			
6	211	1	5	8	33	54	67			
7	336	1	4	13	24	30	87	106		
8	524	1	6	8	33	48	77	183	236	
9	726	1	4	13	18	51	92	163	208	223
10	1016	1	6	8	21	60	93	104	154	378	414

h = 6

k	$n(6,k)$	a_i								
3	52	1	7	12						
4	114	1	4	19	33					
5	216	1	7	12	43	52				
6	388	1	7	11	48	83	115			
7	638	1	4	18	31	104	145	170		
8	1007	1	5	18	29	97	170	219	308	
9	1545	1	6	10	32	77	114	284	447	471

k = 2

Separate formulae are given for h even and h odd:

$$\begin{aligned} n(2t,2) &= t(t+3) && \text{with } a_2 = t+1 \text{ or } t+2 \\ n(2t+1,2) &= t(t+4)+2 && \text{with } a_2 = t+2 \end{aligned}$$

k = 3

h	$n(h,3)$	a_1	a_2	a_3	h	$n(h,3)$	a_1	a_2	a_3
7	69	1	8	13	15	354	1	12	52
8	89	1	9	14	16	418	1	15	54
9	112	1	9	20	17	476	1	14	61
10	146	1	10	26	18	548	1	15	80
11	172	1	9	30	19	633	1	18	65
11	172	1	10	26	20	714	1	17	91
12	212	1	11	37	21	805	1	17	91
13	259	1	13	34	22	902	1	19	102
14	302	1	12	52	22	902	1	20	92

For $h \geq 23$, $n(h,3)$ and a_i are given by the formulae:

$$\begin{aligned} a_2 &= (6t + c_{21}) \\ a_3 &= (2t + c_{31}) + (2t + c_{32})a_2 \\ n(h,3) &= (4t + c_{41}) + (2t + c_{42})a_2 \\ &\quad + (3t + c_{43})a_3 \end{aligned}$$

where $h = 9t + r$, $0 \leq r \leq 8$, and c_{ij} are given by:

r	c_{21}	c_{31}	c_{32}	c_{41}	c_{42}	c_{43}
0	3	1	1	0	0	0
1	3	1	1	0	0	1
2	5	2	1	1	0	1
3	5	2	1	1	0	2
4	7	3	1	2	0	2
5	6	2	2	2	1	2
6	8	3	2	3	1	2
7	8	3	2	3	1	3
8	10	4	2	4	1	3

k = 4

h	$n(h,4)$	a_1	a_2	a_3	a_4
7	165	1	5	24	37
8	234	1	6	25	65
9	326	1	5	34	60
10	427	1	6	41	67
11	547	1	7	48	85
12	708	1	7	48	126
13	873	1	9	56	155
14	1094	1	8	61	164
15	1383	1	12	65	240
16	1650	1	11	78	216
17	1935	1	11	90	252
18	2304	1	16	73	338
19	2782	1	10	99	360
20	3324	1	16	103	488
21	3812	1	16	103	488
22	4368	1	12	121	561
23	5130	1	14	142	659
24	5892	1	16	163	757
25	6745	1	20	149	860
26	7880	1	16	194	734
27	8913	1	21	177	1006
28	9919	1	21	177	1006
29	11081	1	19	230	870
30	12376	1	18	254	969
31	13932	1	25	211	1410

h	$n(h,4)$	a_1	a_2	a_3	a_4
32	15657	1	25	236	1585
33	17242	1	25	236	1585
34	18892	1	24	225	1734
35	21061	1	28	264	1773
36	23445	1	22	355	1700
37	25553	1	29	303	2346
38	27978	1	22	355	2361
39	31347	1	30	343	2634
40	33981	1	30	343	2634
41	36806	1	31	353	3092
42	39914	1	27	465	2692
43	43592	1	34	389	3376
44	47536	1	34	423	3682
45	51218	1	34	423	3682
46	54900	1	28	564	3261
46	54900	1	34	423	3682
47	59702	1	37	460	4004
48	63891	1	38	473	4590
49	69362	1	38	509	4986
50	74348	1	38	509	4986
51	81303	1	39	563	5448
52	86751	1	39	563	5448
53	92199	1	39	563	5448
54	97836	1	41	630	6147

For $55 \le h \le 254$, $n(h,4)$ and a_i are given by one of the following three sets of formulae:

$$(A): \quad \begin{aligned} a_2 &= (9t + c_{21}) \\ a_3 &= (4t + c_{31}) + (3t + c_{32})a_2 \\ a_4 &= (7t + c_{41}) + (2t + c_{42})a_2 + (2t + c_{43})a_3 \\ n(h,4) &= (2t + c_{51}) + (t + c_{52})a_2 + (6t + c_{53})a_3 + (3t + c_{54})a_4 \end{aligned}$$

$$(B): \quad \begin{aligned} a_2 &= (9t + c_{21}) \\ a_3 &= (2t + c_{31}) + (3t + c_{32})a_2 \\ a_4 &= (7t + c_{41}) + (2t + c_{42})a_2 + (2t + c_{43})a_3 \\ n(h,4) &= (4t + c_{51}) + (3t + c_{52})a_2 + (2t + c_{53})a_3 + (3t + c_{54})a_4 \end{aligned}$$

$$(C): \quad \begin{aligned} a_2 &= (9t + c_{21}) \\ a_3 &= (4t + c_{31}) + (3t + c_{32})a_2 \\ a_4 &= (7t + c_{41}) + (2t + c_{42})a_2 + (2t + c_{43})a_3 \\ n(h,4) &= (t + c_{51}) + (4t + c_{52})a_2 + (6t + c_{53})a_3 + (3t + c_{54})a_4 \end{aligned}$$

where $h = 12t + r$, $0 \le r \le 11$, and c_{ij} are given in the following table:

k = 4 (continued)

r		c_{21}	c_{31}	c_{32}	c_{41}	c_{42}	c_{43}	c_{51}	c_{52}	c_{53}	c_{54}	Valid for:
0	A	2	1	0	1	0	1	−3	0	4	−1	$4 \le t \le 5$
0	A	1	0	0	0	0	0	−2	0	1	1	$6 \le t \le 11$
0	B	2	2	−1	3	−1	0	−1	−2	−1	4	$12 \le t \le 21$
1	A	1	0	2	1	1	0	0	0	1	0	$5 \le t \le 21$
2	A	2	1	1	1	1	1	−3	1	4	0	$5 \le t \le 6$
2	A	1	0	2	1	1	0	0	0	1	1	$7 \le t \le 20$
2	B	5	3	−1	6	−1	0	0	−2	−1	5	$21 \le t \le 21$
3	A	3	1	2	2	1	1	−1	0	4	0	$1 \le t \le 20$
4	A	3	1	2	2	1	1	−1	0	4	1	$2 \le t \le 20$
5	A	3	1	2	2	1	1	−1	0	4	2	$4 \le t \le 20$
6	A	3	1	2	2	1	1	−1	0	4	3	$5 \le t \le 20$
7	A	7	3	2	5	1	2	−1	0	7	1	$2 \le t \le 11$
7	A	8	4	1	7	1	0	0	1	1	5	$12 \le t \le 20$
8	A	7	3	3	5	2	2	−1	1	7	1	$1 \le t \le 16$
8	A	8	4	1	7	1	0	0	1	1	6	$17 \le t \le 20$
9	A	7	3	3	5	2	2	−1	1	7	2	$1 \le t \le 20$
10	A	7	3	3	5	2	2	−1	1	7	3	$4 \le t \le 19$
10	C	11	6	1	10	1	0	0	3	0	7	$20 \le t \le 20$
11	A	10	4	3	7	2	2	0	1	7	3	$2 \le t \le 7$
11	B	11	4	2	10	1	2	3	1	1	6	$8 \le t \le 20$

The extremal bases of type A were independently investigated by Mossige [4], who developed a procedure for determining the corresponding h-range. Selmer [7] showed the others to be given by similar formulae of type B. Type C is a variant of type A.

k = 5

h	$n(h,5)$	a_1	a_2	a_3	a_4	a_5
7	345	1	8	11	64	102
8	512	1	9	15	78	115
8	512	1	9	15	80	118
9	797	1	9	23	108	181
10	1055	1	8	27	119	194
11	1475	1	10	34	165	270
12	2047	1	10	26	195	320
13	2659	1	13	34	242	409
14	3403	1	11	48	278	720
15	4422	1	14	50	325	782
16	5629	1	14	61	381	984
17	6865	1	13	67	326	1191
18	8669	1	14	75	500	1306
19	10835	1	14	89	523	1892
20	12903	1	14	102	589	1912
21	15785	1	14	88	727	2060
22	18801	1	18	97	858	2156

k = 5 (continued)

h	$n(h,5)$	a_1	a_2	a_3	a_4	a_5
23	22456	1	20	91	894	3330
24	26469	1	16	148	843	3894
25	31108	1	16	148	975	4554
26	36949	1	22	136	1168	4227
27	42744	1	22	162	1372	4889
28	49436	1	25	139	1510	5657
29	57033	1	23	170	1610	5811
30	66771	1	24	201	1718	7596
31	75558	1	23	192	1976	7018
32	86303	1	25	180	1916	8793
33	96852	1	28	202	2150	9867
34	110253	1	29	209	2434	11256
35	123954	1	27	231	2495	11464
36	140688	1	30	227	2839	12993
37	158389	1	31	234	2926	13391
38	178811	1	30	275	2947	16472
39	197293	1	29	300	3671	16677
40	223580	1	29	266	3382	18856
41	247194	1	32	294	3739	20847
42	273443	1	34	325	4133	23063
43	300747	1	33	342	4560	25414
44	331461	1	32	393	4562	25751
45	368894	1	32	457	5137	28671
46	401350	1	37	421	5602	31205
47	443231	1	33	336	5224	34431
48	490325	1	36	515	6304	35400
49	536399	1	34	422	6065	38741
50	586322	1	42	444	6906	45542
51	634430	1	36	482	7132	40052
52	699698	1	35	570	6602	50446
53	754166	1	40	462	7666	50721
54	823136	1	39	474	7840	51893
55	892139	1	42	511	8494	56238
56	968914	1	39	489	8580	65052
57	1052562	1	43	617	10091	66380
58	1150377	1	46	606	9531	72397
59	1236682	1	44	552	10237	77846
60	1325927	1	41	631	11205	74232
61	1420882	1	44	623	10432	89278
62	1547688	1	49	646	12050	91649
63	1678695	1	49	664	12338	93848
64	1782370	1	52	705	13100	99644

k = 5 (continued)

h	$n(h,5)$	a_1	a_2	a_3	a_4	a_5
65	1888725	1	48	698	12988	111755
66	2036874	1	48	746	13252	113747
67	2165553	1	51	793	14087	120914

k = 6

h	$n(h,6)$	a_1	a_2	a_3	a_4	a_5	a_6
7	664	1	7	12	64	113	193
8	1045	1	9	14	65	170	297
9	1617	1	6	31	48	256	373
10	2510	1	9	31	96	366	411
11	3607	1	7	41	105	490	815
12	5118	1	6	47	120	565	946
13	7066	1	10	35	133	759	1304
14	9748	1	11	49	188	810	2109
15	12793	1	8	71	192	1215	1993
16	17061	1	15	49	285	1292	3043
17	22342	1	13	82	387	1723	4789
18	28874	1	13	94	354	1968	5062
19	36560	1	16	87	408	2351	6452
20	45754	1	17	93	436	2898	6897
21	57814	1	14	129	469	3585	8757
22	72997	1	17	109	624	3998	9618
23	87555	1	12	117	541	4487	11496
24	106888	1	19	138	782	5346	13991
25	129783	1	19	157	896	5656	19313

k = 7

h	$n(h,7)$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
7	1137	1	7	18	62	104	244	259
7	1137	1	8	13	66	115	254	415
8	2001	1	6	28	47	127	412	602
9	3191	1	7	30	86	189	607	920
10	5047	1	6	29	96	246	857	1179
11	7820	1	10	34	153	380	1342	1487
12	11568	1	8	49	127	419	1566	2604
13	17178	1	12	40	223	544	2479	3253

k = 8

h	$n(h,8)$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
7	1911	1	4	17	31	117	209	513	550
7	1911	1	6	20	41	109	228	509	580
8	3485	1	8	13	58	169	295	831	1036

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2000 *Mathematics Subject Classification*: Primary 11B13.
Keywords: h-basis, extremal h-basis.

(Concerned with sequences A001210, A001211, A001216, A001215, A001216, A053346, and A053348.)

Received September 23 2009; revised version received November 18 2009; January 27 2010.
Published in *Journal of Integer Sequences*, January 30 2010.
