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A New Solution to the Equation $\tau(p) \equiv 0 \pmod{p}$

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Dedicated to Jean-Pierre Serre.

Abstract

The known solutions to the equation $\tau(p) \equiv 0 \pmod{p}$ were $p = 2, 3, 5, 7$, and 2411. Here we present our method to compute the next solution, which is $p = 7758337633$. There are no other solutions up to 10^{10}.

1 Introduction

Our study of the Ramanujan's tau function was inspired by the reading of [20]. Lygeros's interest in the congruences of the modular function led him to a correspondence [22] with Serre, in 1988. We identified a few research topics, one of them being the particular congruence $\tau(p) \equiv 0 \pmod{p}$. For some time, the prime integers 2, 3, 5 and 7 were regarded as the only solutions. Serre wrote us about the solution $p = 2411$ found by Newman [14] in 1972. In another letter, Serre exposed the principle of a "log log philosophy", already introduced by Atkin. After having consulted the latter, it appeared that the next solution could be of the order of 1 billion, if it existed. Several numerical approaches were considered. However, computers capabilities were still inadequate to reach 1 billion.

Nearly ten years later, Henri Cohen told us about a faster method based on the computation of Hurwitz class numbers [5, 6]. Therefore we wrote two programs: the first one generated the Hurwitz tables, and the other computed the congruences $\tau(p) \bmod p$, for successive primes p. After several months of numerical investigations on Rozier's computers, we found the next solution on March 15 2010 (see A007659 of *The On-Line Encyclopedia of Integer Sequences*).

Here we describe the algorithm, derived from the Eichler-Selberg trace formula, and give some indications on how obtaining an optimized implementation. We also establish a formula related to the Catalan triangle (A008315) and used to efficiently compute arbitrary $\tau(p)$ values for primes p up to 10^{10}. This was necessary to check the consistency of our results.

2 Ramanujan's tau function

The tau function (A000594) is defined as the Fourier coefficients of the modular discriminant

$$\Delta(z) = q\prod_{n=1}^{+\infty}(1-q^n)^{24} = \sum_{n=1}^{+\infty}\tau(n)q^n \text{ , where } q = e^{2\pi i z}. \tag{1}$$

It attracted a lot of interest from Indian mathematician Srinivasa Ramanujan. He discovered some arithmetical properties, later proved by Mordell [12]:

$$\begin{array}{ll} \tau(nm) = \tau(n)\tau(m) & \text{for } n,\, m \text{ relatively prime integers;} \\ \tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}) & \text{for } p \text{ prime and } r \text{ an integer} \geq 1. \end{array}$$

It turns out that the value of $\tau(n)$ for an integer n can be easily derived from the values $\tau(p)$ for all prime divisors p of n.

Moreover, the tau function has well-known congruences modulo 2^{11}, 3^6, 5^3, 7, 23 and 691 [2, 17, 21] . For instance, any computed value $\tau(n)$ must verify

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691} \tag{2}$$

where $\sigma_{11}(n)$ is the sum of the 11-th powers of the divisors of n.

An upper bound, also conjectured by Ramanujan and proved by Deligne in 1974, is given by

$$|\tau(p)| \leq 2p^{\frac{11}{2}} \text{ , for } p \text{ prime.} \tag{3}$$

The theory of Galois representations attached to modular forms offers deeper understanding of the congruences of Fourier coefficients [19]. In particular, the major achievements of Serre and Deligne on the subject provide the asymptotic density of primes p such that $\tau(p) \equiv 0 \pmod{l}$ for a given prime l. Recent advances even establish that $\tau(p) \bmod l$ can be computed in polynomial time in $\log(p)$ and l [7]. Nevertheless, most of the related results do not apply when $l = p$, and few are known in that case. The question whether the equation $\tau(p) \equiv 0 \pmod{p}$ has infinitely many solutions remains open [4, 13]. Such primes p are said to be not-ordinary for the τ function [8].

3 The log log philosophy

If we assume that the values $\tau(p)$ are randomly distributed modulo p for all primes p, then we can roughly evaluate the number of non-ordinary primes less than N, as follows:

$$\sum_{p\in\Pi, p\leq N} \frac{1}{p} \sim \int_e^N \frac{dt}{t\log t} = \log\log N$$

Heuristically, this number should grow very slowly to infinity as $\log\log N$. This is referred to as the "log log philosophy" [18, 22]. Furthermore, the probability that the interval $\left[10^a, 10^b\right]$, $a \leq b$, contains at least a non-ordinary prime is approximately $(b-a)/b$. Thus, if we conduct an exhaustive search between 10^6 and 10^{10}, the estimated probability of success is only $2/5$.

4 Eichler-Selberg trace formula

The modular discriminant Δ is known to be a cusp form of weight 12, in the upper-half of the complex plane. The space of such modular forms has dimension 1. Hence Δ is an eigenform of the Hecke operators $T_{12}(n)$ applying in this modular space, and $\tau(n) = TrT_{12}(n)$ for all integers n.

Let $k \geq 4$ be an even integer. We recall the Eichler-Selberg trace formula in the space of cusp forms of weight k and level 1 [3]:

$$TrT_k(n) = -\frac{1}{2}\sum_{|t|\leq 2\sqrt{n}} P_k(t,n)H(4n-t^2) - \frac{1}{2}\sum_{dd'=n}\min(d,d')^{k-1}$$

where $H(n)$ is the Hurwitz class number of the integer n and P_k is the polynomial in two variables defined by the equation $P_k(t,n) = \frac{\rho^{k-1}-\bar{\rho}^{k-1}}{\rho-\bar{\rho}}$ with $\rho+\bar{\rho} = t$ and $\rho\bar{\rho} = n$.

The computation of Eichler-Selberg trace formula is not straightforward, and we would like an expression that gives the explicit coefficients of $P_k(t,n)$ polynomials.

From previous definition, it follows that $\rho = re^{i\theta}$ with $r = \sqrt{n}$, $\cos\theta = \frac{t}{2\sqrt{n}}$ and

$$P_k(t,n) = n^{\frac{k-2}{2}}\frac{\sin\left((k-1)\theta\right)}{\sin\theta}.$$

It becomes obvious that $P_k(t,n)$ is closely related to Chebyshev polynomials U_k of the second kind and degree k:

$$P_k(t,n) = n^{\frac{k-2}{2}}U_{k-2}\left(\cos\theta\right) = n^{\frac{k-2}{2}}U_{k-2}\left(\frac{t}{2\sqrt{n}}\right)$$

By applying the general expression [1] of U_k with binomial coefficients, we get

$$P_k(t,n) = \sum_{i=0}^{\frac{k}{2}-1}(-1)^i\binom{k-2-i}{i}n^i t^{k-2-2i}.$$

Now we introduce the Hurwitz sums $s_j(n)$ defined by

$$s_j(n) = \frac{1}{2} \sum_{|t| \leq 2\sqrt{n}} t^j H(4n - t^2).$$

Thus we obtain

$$TrT_k(n) = -\sum_{i=0}^{\frac{k}{2}-1} (-1)^i \binom{k-2-i}{i} n^i s_{k-2-2i}(n) - \frac{1}{2} \sum_{dd'=n} \min(d, d')^{k-1}. \tag{4}$$

For $k = 12$ and p a prime, we get the explicit formula [3]:

$$\tau(p) = -s_{10}(p) + 9ps_8(p) - 28p^2 s_6(p) + 35p^3 s_4(p) - 15p^4 s_2(p) + p^5 s_0(p) - 1.$$

A congruence modulo p can be derived, as follows:

$$\tau(p) \equiv -s_{10}(p) - 1 \pmod{p}. \tag{5}$$

5 Hurwitz sums and Catalan numbers

For every prime p, it is known [5, 6], since Kronecker, that

$$s_0(p) = p. \tag{6}$$

Now we consider the linear system composed of equation (6) and trace formulas (4) for $k = 4, 6, 8, 10, 12$ and prime p.

The theory of cusp forms asserts that

$$TrT_k(p) = \begin{cases} 0, & \text{if } k = 4, 6, 8, 10; \\ \tau(p), \text{if } k = 12. \end{cases}.$$

This yields a triangular system of six equations and six unknown values $s_j(p)$, for $j = 0, 2, 4, 6, 8, 10$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -p & 1 & 0 & 0 & 0 & 0 \\ p^2 & -3p & 1 & 0 & 0 & 0 \\ -p^3 & 6p^2 & -5p & 1 & 0 & 0 \\ p^4 & -10p^3 & 15p^2 & -7p & 1 & 0 \\ -p^5 & 15p^4 & -35p^3 & 28p^2 & -9p & 1 \end{pmatrix} \begin{pmatrix} s_0(p) \\ s_2(p) \\ s_4(p) \\ s_6(p) \\ s_8(p) \\ s_{10}(p) \end{pmatrix} = \begin{pmatrix} p \\ -1 \\ -1 \\ -1 \\ -1 \\ -\tau(p) - 1 \end{pmatrix}$$

This system has determinant 1, and its inverse matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p & 1 & 0 & 0 & 0 & 0 \\ 2p^2 & 3p & 1 & 0 & 0 & 0 \\ 5p^3 & 9p^2 & 5p & 1 & 0 & 0 \\ 14p^4 & 28p^3 & 20p^2 & 7p & 1 & 0 \\ 42p^5 & 90p^4 & 75p^3 & 35p^2 & 9p & 1 \end{pmatrix}.$$

In the first column appear the Catalan numbers A000108 $C_n = \frac{1}{n+1}\binom{2n}{n}$, which have generating function $c(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \ldots$.

In the remaining columns, the non-zero coefficients are the first terms of the number sequences $C_n(m) = [x^n]c^m(x)$, generated by the m-th powers of $c(x)$, for $m = 3, 5, 7, 9, 11$ (see A000245, A000344). They are given by the general formula $C_n(m) = \frac{m}{m+2n}\binom{m+2n}{n}$ [9, 10].

The sequences $C_n(m)$ are widely referenced in the literature with a great variety of definitions, notations, and denominations. For instance, this number family also appears in the Catalan triangle A008315, which may also have other representations (e.g., A009766, A108786, [1] p796). Moreover, there are several combinatorial results involving Chebyshev polynomials U_k and $C_n(m)$ numbers, and we only give an inversion formula, following [1, 11, 16]:

$$\sum_{i=0}^{n} C_{n-i}(2i+1)U_{2i}(x/2) = x^{2n}.$$

The resolution of the previous system implies the Hurwitz sums below:

$$\begin{aligned}
s_2(p) &= p^2 - 1\\
s_4(p) &= 2p^3 - 3p - 1\\
s_6(p) &= 5p^4 - 9p^2 - 5p - 1\\
s_8(p) &= 14p^5 - 28p^3 - 20p^2 - 7p - 1\\
s_{10}(p) &= 42p^6 - 90p^4 - 75p^3 - 35p^2 - 9p - 1 - \tau(p).
\end{aligned}$$

As a consequence of the functional equation $c(x) = 1 + xc^2(x)$, the polynomial parts in p have the common factor $(p+1)$:

$$\begin{aligned}
s_2(p) &= (p+1)(p-1)\\
s_4(p) &= (p+1)(2p^2 - 2p - 1)\\
s_6(p) &= (p+1)(5p^3 - 5p^2 - 4p - 1)\\
s_8(p) &= (p+1)(14p^4 - 14p^3 - 14p^2 - 6p - 1)\\
s_{10}(p) &= (p+1)(42p^5 - 42p^4 - 48p^3 - 27p^2 - 8p - 1) - \tau(p)
\end{aligned}$$

Again, the above coefficients are issued from the number sequences $C_n(m)$ for $m = 1, 2, 4, 6, 8, 10$ (see A002057, A003517), with a minus sign if $m > 1$. These are essentially the same sequences as in Shapiro's Catalan triangle [23].

We have used the very last equation to compute $\tau(p)$ numerically from $s_{10}(p)$. It avoids the computation of $s_k(p)$ for any $k < 10$, except for verification.

6 Computing Hurwitz tables

The Hurwitz numbers $H(n)$ (see A058305, A058306), for integers $n > 0$, are closely related to the class numbers of binary quadratic forms $ax^2 + bxy + cy^2$ of negative discriminant $-n$. It can be proved [6] that $H(n)$ is equal to the number of integer triplets (a, b, c) determined by the four conditions

1. $4ac - b^2 = n$,
2. $|b| \leq a \leq c$,
3. if $a = |b|$ or $a = c$, then $b \geq 0$,
4. the triplets $(a, 0, a)$ (resp. (a, a, a)) are counted with weight 1/2 (resp. 1/3).

Hence $H(n)$ is a non-negative rational value p/q with denominator $q \leq 3$. Moreover, the first condition implies $H(n) = 0$ for $n \equiv 1$ or 2 (mod 4).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$H(n)$	0	0	1/3	1/2	0	0	1	1	0	0	1	4/3	0	0	2	3/2

Our algorithm is directly derived from previous properties of $H(n)$, and was written in the C language. It is indeed straightforward to compute the Hurwitz numbers for all integers of a given interval together, by generating all the triplets (a, b, c) satisfying conditions 2 and 3, and such that $4ac - b^2$ falls into the interval. Somehow, this method recalls the sieve of Eratosthenes.

During and after calculations, it is convenient to store only the integer values $6H(n)$ for $n \equiv 0$ or 3 (mod 4), $N_1 \leq n < N_2$, where N_1 and N_2 are arbitrary integers. These two parameters N_1 and N_2 control the granularity of computations. We suggest adjusting them in such a manner that the temporary tables in main memory have a smaller size than the cache memory of the CPU. Since $6H(n)$ has a value of the order of $\sqrt{n}$, if not zero, we can use 4-bytes integers to handle those data. Empirically, there exists some optimal value for the difference $N_2 - N_1$, typically between 200000 and 1000000, depending on hardware specifications. The access time in memory appears to be critical in our case.

As expected, we checked that the computation time of the Hurwitz tables for all integers in the interval $[N; N + K)$, for a constant K such that $N \gg K \gg 1$, grows like $N^{1/2}$, approximately (Table 1.).

N	time
0	0.28 s
10^6	0.49 s
10^7	01.27 s
10^8	03.83 s
10^9	12.23 s
10^{10}	43.60 s

Table 1: Computation time of Hurwitz tables between N and $N + 10^6$

In our study, we generated the Hurwitz tables up to 40 billion, with an overall size of 75 Gb in binary format. This made possible the calculation of $\tau(p)$ mod p for primes $p < 10^{10}$. We verified the validity of our tables by applying equation (6), for all primes within several intervals of 10^6 integers.

7 Computing the tau function

We computed the values $\tau(p) \bmod p$, for successive primes p, with another program also written in the language C that uses formula (5)

$$\tau(p) \equiv -s_{10}(p) - 1 \pmod{p}, \qquad \text{where } s_{10}(p) = \sum_{0<t<2\sqrt{p}} t^{10} H(4p - t^2).$$

Since all the Hurwitz tables do not fit in RAM memory, we had to load them dynamically. Obviously, it is recommended to compute many congruences $\tau(p) \bmod p$ at once. We suggest dividing the Hurwitz tables in arrays that fit in CPU cache memory and performing calculations on each array sequentially. This optimization improves the efficiency of the algorithm in case of an exhaustive search.

Similarly, we developed a program that gives the exact $\tau(p)$ values with the formula

$$\tau(p) = (p+1)(42p^5 - 42p^4 - 48p^3 - 27p^2 - 8p - 1) - s_{10}(p).$$

The upper bound (3) provides a fair estimation of the order of $\tau(p)$. Thus, we had to handle numbers with more than 50 digits. It can be achieved with the use of the multiprecision library PARI.

Clearly, both computation times grow roughly like $\sqrt{p}$, provided that the proportion of time spent to load Hurwitz tables is small. Practically, it is the case if we compute tau function for a sufficiently large set of consecutive primes (Table 2). Hence the computation time for all primes p up to a given integer N evolves like $N^{3/2}$, approximately.

N	time for $\tau(p) \bmod p$	time for $\tau(p)$
0	05 s	10 s
10^6	08 s	16 s
10^7	22 s	39 s
10^8	72 s	112 s
10^9	306 s	489 s

Table 2: Computation time of $\tau(p) \bmod p$ and $\tau(p)$ for all primes p between N and $N + 10^6$

First investigations were conducted on a 32-bit computer with a Pentium 4 processor (2.6 GHz) and 1 Gb of RAM, between August and October, 2009. We generated the values of $\tau(p) \bmod p$ for primes $p < 1.5 \cdot 10^9$. Not surprisingly, we were facing increasing technical constraints. Then we acquired a 64-bit computer with a Core i7 processor (2.93 GHz) and 6 Gb of RAM, and we installed a Linux operating system. Thanks to the presence of 4 cores (8 execution units), we could launch up to three processes for the Hurwitz tables and four processes for the congruences simultaneously.

We generated the congruences modulo p for all primes p below 10 billion. The overall computation time for the Hurwitz tables and $\tau(p) \bmod p$ values was approximately two months, on a Core i7 processor. The Hurwitz tables computation represents a third of total CPU time.

We also computed the exact $\tau(p)$ values for all primes p up to 1 billion, with a CPU time of about 35 hours, by launching two processes. This does not include the computation time of the required Hurwitz numbers, which was relatively short anyway. Finally, thousands of $\tau(p)$ values for arbitrary primes p between 10^9 and 10^{10} were computed and verified by using some of the known congruences, and systematically the congruence (2) modulo 691. It was indeed necessary to check the consistency of $\tau(p)$ mod p values using

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

Here we provide a very simple PARI/GP implementation of tau function which takes as input a prime number.

```
tau(p) = {
  tmax=floor(2*sqrt(p)); s10=0;
  for(t=1, tmax, s10+=(t^10)*qfbhclassno(4*p-t*t));
  return (p+1)*(42*p^5-42*p^4-48*p^3-27*p^2-8*p-1)-s10;
}
```

8 Results

Our first significant result was the finding of a new solution to the equation $\tau(p) \equiv -1 \pmod{p}$ for prime $p = 692881373$, on September 6 2009. The only other known solution was 5807.

$$\begin{aligned}\tau(692881373) &= -2134035447986554588547794684277099135915023500378 \\ &= -2 \cdot 3 \cdot 16183 \cdot 45826933447 \cdot 479590473338104688515299840883663\end{aligned}$$

We checked the congruences modulo 2^{12}, 3^7, 5^3, 7, 23 and 691.

Our main result is the discovery of a new prime $p = 7758337633$ such that $\tau(p)$ is divisible by p, on March 15 2010. Indeed we found that

$$\begin{aligned}\tau(7758337633) &= 3634118031125820057253378550628821747860472052772622882 \\ &= 2 \cdot 31481 \cdot 7758337633 \cdot 7439638579196209777834920016764711229817.\end{aligned}$$

We checked the congruences modulo 2^{11}, 3^6, 5^3, 7, 23 and 691. The latter result was announced on the mailing list NMBRTHRY of North Dakota HECN, and added to the sequence A007659.

Assuming the correctness of all computations, our study shows that 2, 3, 5, 7, 2411 and 7758337633 are the only prime solutions to the equation $\tau(p) \equiv 0 \pmod{p}$ less than 10^{10}.

Moreover, we give all solutions to the equations $\tau(p) \equiv q \pmod{p}$ where $|q| \leq 100$, and p a prime, $10^8 < p < 10^{10}$ (Table 3).

p	q	p	q	p	q	p	q	p	q
108306623	64	249317993	47	423822691	28	1052035739	-84	2491075429	14
117138737	-91	254519539	47	459728417	-12	1078801037	-77	3586202561	68
117718969	-40	261550153	56	463203047	80	1155439651	25	3801305863	-26
138395681	-18	315713759	31	658562939	24	1321424171	85	3981602959	-12
138576103	57	316254821	21	663781537	-7	1322068141	-89	5029365641	-94
149929751	-87	322980089	-58	692881373	-1	1433436523	-12	5267222287	-53
153096653	-36	332139911	-15	734238613	-18	1500848449	35	5825117047	-21
165453721	-31	337092443	76	781380671	-95	1818264659	-20	6606460087	-55
196770907	8	340972243	14	825440597	-66	1854155309	55	7076349307	33
217732523	-97	349624213	58	1001976247	54	1932306841	19	7289754107	10
221148401	49	359657993	-42	1044587639	-92	2338478239	93	7758337633	0

Table 3: List of pairs (p, q) verifying $\tau(p) \equiv q \pmod p$, $|q| \leq 100$, p is prime, $10^8 < p < 10^{10}$

Finally, we recall that the equation $\tau(p) \equiv 1 \pmod p$ has known solutions 11, 23 and 691. There are no other solutions up to 10^{10}.

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