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Investigating Geometric and Exponential Polynomials with Euler-Seidel Matrices

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Abstract

In this paper we use the Euler-Seidel matrix method to obtain some properties of geometric and exponential polynomials and numbers. Some new results are obtained and some known results are reproved.

1 Introduction

This work is based on the Euler-Seidel matrix method [11] which is related to algorithms, combinatorics and generating functions. This method is quite useful to investigate properties of some special numbers and polynomials.

In this paper we consider the Euler-Seidel matrix method for some combinatorial numbers and polynomials. This method is relatively easier than the most of combinatorial methods to investigate the structure of such numbers and polynomials.

We obtain new properties of geometric (or Fubini) polynomials and numbers. In addition, we use this method to find out some equations and recurrence relations of exponential (or Bell) polynomials and numbers, and Stirling numbers of the second kind. Although some results in this paper are known, this method provides different proofs as well as new identities.

We first consider a given sequence $(a_n)_{n\geq 0}$, and then determine the Euler-Seidel matrix corresponding to this sequence is recursively by the formulae

$$\begin{aligned} a_n^0 &= a_n, \quad (n \geq 0), \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1}, \quad (n \geq 0,\, k \geq 1) \end{aligned} \tag{1}$$

where a_n^k represents the kth row and nth column entry. From relation (1), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as,

$$a_0^n = \sum_{k=0}^{n} \binom{n}{k} a_k^0, \tag{2}$$

and

$$a_n^0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_0^k. \tag{3}$$

Euler [12] deduced the following proposition.

Proposition 1 (Euler)**.** *Let*

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence $(a_n^0)_{n\geq 0}$*. Then the generating function of the sequence* $(a_0^n)_{n\geq 0}$ *is*

$$\overline{a}(t) = \sum_{n=0}^{\infty} a_0^n t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right). \tag{4}$$

A similar statement was proved by Seidel [19] with respect to the exponential generating function.

Proposition 2 (Seidel)**.** *Let*

$$A(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence $(a_n^0)_{n\geq 0}$*. Then the exponential generating function of the sequence* $(a_0^n)_{n\geq 0}$ *is*

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = e^t A(t). \tag{5}$$

Dumont [11] presented several examples of Euler-Seidel matrices, mainly using Bernoulli, Euler, Genocchi, exponential (Bell) and tangent numbers. He also attempted to give a polynomial extension of the Euler-Seidel matrix method. In [9], Dil et al. obtained some identities on Bernoulli and allied polynomials by introducing polynomial extension of this method. Mező and Dil [16] gave a detailed study of the harmonic and hyperharmonic numbers using the Euler-Seidel matrix method. Moreover, some results on $r-$Stirling numbers and a new characterization of the Fibonacci sequence have been presented. In [10] Dil and Mező presented another algorithm which depends on a recurrence relation and two initial sequences. Using the algorithm which is symmetric respect to the rows and columns, they obtained some relations between Lucas sequences and incomplete Lucas sequences. Hyperharmonic numbers have been investigated as well.

2 Definitions and Notation

Now we give a summary about some special numbers and polynomials which we will need later.

Stirling numbers of the second kind.

Stirling numbers of the second kind $\left\{ n \atop k \right\}$ are defined by means of generating functions as follows [1, 6]:

$$\sum_{n=0}^{\infty} \left\{ n \atop k \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}. \tag{6}$$

2.1 Exponential polynomials and numbers

Exponential polynomials (or single variable Bell polynomials) $\phi_n(x)$ are defined by [2, 17, 18]

$$\phi_n(x) := \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k. \tag{7}$$

We refer [5] for comprehensive information on the exponential polynomials.

The first few exponential polynomials are:

$$\begin{array}{|l|} \hline \phi_0(x) = 1, \\ \hline \phi_1(x) = x, \\ \hline \phi_2(x) = x + x^2, \\ \hline \phi_3(x) = x + 3x^2 + x^3, \\ \hline \phi_4(x) = x + 7x^2 + 6x^3 + x^4. \\ \hline \end{array}. \tag{8}$$

The exponential generating function of the exponential polynomials is given by [6]

$$\sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} = e^{x\left(e^t - 1\right)}. \tag{9}$$

The well known exponential numbers (or Bell numbers) ϕ_n ([3, 6, 7, 20]) are obtained by setting $x = 1$ in (7), i.e,

$$\phi_n := \phi_n(1) = \sum_{k=0}^{n} \left\{ n \atop k \right\}. \tag{10}$$

The first few exponential numbers are:

$$\phi_0 = 1,\ \phi_1 = 1,\ \phi_2 = 2,\ \ \phi_3 = 5,\ \ \phi_4 = 15. \tag{11}$$

They form sequence A000110 in Sloane's *Encyclopedia*.

Readers may also consult the lengthy bibliography of Gould [14], where several papers and books are listed about the exponential numbers.

The following recurrence relations that we reprove with the Euler-Seidel matrix method hold for exponential polynomials [18];

$$\phi_{n+1}(x) = x\left(\phi_n(x) + \phi_n'(x)\right) \tag{12}$$

and

$$\phi_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}\phi_k(x). \tag{13}$$

2.2 Geometric polynomials and numbers

Geometric polynomials (also known as Fubini polynomials) are defined as follows [4]:

$$F_n(x) = \sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!x^k. \tag{14}$$

By setting $x = 1$ in (14) we obtain geometric numbers (or preferential arrangement numbers, or Fubini numbers) F_n ([8, 15]) as

$$F_n := F_n(1) = \sum_{k=0}^{n}\left\{ {n \atop k} \right\} k!. \tag{15}$$

The exponential generating function of the geometric polynomials is given by [4]

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}. \tag{16}$$

Let us give a short list of these polynomials and numbers as follows

$F_0(x) = 1,$
$F_1(x) = x,$
$F_2(x) = x + 2x^2,$
$F_3(x) = x + 6x^2 + 6x^3,$
$F_4(x) = x + 14x^2 + 36x^3 + 24x^4,$

and

$$F_0 = 1,\ \ F_1 = 1,\ \ F_2 = 3,\ \ F_3 = 13,\ \ F_4 = 75.$$

They form sequence A000670 in Sloane's *Encyclopedia*

Geometric and exponential polynomials are connected by the relation ([4])

$$F_n(x) = \int_0^{\infty}\phi_n(x\lambda)e^{-\lambda}d\lambda. \tag{17}$$

Now we state our results.

3 Results obtained by the matrix method

Although we define Euler-Seidel matrices as matrices of numbers, we can also consider entries of these matrices as polynomials [9]. Thus the generating functions that we mention in the statement of Seidel's proposition turn out to be two variables generating functions. Therefore from now on when we consider these generating functions as exponential generating functions of polynomials we use the notations $A(t,x)$ and $\overline{A}(t,x)$. Using these notations relation (5) becomes

$$\overline{A}(t,x) = e^t A(t,x). \tag{18}$$

3.1 Results on geometric numbers and polynomials

This part of our work contains some relations on the geometric numbers and polynomials, most of which seems to be new.

Proposition 3. *We have*

$$2F_n = \sum_{k=0}^{n} \binom{n}{k} F_k \tag{19}$$

and

$$F_n = 2\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_k \tag{20}$$

where $n \geq 1$.

Proof. Let us set the initial sequence of an Euler-Seidel matrix as the sequence of geometric numbers, i.e., $(a_n^0)_{n\geq 0} = (F_n)_{n\geq 0}$. Then we have

$$\begin{bmatrix} 1 & 1 & 3 & 13 & 75 & \cdots \\ 2 & 4 & 16 & 88 & \cdots & \\ 6 & 20 & 104 & \cdots & & \\ 26 & 124 & \cdots & & & \\ 150 & \cdots & & & & \\ \cdots & & & & & \end{bmatrix}.$$

Observing the first row and the first column we see that $a_0^n = 2F_n$, $n \geq 1$. Firstly we need a proof of this fact. Proposition 2 enables us to write

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = \frac{e^t}{2-e^t} = 2\frac{1}{2-e^t} - 1 = \sum_{n=1}^{\infty} 2F_n \frac{t^n}{n!} + 1.$$

Now comparison of the coefficients of the both sides gives $a_0^n = 2F_n$ where $n \geq 1$. Using this result with equations (2) and (3) we get (19) and (20). ∎

Proposition 4. *Geometric polynomials satisfy the following recurrence relation*

$$F_n(x) = x\sum_{k=0}^{n-1} \binom{n}{k} F_k(x). \tag{21}$$

Proof. Let us set the initial sequence of an Euler-Seidel matrix as the sequence of geometric polynomials, i.e. $(a_n^0)_{n\geq 0} = (F_n(x))_{n\geq 0}$. Thus we obtain from (18)

$$A(t,x) = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!} = \frac{1}{1 - x(e^t - 1)}$$

and

$$\overline{A}(t,x) = \frac{e^t}{1 - x(e^t - 1)}. \tag{22}$$

Then differentiation with respect to t yields

$$\overline{A}(t,x) = \left[\frac{1}{x} - \left(e^t - 1\right)\right] \frac{d}{dt} A(t,x)$$

which can equally well be written as

$$\overline{A}(t,x) = \sum_{n=0}^{\infty} \left[\frac{F_{n+1}(x)}{x} + F_{n+1}(x) - \sum_{k=0}^{n} \binom{n}{k} F_{k+1}(x)\right] \frac{t^n}{n!}.$$

Equating coefficients of $\frac{t^n}{n!}$ in the preceding equation yields

$$a_0^n = \frac{F_{n+1}(x)}{x} - \sum_{k=1}^{n} \binom{n}{k-1} F_k(x). \tag{23}$$

In view of (2) equation (23) shows the validity of (21). ∎

Remark 5. As a special case we get (19) by setting $x = 1$ in (21).

Corollary 6.

$$F_{n+1}(x) = \frac{x}{1+x} \sum_{k=0}^{n} \binom{n}{k} \left[F_k(x) + F_{k+1}(x)\right]. \tag{24}$$

Proof. With the aid of Proposition 4 we can write

$$\begin{aligned} F_{n+1}(x) &= x \sum_{k=0}^{n} \binom{n}{k} F_k(x) + x \sum_{k=1}^{n} \binom{n}{k-1} F_k(x) \\ &= x \sum_{k=0}^{n-1} \binom{n}{k} \left[F_k(x) + F_{k+1}(x)\right] + x F_n(x) \end{aligned} \tag{25}$$

Then (25) yields equation (24). ∎

Now we give an important recurrence relation for the geometric polynomials.

Proposition 7. *The following recurrence relation holds for the geometric polynomials:*

$$F_{n+1}(x) = x \frac{d}{dx} \left[F_n(x) + x F_n(x)\right]. \tag{26}$$

Proof. We may use (13) and (17) to conclude that

$$F_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}\int_0^{\infty}\phi_k(x\lambda)\lambda e^{-\lambda}d\lambda.$$

Using (7) this becomes

$$F_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}\sum_{r=0}^{k}\left\{{k \atop r}\right\}x^r\int_0^{\infty}\lambda^{r+1}e^{-\lambda}d\lambda$$

from which it follows that

$$F_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}\sum_{r=0}^{k}\left\{{k \atop r}\right\}(r+1)!x^r.$$

This can equally well be written by means of derivative as

$$F_{n+1}(x) = x\frac{d}{dx}x\sum_{k=0}^{n}\binom{n}{k}F_k(x).$$

Now equation (21) permits us to write

$$F_{n+1}(x) = x\frac{d}{dx}\left[F_n(x) + xF_n(x)\right].$$

∎

We have the following relation between the geometric polynomials and their derivatives.

Corollary 8.

$$\sum_{k=0}^{n}\binom{n}{k}xF_k'(x) = \sum_{k=1}^{n}\binom{n}{k-1}F_k(x) \tag{27}$$

Proof. Combining results of Proposition 4 and Proposition 7 gives (27). ∎

4 Some other applications of the method

In this section, using the Euler-Seidel matrix method, we are able to reprove some known identities of Stirling numbers of second kind, exponential numbers and polynomials. Using this method one can extend these results to the other sequences.

4.1 Applications to the Stirling numbers of the second kind

Setting the initial sequence of an Euler-Seidel matrix as the sequence of the Stirling numbers of the second kind, i.e., $a_n^0 = \left\{ {n \atop m} \right\}$ where m is a fixed nonnegative integer, we get the exponential generating function of the first row as

$$A(t) = \frac{(e^t - 1)^m}{m!}.$$

We obtain from (2)

$$a_0^n = \sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop m} \right\}. \tag{28}$$

Equation (5) yields

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = e^t \frac{(e^t - 1)^m}{m!}$$

which can equally be written as

$$\overline{A}(t) = \frac{d}{dt} \frac{(e^t - 1)^{m+1}}{(m+1)!}. \tag{29}$$

Comparison of the coefficients of t^n in equation (29) and consideration (28) yield the following result:

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} = \left\{ {n+1 \atop m+1} \right\}. \tag{30}$$

Hence with the help of (30) and binomial inversion formula (3) we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left\{ {k+1 \atop m+1} \right\} = \left\{ {n \atop m} \right\}. \tag{31}$$

Relations (30) and (31) can be found in [13] respectively as the equations (6.15) and (6.17) on page 265.

4.2 Applications to the exponential numbers and polynomials

4.2.1 Applications to the exponential numbers

Let us construct an Euler-Seidel matrix with the initial sequence $(a_n^0)_{n\geq 0} = (\phi_n)_{n\geq 0}$. Then we get the following Euler-Seidel matrix

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 15 & 52 & \cdots \\ 2 & 3 & 7 & 20 & \cdots & & \\ 5 & 10 & 27 & \cdots & & & \\ 15 & 37 & \cdots & & & & \\ 52 & \cdots & & & & & \\ \cdots & & & & & & \end{bmatrix}.$$

From this matrix we observe that $a_0^n = \phi_{n+1}$. Now we prove this observation using generating functions. Since $(a_n^0)_{n\geq 0} = (\phi_n)_{n\geq 0}$ we have

$$A(t) = e^{e^t - 1}.$$

Equation (5) enables us to write

$$\overline{A}(t) = e^{e^t + t - 1} = \frac{d}{dt}\left(e^{e^t - 1}\right) = \sum_{n=0}^{\infty} \phi_{n+1} \frac{t^n}{n!}. \tag{32}$$

Comparison of the coefficients of both sides in (32) gives

$$a_0^n = \phi_{n+1} \tag{33}$$

the desired result. From (2) and (33) it follows that

$$\phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k. \tag{34}$$

Now considering (3) and (34) together we obtain

$$\phi_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \phi_{k+1}. \tag{35}$$

The identity (34) can be found in [13] on page 373 and (35) is inverse binomial transform of the identity (34).

4.2.2 Applications to the exponential polynomials

Setting the initial sequence of an Euler-Seidel matrix as the sequence of exponential polynomials, i.e., $(a_n^0)_{n\geq 0} = (\phi_n(x))_{n\geq 0}$ we get following Euler-Seidel matrix

$$\begin{bmatrix} 1 & x & x + x^2 & x + 3x^2 + x^3 & \cdots \\ 1 + x & 2x + x^2 & 2x + 4x^2 + x^3 & \cdots & \\ 1 + 3x + x^2 & 4x + 5x^2 + x^3 & \cdots & & \\ 1 + 7x + 6x^2 + x^3 & \cdots & & & \\ \cdots & & & & \end{bmatrix}.$$

We claim that $xa_0^n = \phi_{n+1}(x)$. Now we prove this fact. With the aid of Proposition 2 we can write

$$\overline{A}(t, x) = e^t e^{x\left(e^t - 1\right)} = \frac{1}{x}\frac{d}{dt} e^{x\left(e^t - 1\right)}. \tag{36}$$

Comparison of the coefficients of the both sides in (36) gives

$$xa_0^n = \phi_{n+1}(x) \tag{37}$$

the desired result. Hence equalities (2) and (37) constitute a new proof of the equation (13).

Equation (3) and equation (13) show the validity of the following equation

$$x\phi_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \phi_{k+1}(x). \tag{38}$$

It is clear that equations (13) and (38) are the generalizations of equations (34) and (35), respectively.

With the help of generating functions technique we derive more results for the exponential polynomials. The following remark is a new proof of the equation (12).

Remark 9. Differentiation both sides of the equation (9) with respect to x we get

$$\sum_{n=0}^{\infty} \phi_n^{'}(x) \frac{t^n}{n!} = e^t e^{x\left(e^t-1\right)} - e^{x\left(e^t-1\right)},$$

which combines with (36) to give

$$\sum_{n=0}^{\infty} \phi_n^{'}(x) \frac{t^n}{n!} = \overline{A}(t,x) - A(t,x).$$

Then the last equation gives equation (12) by comparing coefficients.

Our last remark is about a relation between the exponential polynomials and their derivatives.

Remark 10. Employing (12) in the equation (38) we obtain

$$\sum_{k=0}^{n-1} \binom{n}{k} (-1)^k \phi_k(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \phi_k^{'}(x). \tag{39}$$

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